



# General Relativity

Gravitational Waves - Theory and Analysis

Sreekanth H

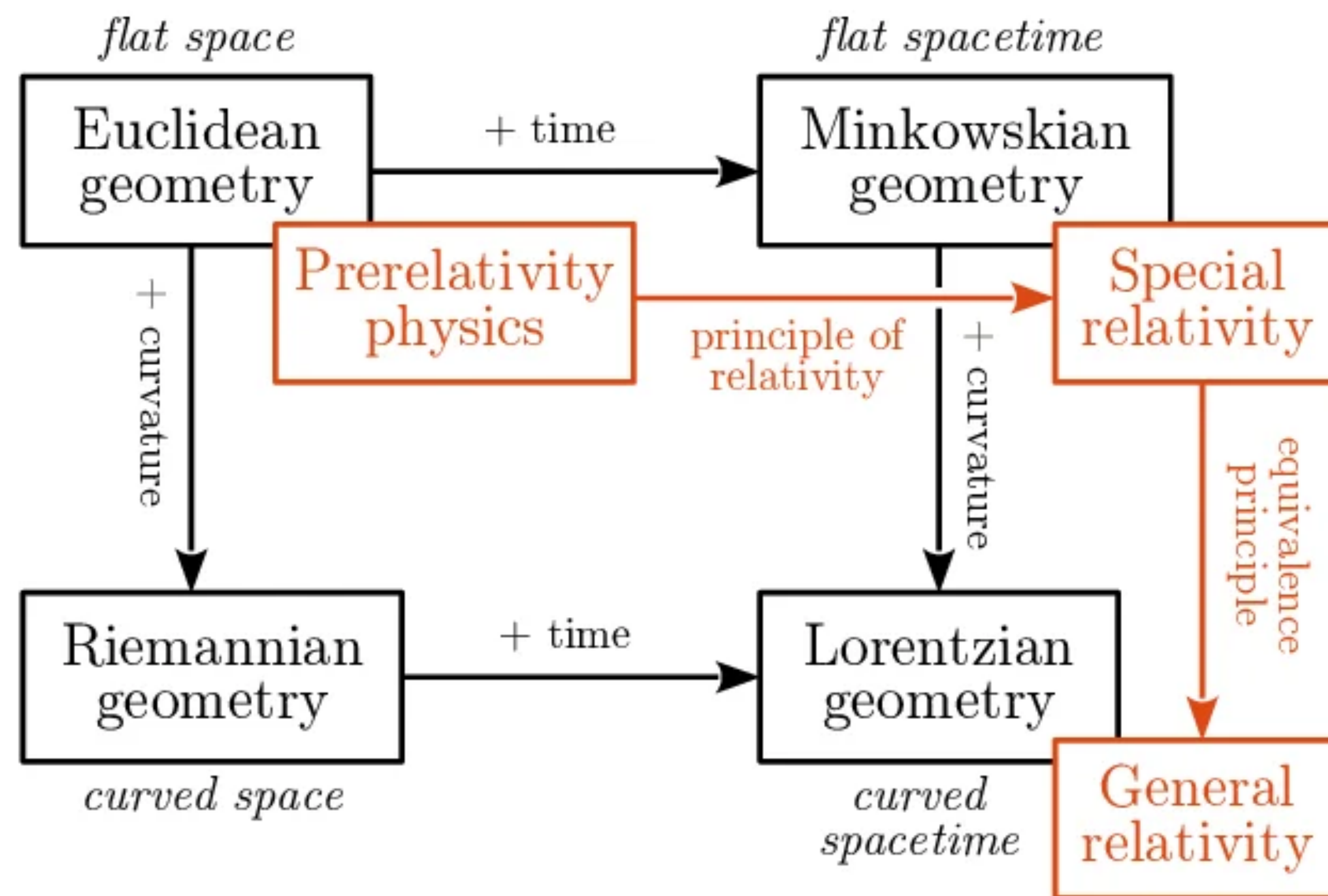
# Overview

- Galilean Relativity
- Special theory and Minkowski Spacetime
- Tensors
- Christoffel symbols
- Covariant Derivative
- Parallel Transport
- Geodesic equation
- Riemann curvature tensor
- Einstein Field equations

