# Gravitational Wave Theory

## Lecture 3

**Sreekanth Harikumar**

# Overview

- General Relativity ( a warm up…)
- Linearized gravity
- Geodesic deviation
- Effect of GW on test masses
- Generation of gravitational waves
- Energy of gravitational waves
- Propagation in curved spacetime

# A warm up on General Relativity

## Einstein Hilbert Action

$$\mathcal{S} = \frac{1}{2\kappa^2}\int \sqrt{-g} R d^4x + \mathcal{S}_m$$

where $\kappa^2 = 8\pi G/c^4$, $R$ is the Ricci scalar which depends on the metric tensor $g_{\mu\nu}$

Using the variational principle one can minimize the action and obtain the field equations

## Einstein Field Equations

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}}$$

$R_{\mu\nu}$ - *Ricci tensor*

$R$ - *Ricci scalar*

$T_{\mu\nu}$- *Energy momentum Tensor*

$$R_{\alpha\beta} = R_{\alpha\mu\beta}{}^{\mu}$$

$$R = g^{\mu\nu}R_{\mu\nu}$$

Spacetime and Geometry by Sean.M.Carroll (2019)

# Geodesics

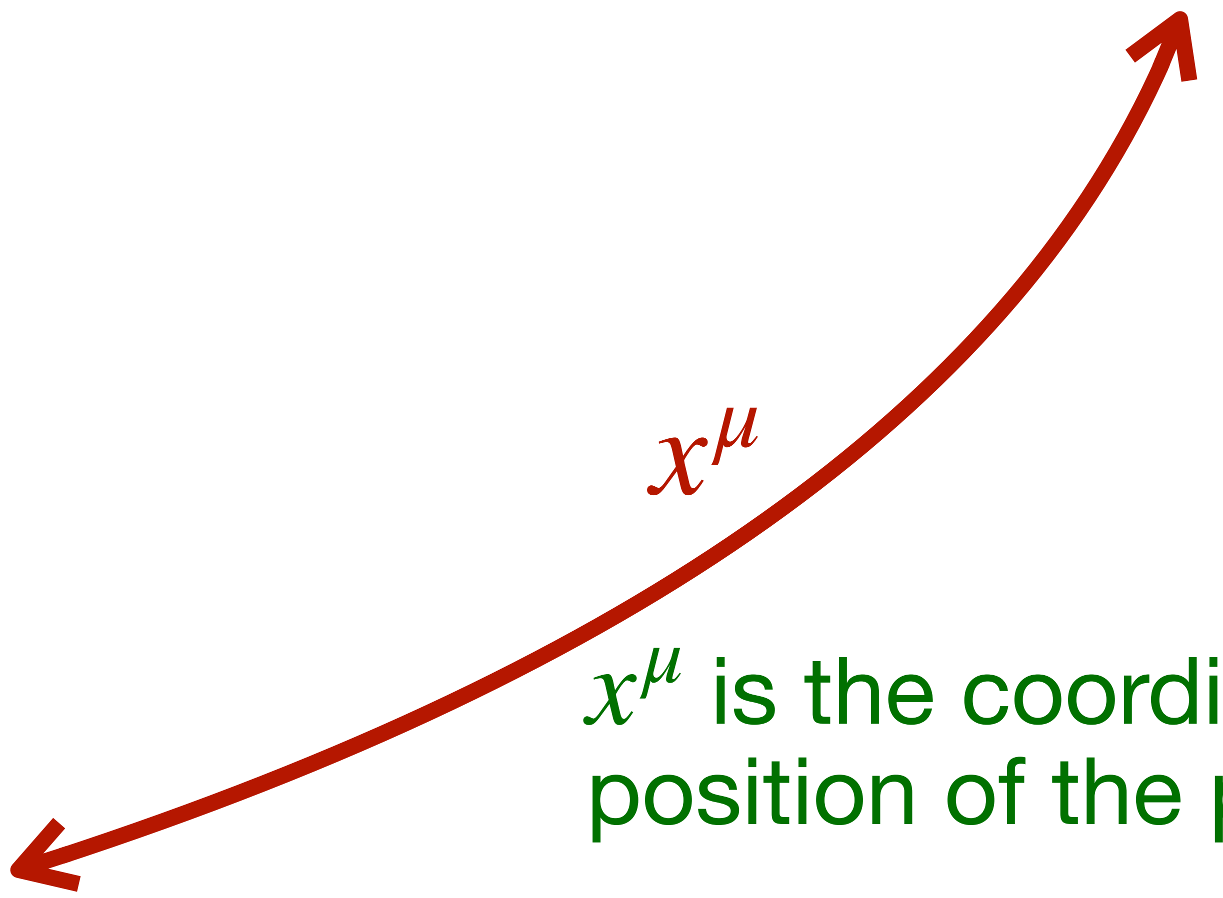
**Geodesic equation describes the path of objects in curved spacetime.**

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}=0$$

$$\Gamma^\alpha_{\mu\nu}=\frac{1}{2}g^{\alpha\gamma}(\partial_\mu g_{\gamma\nu}+\partial_\nu g_{\gamma\mu}-\partial_\gamma g_{\mu\nu})$$

**Massive particles follow time-like geodesics**

**Massless particles like photons follow null geodesics**



$x^\mu$ is the coordinate position of the particle

**Gravitational waves also follow null geodesics**

$$\frac{d^2x^\mu}{d\lambda^2}+\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}=0$$

**A third class of space-like geodesics exits which are purely mathematical**

# Linearized gravity

**Metric perturbation**

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

perturbation (GW)

Flat Minkowski background

- The perturbation is assumed to be very small $|h_{\mu\nu}| << 1$
- This conditions ensures the field is weak and the background is almost flat.
- In this linearized limit the tensor indices are raised and lowered with the background metric $\eta_{\mu\nu}$
- The weakness of this field (our GW ) ensures $h^2 \sim h_{\mu\nu}\,.\,h_{\alpha\beta}$ can be neglected

# Linearized gravity

**Metric perturbation**

$$g_{\mu\nu} = \underbrace{\eta_{\mu\nu}}_{\text{Flat Minkowski background}} + h_{\mu\nu} \longleftarrow \text{perturbation (GW)}$$