**General Relativity** is

...a theory of **space, time and gravity**

**Gravity is Geometry**

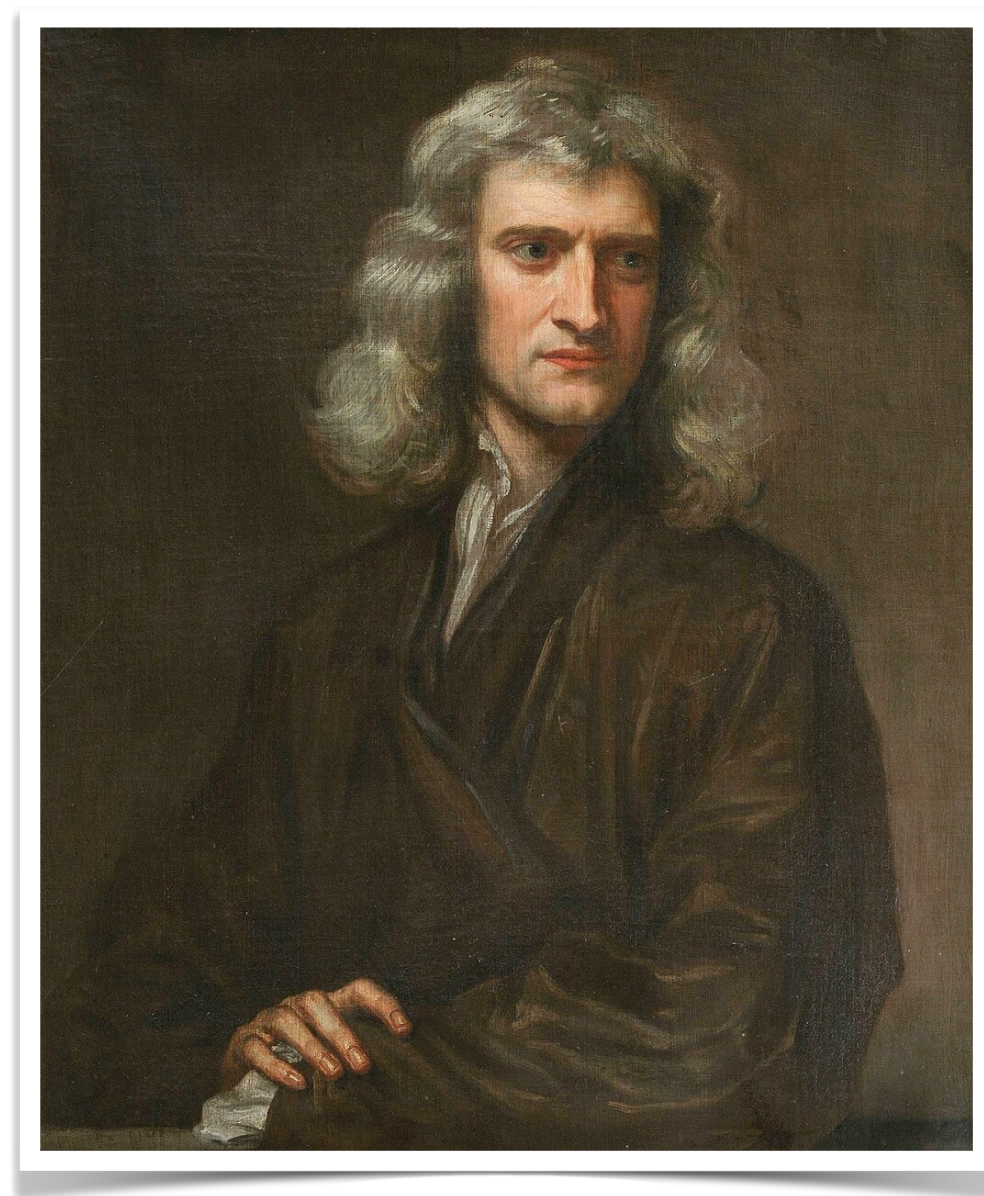
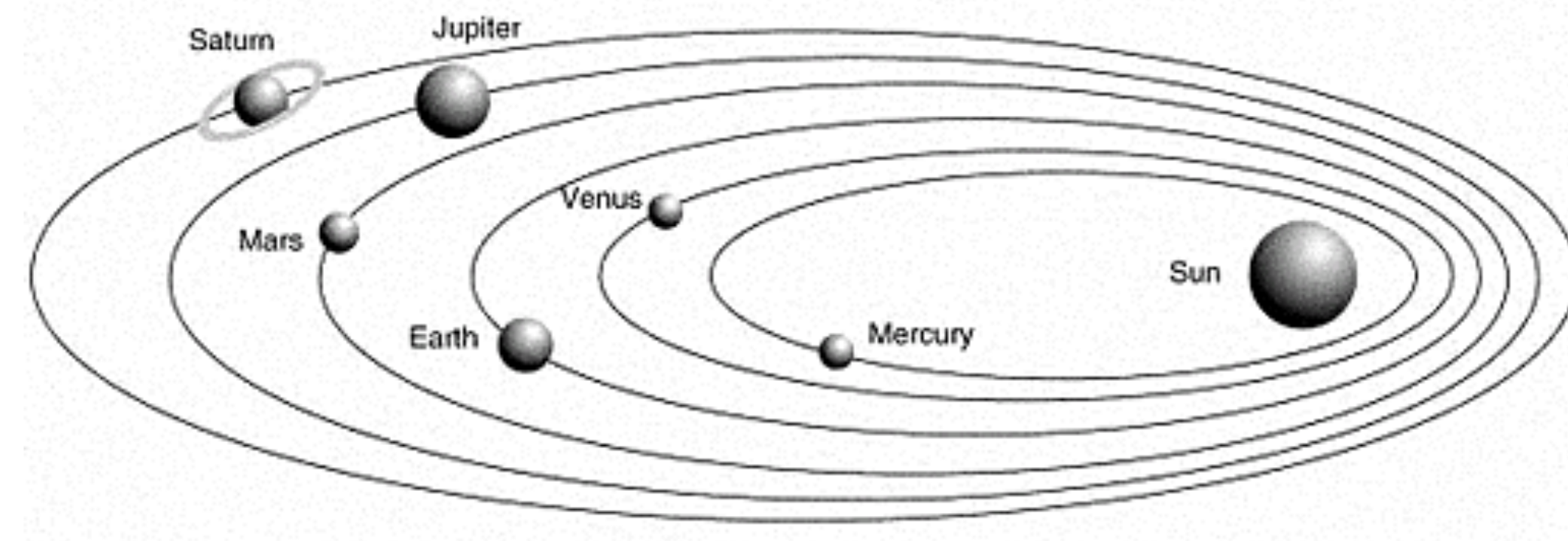