## Trace

$$h = \eta^{\mu\nu} h_{\mu\nu} = h^{\mu}_{\mu} = h^{\nu}_{\nu}$$

## Inverse metric

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

## Proof

$$\begin{aligned}
g_{\mu\nu} g^{\nu\rho} &= (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\rho} - h^{\nu\rho}) \\
&= \eta_{\mu\nu}\eta^{\nu\rho} + \eta_{\mu\nu} h^{\nu\rho} - h_{\mu\nu}\eta^{\nu\rho} - \mathcal{O}(h^2) \\
&= \delta_{\mu}^{\rho} - h_{\mu}^{\rho} + h_{\mu}^{\rho} \\
&= \delta_{\mu}^{\rho}
\end{aligned}$$

# How to obtain the field equations in this limit ?

**Linearized Einstein field equations can be obtained using the linearized metric into the Einstein Field equation and neglecting the higher order terms**

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

**Riemannian curvature tensor**

$$R^{\delta}_{\alpha\beta\gamma} = -\frac{\partial}{\partial x^{\alpha}} \Gamma^{\delta}_{\beta\gamma} + \frac{\partial}{\partial x^{\beta}} \Gamma^{\delta}_{\alpha\gamma} - \Gamma^{\delta}_{\alpha\mu} \Gamma^{\mu}_{\beta\gamma} + \Gamma^{\delta}_{\beta\mu} \Gamma^{\mu}_{\alpha\gamma}$$

**Christoffel symbol or connection**

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\gamma} (\partial_{\mu} g_{\gamma\nu} + \partial_{\nu} g_{\gamma\mu} - \partial_{\gamma} g_{\mu\nu})$$

**Ricci curvature tensor be obtained by contracting them**

$$R_{\alpha\beta} = R_{\alpha\mu\beta}{}^{\mu}$$

# Deriving the linearized field equations

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

**Let us begin with the connection**

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$$

$$= \frac{1}{2}(\eta^{\rho\sigma} - h^{\rho\sigma})\Big[\partial_\mu(\eta_{\sigma\nu} + h_{\sigma\nu}) + \partial_\nu(\eta_{\sigma\mu} + \eta_{\sigma\mu}) - \partial_\sigma(\eta_{\mu\nu} + h_{\mu\nu})\Big]$$

We can kill all derivatives of Minkowski metric $\partial_\rho \eta_{\mu\nu} = 0$

$$= \frac{1}{2}(\eta^{\rho\sigma} - h^{\rho\sigma})\Big[\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\Big]$$

$$= \frac{1}{2}\eta^{\rho\sigma}\Big[\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\Big]$$

To be still in the linearized limit we have killed the $h$ term in the first bracket.

# Field equations……first the Ricci tensor

$$R_{\mu\nu} = \partial_\mu \Gamma^\rho_{\rho\nu} - \partial_\rho \Gamma^\rho_{\mu\nu} + \cancel{\Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu}} - \cancel{\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\rho}} \,.$$

$$= \partial_\mu \Gamma^\rho_{\rho\nu} - \partial_\rho \Gamma^\rho_{\mu\nu}$$

The connection is already first order in metric perturbation $\Gamma(h)$ and therefore $\Gamma\,.\,\Gamma \sim \mathcal{O}(h^2)$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}\left[\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right]$$

$$\boxed{\begin{aligned} R_{\mu\nu} &= \partial_\mu \Gamma^\rho_{\rho\nu} - \partial_\rho \Gamma^\rho_{\mu\nu} \\ &= \frac{1}{2}\eta^{\rho\sigma}\partial_\mu\left[\partial_\rho h_{\sigma\nu} + \partial_\nu h_{\sigma\rho} - \partial_\sigma h_{\rho\nu}\right] - \frac{1}{2}\eta^{\rho\sigma}\partial_\rho\left[\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right] \\ &= \frac{1}{2}\left[\partial_\mu\partial_\nu h - \partial_\mu\partial^\rho h_{\rho\nu} - \partial^\sigma\partial_\nu h_{\sigma\mu} + \Box h_{\mu\nu}\right] \end{aligned}}$$

# Field equations……then Ricci scalar…

$$
\begin{aligned}
R &= g^{\mu\nu}R_{\mu\nu} \\
&= \eta^{\mu\nu}R_{\mu\nu} \\
&= \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\partial_\nu h - \partial_\mu\partial^\rho h_{\rho\nu} - \partial^\sigma\partial_\nu h_{\sigma\mu} + \Box h_{\mu\nu}) \\
&= \frac{1}{2}(\Box h - \partial^\nu\partial^\rho h_{\rho\nu} - \partial^\sigma\partial^\mu h_{\sigma\mu} + \Box h) \\
&= \Box\, h - \partial^\nu\partial^\rho h_{\rho\nu}
\end{aligned}
$$

( Remember all manipulation of tensor indices are done with flat metric $\eta_{\mu\nu}$ )

**Ready to patch up..**

$$R_{\mu\nu} = \frac{1}{2}\left[\partial_\mu\partial_\nu h - \partial_\mu\partial^\rho h_{\rho\nu} - \partial^\sigma\partial_\nu h_{\sigma\mu} + \Box h_{\mu\nu}\right]$$

$$R = \Box\, h - \partial^\nu\partial^\rho h_{\rho\nu}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

This is a system of partial differential equations second order in $h_{\mu\nu}$

$$\partial_\mu\partial_\nu h - \partial_\mu\partial^\rho h_{\rho\nu} - \partial^\sigma\partial_\nu h_{\sigma\mu} + \Box h_{\mu\nu} - \eta_{\mu\nu}(\Box h - \partial^\sigma\partial^\tau h_{\sigma\tau}) = 2\kappa T_{\mu\nu}$$

# Trace reversed tensor

Full linearized field equation

$$\partial_\mu\partial_\nu h - \partial_\mu\partial^\rho h_{\rho\nu} - \partial^\sigma\partial_\nu h_{\sigma\mu} + \Box h_{\mu\nu} - \eta_{\mu\nu}(\Box h - \partial^\sigma\partial^\tau h_{\sigma\tau}) = -2\kappa T_{\mu\nu}$$

**Trace reversed tensor** $$\boxed{\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$ is the trace. It is worth noting that the trace of the above equation is $\gamma = \eta^{\mu\nu}\bar{\gamma}_{\mu\nu} = -h$

Substituting this $h_{\mu\nu} = \gamma_{\mu\nu} - \frac{\gamma}{2}\eta_{\mu\nu}$ in the above linearized field equation..

$$\Box\gamma_{\mu\nu} - \partial_\mu\partial^\rho\gamma_{\rho\nu} - \partial_\nu\partial^\rho\gamma_{\rho\mu} + \eta_{\mu\nu}\partial^\sigma\partial^\tau\gamma_{\sigma\tau} = -2\kappa T_{\mu\nu}$$

Still looks complicated!!

# Hilbert gauge or Harmonic gauge ( $\partial^\alpha \gamma_{\alpha\beta} = 0$ )

Like in EM theory one can choose $\partial^\alpha \gamma_{\alpha\beta} = 0$

$$\Box \gamma_{\mu\nu} - \cancel{\partial_\mu \partial^\rho} \gamma_{\rho\nu} - \cancel{\partial_\nu \partial^\rho} \gamma_{\rho\mu} + \eta_{\mu\nu} \cancel{\partial^\sigma \partial^\tau} \gamma_{\sigma\tau} = -2\kappa T_{\mu\nu}$$

$$\boxed{\Box \gamma_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}}$$

**Final wave equation with some conditions to be met**

It is valid only if Harmonic gauge condition is satisfied. What gives us the guarantee that it is true ?

They where Gauge transformations play an important role….

**The Hilbert Gauge gives us four conditions which reduces the 10 independent components to 6**

# Gauge transformations

$$x'^{\alpha} \to x^{\alpha} + \xi^{\alpha}(x^{\beta})$$

$$g'_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}} g_{\mu\nu}$$

$$g'_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \partial_{\alpha}\xi_{\beta} - \partial_{\beta}\xi_{\alpha} + \mathcal{O}(h^2)$$

Therefore, the perturbation transforms as

$$h'_{\alpha\beta} = h_{\alpha\beta} - \partial_{\alpha}\xi_{\beta} - \partial_{\beta}\xi_{\alpha}$$

The trace reversed tensor becomes

$$\gamma'_{\alpha\beta} = \gamma_{\alpha\beta} - \partial_{\alpha}\xi_{\beta} - \partial_{\beta}\xi_{\alpha} + \eta_{\alpha\beta}\partial_{\mu}\xi^{\mu}$$

The function $\xi^{\alpha}$ is chosen in such a way that it satisfies $|\partial_{\beta}\xi^{\alpha}| \ll 1$

$$\frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta} + \partial_{\beta}\xi^{\alpha}$$

$$\frac{\partial x^{\alpha}}{\partial x'^{\beta}} = \delta^{\alpha}_{\beta} + \partial_{\beta}\xi^{\alpha} + \mathcal{O}((\partial\xi)^2)$$

The condition $|\partial_{\beta}\xi^{\alpha}| \ll 1$ guarantee that the new metric perturbation $h'_{\alpha\beta}$ or $\gamma'_{\alpha\beta}$ to be small

# Gauge transformations

$$\gamma'_{\alpha\beta} \longrightarrow \gamma_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha + \eta_{\alpha\beta}\partial_\mu \xi^\mu$$

Apply Harmonic gauge condition here… $\partial^\alpha \gamma'_{\alpha\beta} = 0$

$$\partial^\alpha \gamma'_{\alpha\beta} \longrightarrow \partial^\alpha \gamma_{\alpha\beta} - \Box \xi_\beta$$

If the gauge condition has to be valid in the new frame $\partial^\alpha \gamma'_{\alpha\beta} = 0$

$$\boxed{\Box\, \xi_\beta = f_\beta}$$

This equation always admit solutions because d'Alembertian operator is invertible

$$\xi_\beta(x) = \int d^4x\ G(x-y) f_\beta(y)$$

**Why TT gauge ?**

- It contains only non-gauge information
- No unphysical modes

# Further simplification

$$\Box\gamma_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} \longrightarrow$$

**Vacuum solutions**

$$\Box\gamma_{\mu\nu} = 0$$

- Outside the source, $T_{\mu\nu} = 0$
- A simple solution to this is the Plane Wave
- GWs travel at the speed of light

The coordinate transformation did not completely fix the gauge

$$\partial^\alpha\gamma'_{\alpha\beta} \longrightarrow \partial^\alpha\gamma_{\alpha\beta} - \Box\xi_\beta$$

Choosing $\partial^\alpha\gamma_{\alpha\beta} = 0$ , means $\Box\,\xi_\beta = 0$

$$\boxed{\gamma'_{\alpha\beta} \longrightarrow \gamma_{\alpha\beta} - \partial_\alpha\xi_\beta - \partial_\beta\xi_\alpha + \eta_{\alpha\beta}\partial_\mu\xi^\mu}$$

This means from he remaining six independent components of $\gamma_{\mu\nu}$ we can subtract functions

$$-\partial_\alpha\xi_\beta - \partial_\beta\xi_\alpha + \eta_{\alpha\beta}\partial_\mu\xi^\mu$$

# Transverse -Traceless ( TT ) gauge

We can choose functions so as to impose four conditions on $\gamma_{\mu\nu}$

**Condition 1**

- Choose $\xi^0$ in such a way that $\gamma = 0$
- If trace vanishes $\gamma_{\mu\nu} = h_{\mu\nu}$

**Condition 2**

- Choose $\xi^i$ so that $\gamma^{0i} = 0$
- Now using Lorenz condition we can see that

$$\partial^0 \gamma_{00} + \partial^i \gamma_{0i} = 0$$

$$\partial^0 \gamma_{00} = 0$$

This means $\gamma_{00}$ is constant in time and are related to static Newtonian potential

$$\gamma^{0\mu} = 0$$

**Transverse -Traceless ( TT ) gauge**

$$\boxed{h^{0\mu} = 0, \quad h^i_i = 0, \quad \partial^i h_{ij} = 0}$$

**Important**

TT gauge cannot be imposed inside the source since $T_{\mu\nu} \neq 0$

# Similarities with Maxwell's Electrodynamics

**The linearized Einstein theory is a Lorentz invariant theory of the gravitational field on Minkowski spacetime.**

| **Linearized gravity** | **Electrodynamics** |
|---|---|
| $\gamma_{\mu\nu}$ | $A_\mu$ |
| $T_{\mu\nu}$ | $J_\mu$ |
| Hilbert gauge $\partial^\mu \gamma_{\mu\nu} = 0$ | Lorenz gauge $\partial^\mu A_\mu = 0$ |
| $\Box \gamma_{\mu\nu} = -2\kappa T_{\mu\nu}$ | $\Box A_\mu = \mu_0^{-1} J_\mu$ |

**Important :** Linearized gravity is only approximately Lorentz invariant

**EM field equation** $\partial_\mu F^{\mu\nu} = j^\nu$

$$\partial_\mu(\partial_\mu A^\nu - \partial_\nu A^\mu) = j^\nu$$

$$\Box A^\nu = j^\nu$$

Lorentz gauge is used $\partial_\mu A^\mu = 0$

$$A_\mu \longrightarrow A_\mu - \partial_\mu \theta$$

$\Box \theta = 0$ can be used to set $A^0 = 0$

$$A^i, \quad \partial_i A^i = 0$$

# Polarization

**Transverse -Traceless ( TT ) gauge**

$$\boxed{h^{0\mu} = 0, \quad h^i_i = 0, \quad \partial^i h_{ij} = 0}$$

Two independent components 2 . These are known as the “plus” and “cross” modes

$$h^{TT}_{\mu\nu}(t, \mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(t, \mathbf{x}) & h_\times(t, \mathbf{x}) & 0 \\ 0 & h_\times(t, \mathbf{x}) & -h_+(t, \mathbf{x}) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

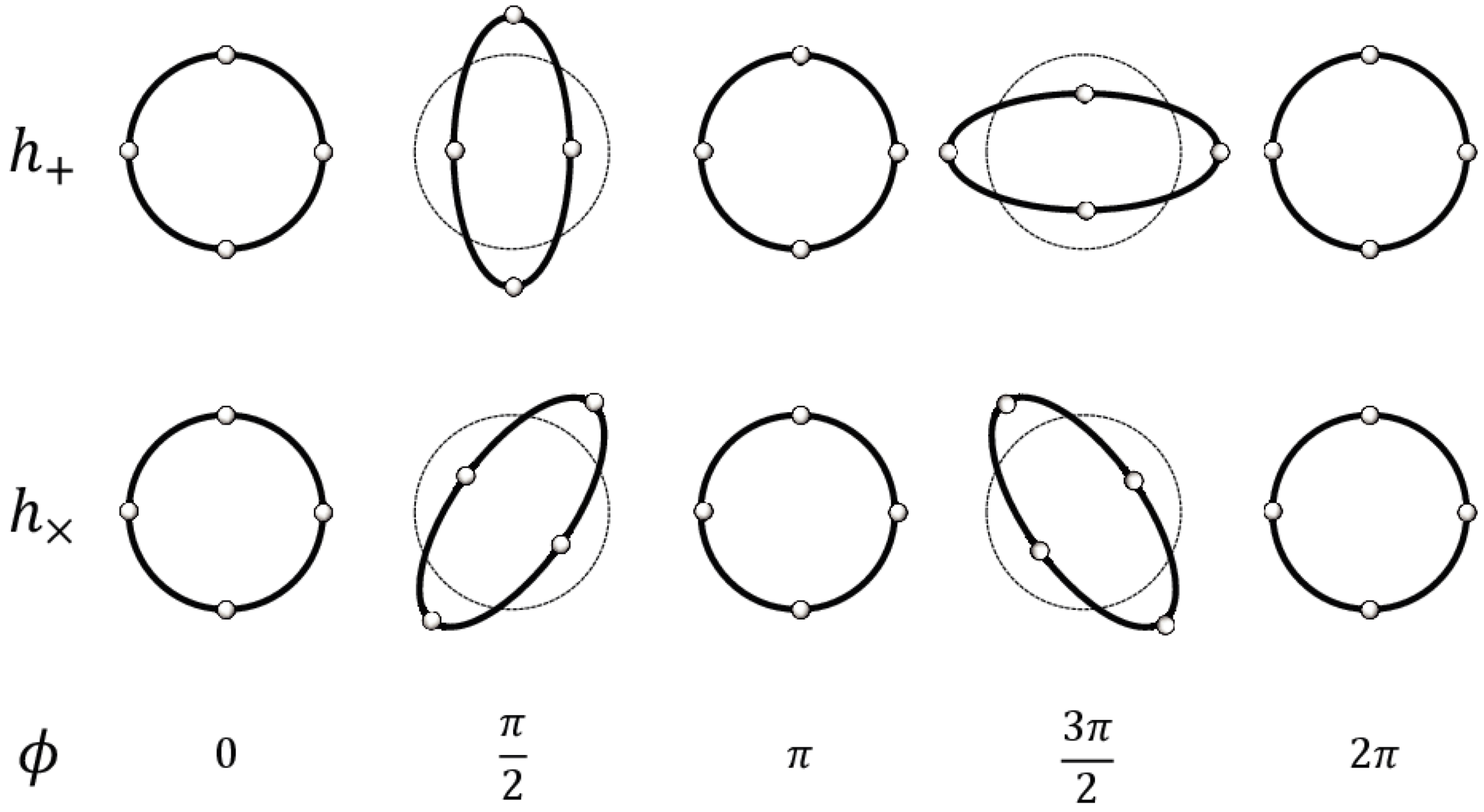


Image credit: Eur. Phys. J. Plus 132 (2017)

# Geodesic deviation

Consider two objects(mirrors) freely falling at points $\mathscr{A}$ and $\mathscr{B}$ having four-velocity $u^\alpha(\mathscr{A})$ and $u^\beta(\mathscr{B})$respectively. Have $\xi^\mu$as the separation vector connecting these two geodesics, one through each objects. Then the rate of change of the separation vector is given by

$$v^\alpha = \frac{d\xi^\alpha}{dt} = \frac{d\xi^\alpha}{dt}\frac{dx^\mu}{dx^\mu} = u^\mu \nabla_\mu \xi^\alpha$$

$$v^\alpha = \xi^\mu \nabla_\mu u^\alpha$$

## Non-commutativity property

$$\boxed{[\nabla_\mu, \nabla_\nu]u^\rho = \nabla_\mu \nabla_\nu u^\rho - \nabla_\nu \nabla_\mu u^\rho = R^\rho_{\sigma\mu\nu}\mu^\sigma}$$

$$\begin{aligned} a^\alpha = \frac{dv^\alpha}{dt} &= u^\alpha \nabla_\rho (\xi^\mu \nabla_\mu u^\alpha) \\ &= (u^\alpha \nabla_\rho \xi^\mu)(\nabla_\mu u^\alpha) + \xi^\mu u^\rho \nabla_\rho \nabla_\mu u^\alpha \\ &= (\xi^\mu \nabla_\rho u^\alpha)(\nabla_\mu u^\alpha) + \xi^\mu u^\rho \nabla_\mu \nabla_\rho u^\alpha + \xi^\mu u^\rho u^\nu R_{\mu\rho\nu}{}^{\alpha} \\ &= (\xi^\mu \nabla_\rho u^\alpha)(\nabla_\mu u^\alpha) + \xi^\mu u^\rho u^\nu R_{\mu\rho\nu}{}^{\alpha} + \xi^\mu \nabla_\mu (u^\rho \nabla_\rho u^\alpha) - (\xi^\mu \nabla_\mu u^\rho)(\nabla_\rho u^\alpha) \end{aligned}$$

The term $u^\rho \nabla_\rho u^\alpha$ vanishes as a consequence of geodesic equations and the last two terms mutually cancel, giving as the equation of geodesic deviation.

$$\boxed{a^\alpha = -R^\alpha_{\mu\rho\nu}u^\mu \xi^\rho u^\nu}$$

# Effect on test masses

## Geodesic deviation

$$a^\alpha = -R^\alpha_{\mu\rho\nu} u^\mu \xi^\rho u^\nu$$

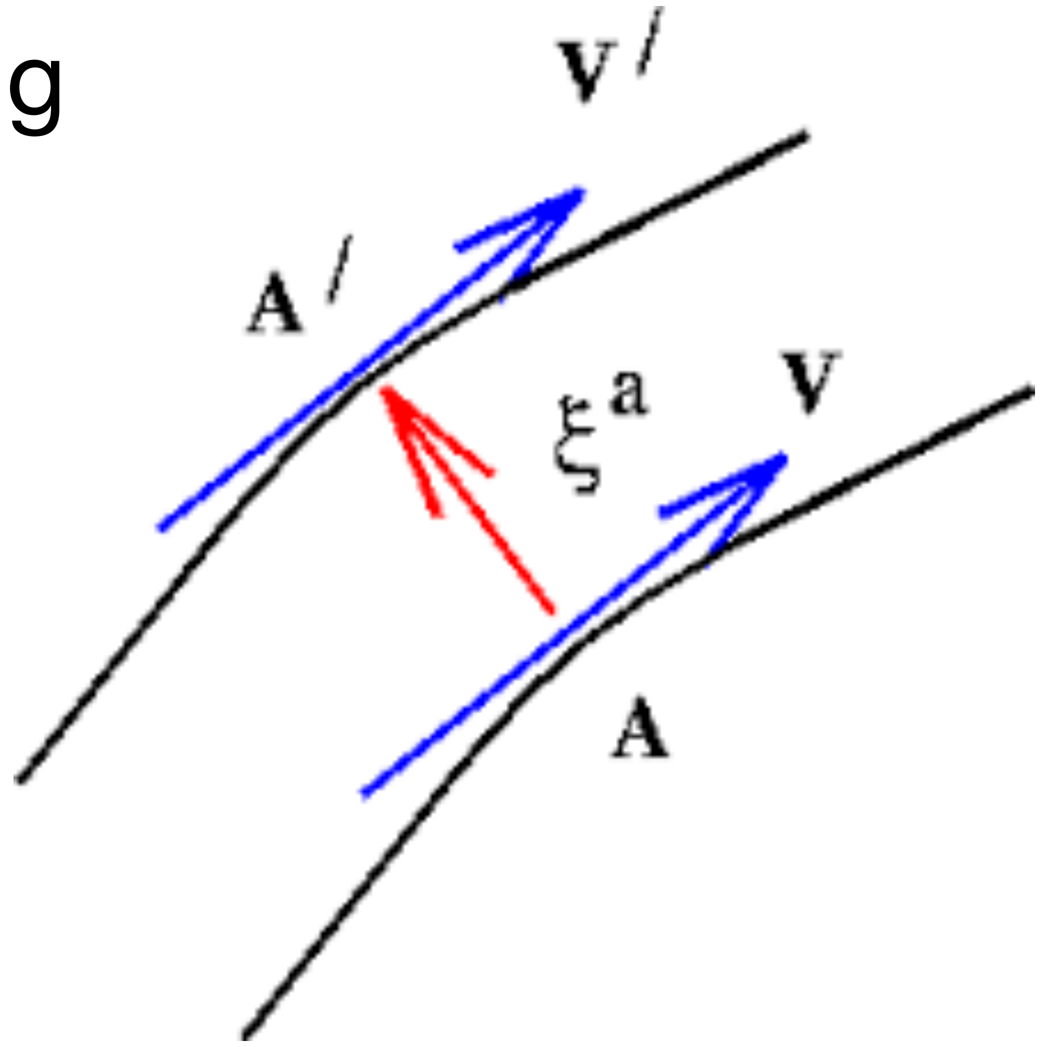
$$\frac{d^2\xi^i}{d\tau^2} = -c^2 R^i_{0j0}\xi^j$$

$$\boxed{\ddot{\xi}^i = \frac{1}{2}\ddot{h}^{TT}_{ij}\xi^j}$$

The separation vector $\xi^\mu$ evolves dynamically according to the wave. This is what detectors measure

Consider test masses to be slowly moving

$$u^\mu \approx (1,0,0,0)$$



### In the TT frame

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}\left(\partial_\alpha\partial_\nu h_{\mu\beta} + \partial_\beta\partial_\mu h_{\nu\alpha} - \partial_\beta\partial_\nu h_{\mu\alpha} - \partial_\alpha\partial_\mu h_{\nu\beta}\right)$$

$$R_{i0j0} = \frac{1}{2}\left(\partial_j\partial_0 h_{i0} + \partial_0\partial_i h_{0j} - \partial_0\partial_0 h_{ij} - \partial_j\partial_i h_{00}\right)$$

**Driving force matrix**

# Ring of test particles and polarization (plus mode)

## Geodesic deviation

$$\frac{d^2\xi^i}{dt^2} = \frac{1}{2}\ddot{h}_{ij}^{TT}(t)\,\xi^j$$

$$\boxed{\begin{aligned} \frac{d^2\xi^x}{dt^2} &= \frac{1}{2}\left(\ddot{h}_+\xi^x + \ddot{h}_\times\xi^y\right) \\ \frac{d^2\xi^y}{dt^2} &= \frac{1}{2}\left(\ddot{h}_\times\xi^x - \ddot{h}_+\xi^y\right) \end{aligned}}$$

## Solution

$$\xi^i(t) = \left(\delta^i_{\ j} + \frac{1}{2}h_{ij}^{TT}(t)\right)\xi_0^j$$

$$h_{ij}^{TT}(t) = \begin{pmatrix} h_+(t) & h_\times(t) & 0 \\ h_\times(t) & -h_+(t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

## Arrange a ring of test particles on a circle

$$\xi_0^x = R\cos\phi, \quad \xi_0^y = R\sin\phi$$

Kill one of the polarization $h_\times = 0$

$$\xi^x(t) = \left(1 + \frac{1}{2}h_+(t)\right)R\cos\phi$$

$$\xi^y(t) = \left(1 - \frac{1}{2}h_+(t)\right)R\sin\phi$$

- Stretches in $x$-direction
- Compresses in $y$-direction
- The ring becomes an ellipse aligned with $x - y$ axis

# Ring of test particles and polarization (corss mode )

## Geodesic deviation

$$\frac{d^2\xi^i}{dt^2} = \frac{1}{2}\ddot{h}^{TT}_{ij}(t)\,\xi^j$$

$$\boxed{\begin{aligned}\frac{d^2\xi^x}{dt^2} &= \frac{1}{2}\left(\ddot{h}_+\xi^x + \ddot{h}_\times\xi^y\right)\\ \frac{d^2\xi^y}{dt^2} &= \frac{1}{2}\left(\ddot{h}_\times\xi^x - \ddot{h}_+\xi^y\right)\end{aligned}}$$

## Solution

$$\xi^i(t) = \left(\delta^i{}_j + \frac{1}{2}h^{TT}_{ij}(t)\right)\xi^j_0$$

$$h^{TT}_{ij}(t) = \begin{pmatrix} h_+(t) & h_\times(t) & 0 \\ h_\times(t) & -h_+(t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

## Arrange a ring of test particles on a circle

$$\xi^x_0 = R\cos\phi, \quad \xi^y_0 = R\sin\phi$$

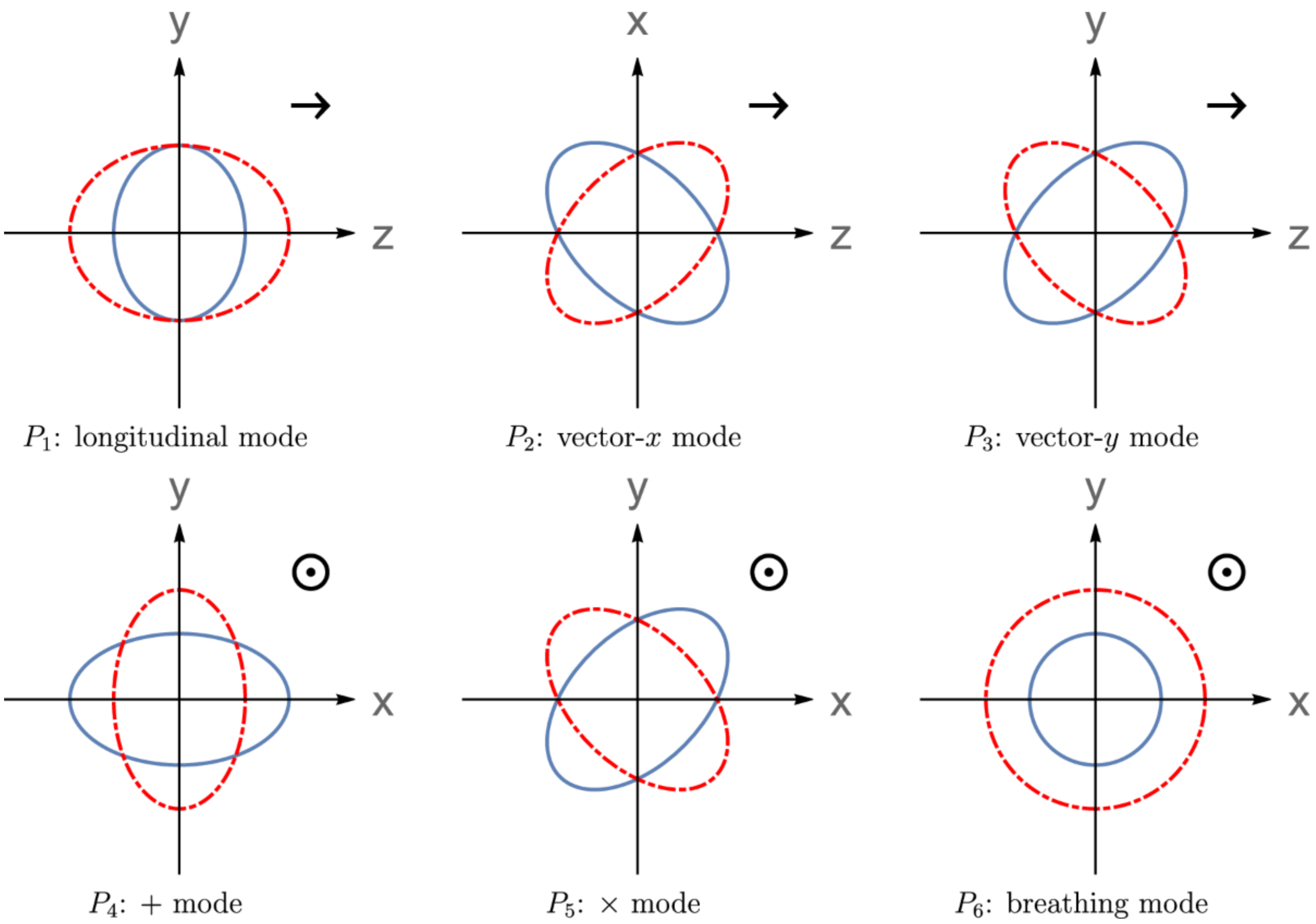
Kill the other polarization $h_+ = 0$

$$\xi^x(t) = R\cos\phi + \frac{1}{2}h_\times(t)\,R\sin\phi$$

$$\xi^y(t) = R\sin\phi + \frac{1}{2}h_\times(t)\,R\cos\phi$$

- Stretching happens along diagonals
- The ellipse is rotated by $45\,\mathrm{deg}$
- The ring gets distorted into a tilted ellipse

# GWs in alternative theories



**Crtedit: Zi-Fan et. Al. Eur. Phys. J. C 85, 65 (2025)**

# Generation of gravitational waves

Analogous to retarded potential in electromagnetism

$$\Box\gamma_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} \quad\longrightarrow\quad \gamma^{\mu\nu}(ct,\vec{r}) = \frac{1}{4\pi}\int_{R^3}\frac{2\kappa T^{\mu\nu}\left(ct-|\vec{r}\,'-\vec{r}\,|,\vec{r}\,'\right)d^3\vec{r}\,'}{|\vec{r}\,'-\vec{r}\,|}$$

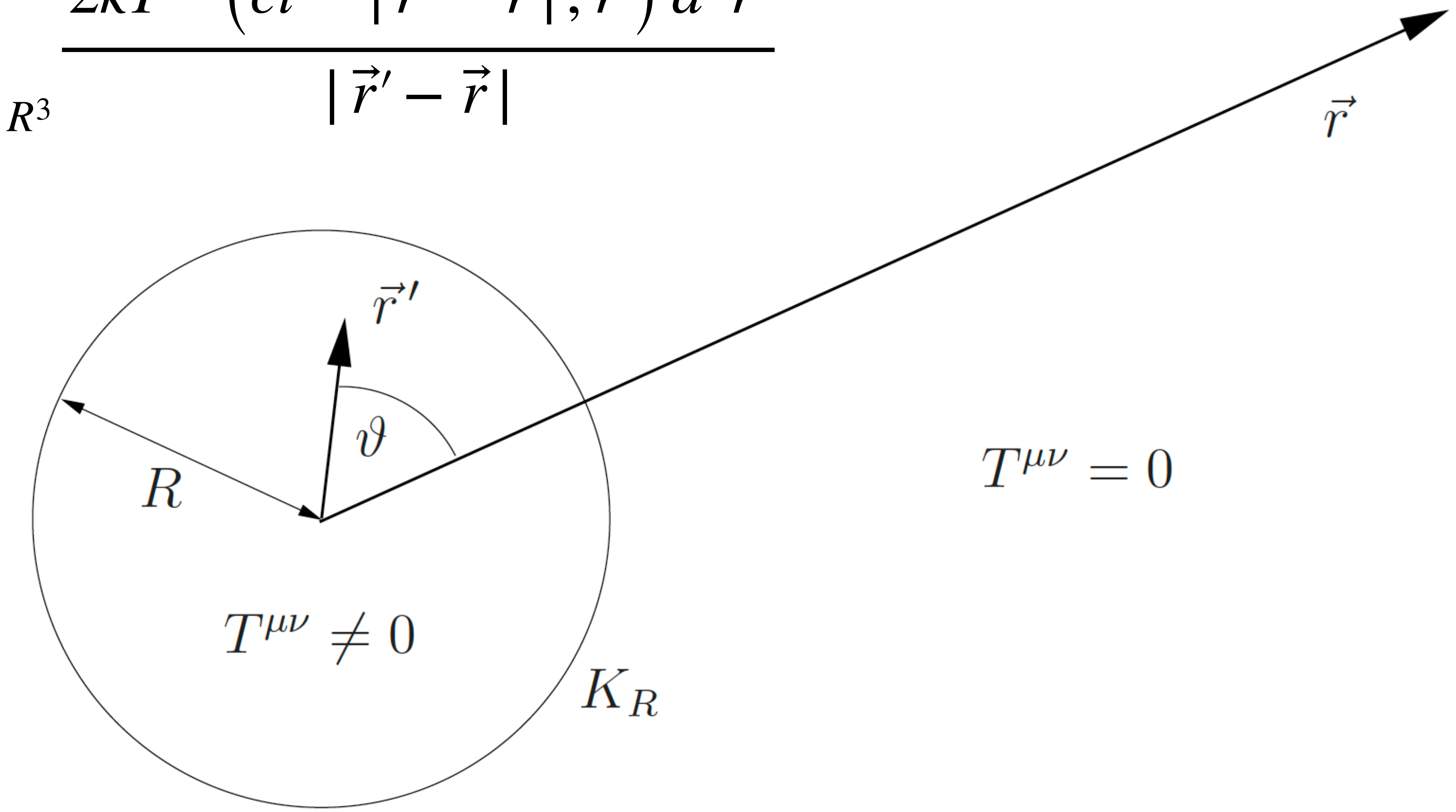


**Image credit: Lecture notes of Claus Lämmerzahl and Volker Perlick**

## Far-field approximation $r \gg R$

$$|\vec{r}\,'-\vec{r}\,| = \sqrt{r'^2+r^2-2rr'\cos\theta}$$

$$= r\sqrt{1-2\frac{r'}{r}\cos\theta+\frac{r'^2}{r^2}}$$

$$= r\left(1+\mathcal{O}\left(\frac{r'}{r}\right)\right)$$

**Note:** The domain of integration is only inside the region $R$ of source. Also when it is so far the wavefronts become plane

$$\boxed{\gamma^{\mu\nu}(ct,\vec{r}) = \frac{\kappa}{2\pi r}\int_{R^3}T^{\mu\nu}\left(ct-r,\vec{r}\,'\right)d^3\vec{r}\,'}$$

# Generation of GWs

Let us work on the far field approximation

$$\gamma^{\mu\nu}(ct,\vec{r}) = \frac{\kappa}{2\pi r}\int_{R^3} T^{\mu\nu}\left(ct - r, \vec{r}'\right) d^3\vec{r}'$$

**One can do a multipole expansion to obtain different features**

- Conservation of mass → absence of **monopole radiation**
- Conservation of momentum → absence of **dipole radiation**
- **No conservation laws force** $\ddot{Q}_{ij} = 0$

**Next Lecture**

**Generation of GWs by different sources**

**Quadrupole formula**

$$\bar{h}_{ij}(t,\mathbf{x}) = \frac{2G}{c^4 r}\frac{d^2 Q_{ij}}{dt^2}\bigg|_{t-r/c}$$

**Quadrupole moment tensor**

$$Q_{ij}(t) = \int \rho(t,\mathbf{x})\, x_i x_j\, d^3x$$

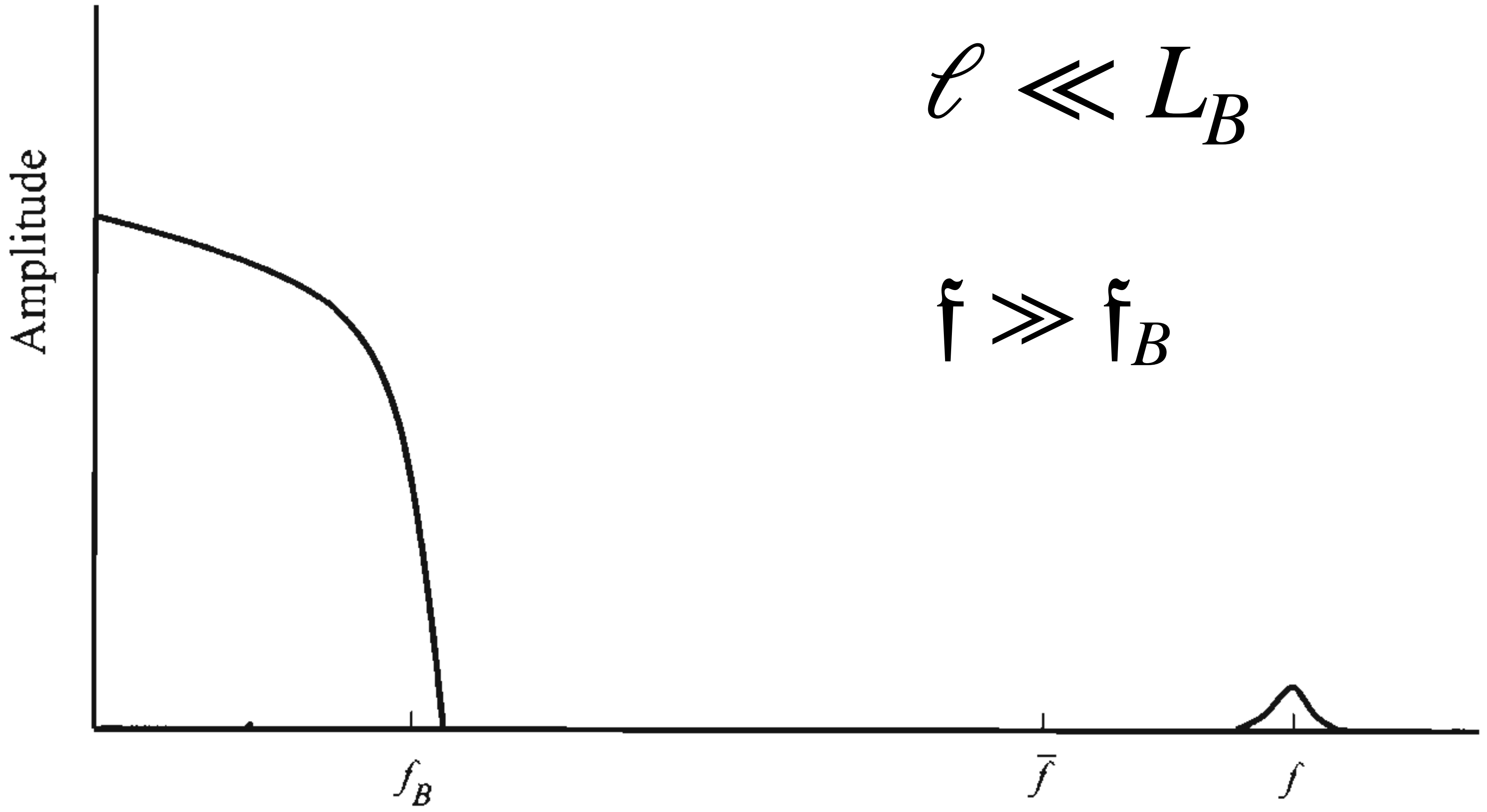
# Energy or Curvature or GWs ?

**So far we have seen expansion over flat space…..**

**BUT**

- **What it GWs are propagating on a non-flat background?**
- **Energy of GWs $\Longrightarrow$ curvature (How to distinguish it from the GW ?)**
- **Geometric optics of GWs**
- **Lensing**

$$g^B_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$$



$$\ell \ll L_B$$

$$\mathfrak{f} \gg \mathfrak{f}_B$$

*Short wave expansion* $\quad R^B_{\mu\nu} = R_{\mu\nu} + R^{[1]}_{\mu\nu} + R^{[2]}_{\mu\nu} + \dots$

- *Background varies very slowly* $\partial g \sim g/L_B$
- *Perturbation varies rapidly* $\partial h \sim h/\ell$

*Complete field equation*

$$R^B_{\mu\nu} = \kappa^2\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T,\right)$$

$$R_{\mu\nu} = -\left(R^{[2]}_{\mu\nu}\right)^{\text{Low}} + \kappa^2\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{Low}}$$

$$R^{[1]}_{\mu\nu} = -\left(R^{[2]}_{\mu\nu}\right)^{\text{High}} + \kappa^2\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{High}}$$

- $R_{\mu\nu}$ contain only slowly varying component
- $R^{[1]}_{\mu\nu}$ contain high frequency
- $R^{[2]}_{\mu\nu}$ contain both high and low

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