# Newtonian View of Space and Time

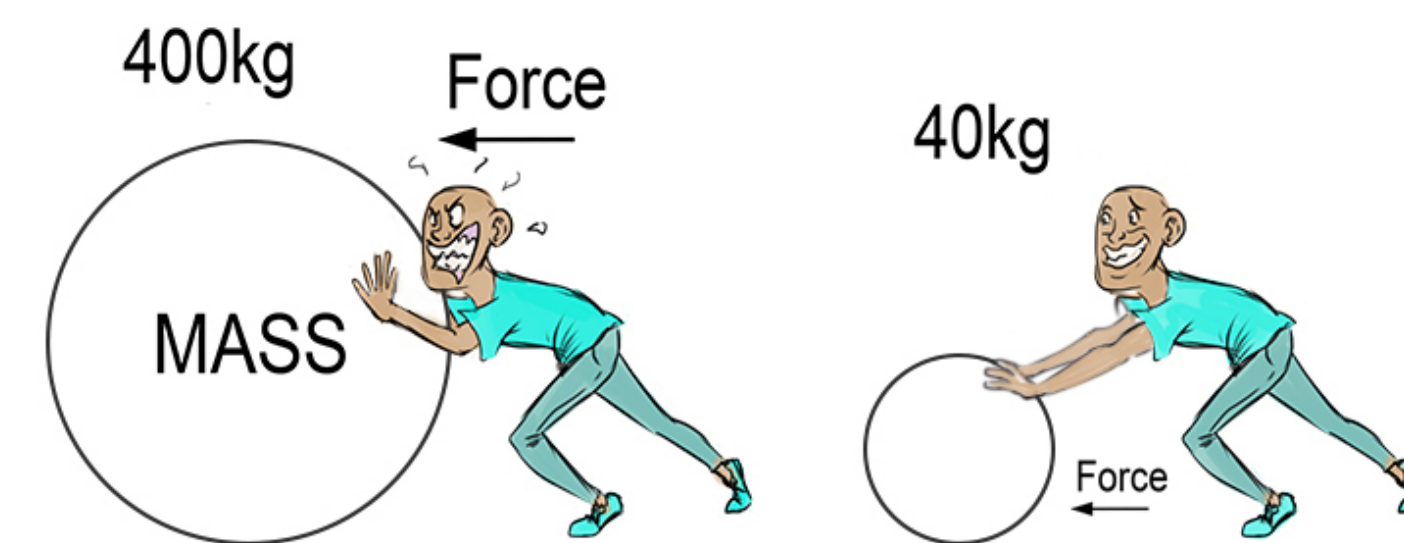
## Newtonian theory - 17th century

- Newton's laws of motion
- Universal law of gravitation

$$F \propto \frac{m_1 m_2}{r^2}$$

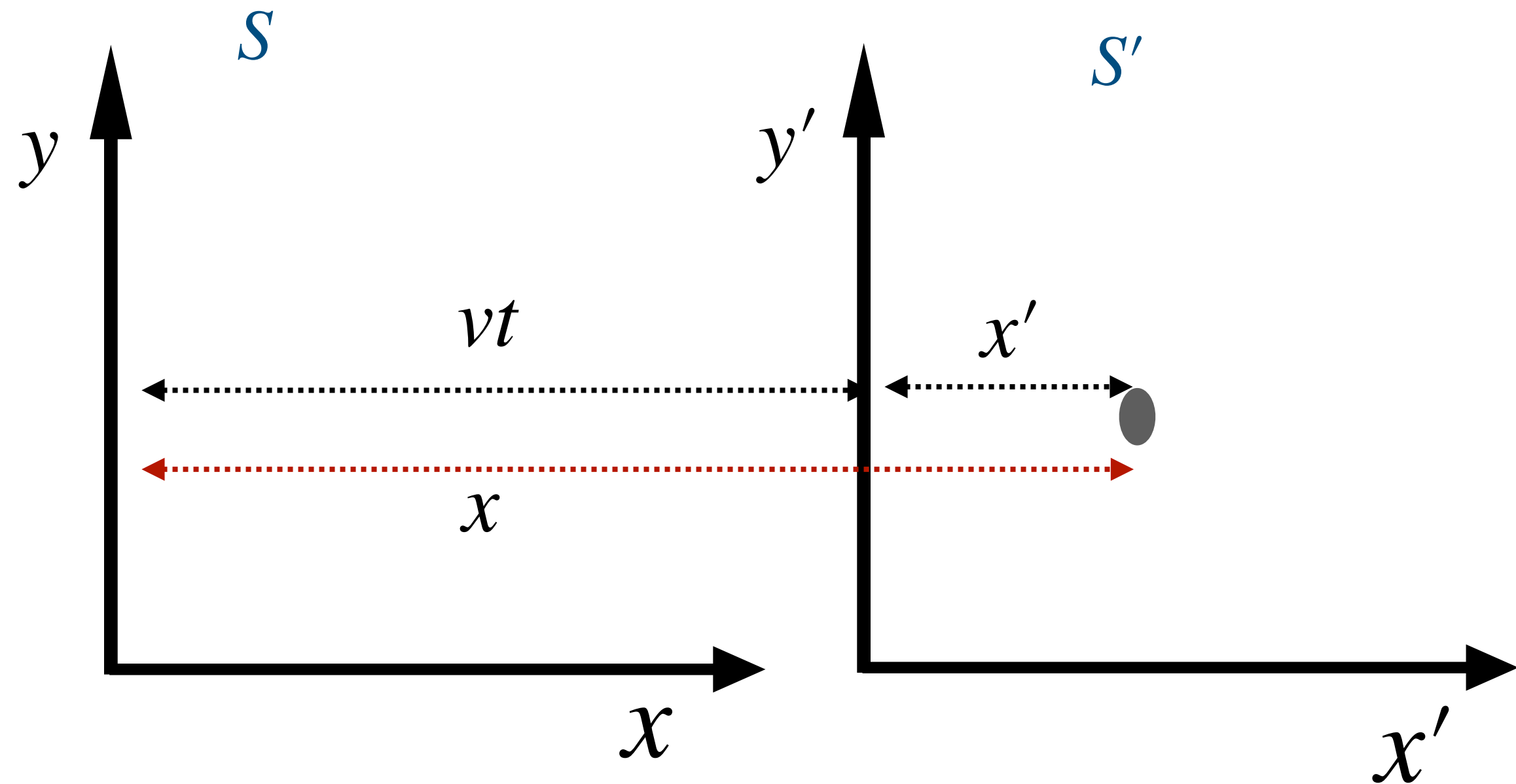


- Gravitation is an instant phenomenon
- Time is absolute
- Gravity is a Force
- The concept of gravitational waves doesn't exist
- Unable to explain Mercury's perihelion precession
- Theory fails in strong gravity regimes



\* Portrait of Newton, by German painter Godfrey Kneller

# Galelian Relativity



## Galelian Transformation

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

**Inertial Frames !!!**

**RELATIVITY** in Newtonian Mechanics

Laws of physics  
remain same in  
these frames

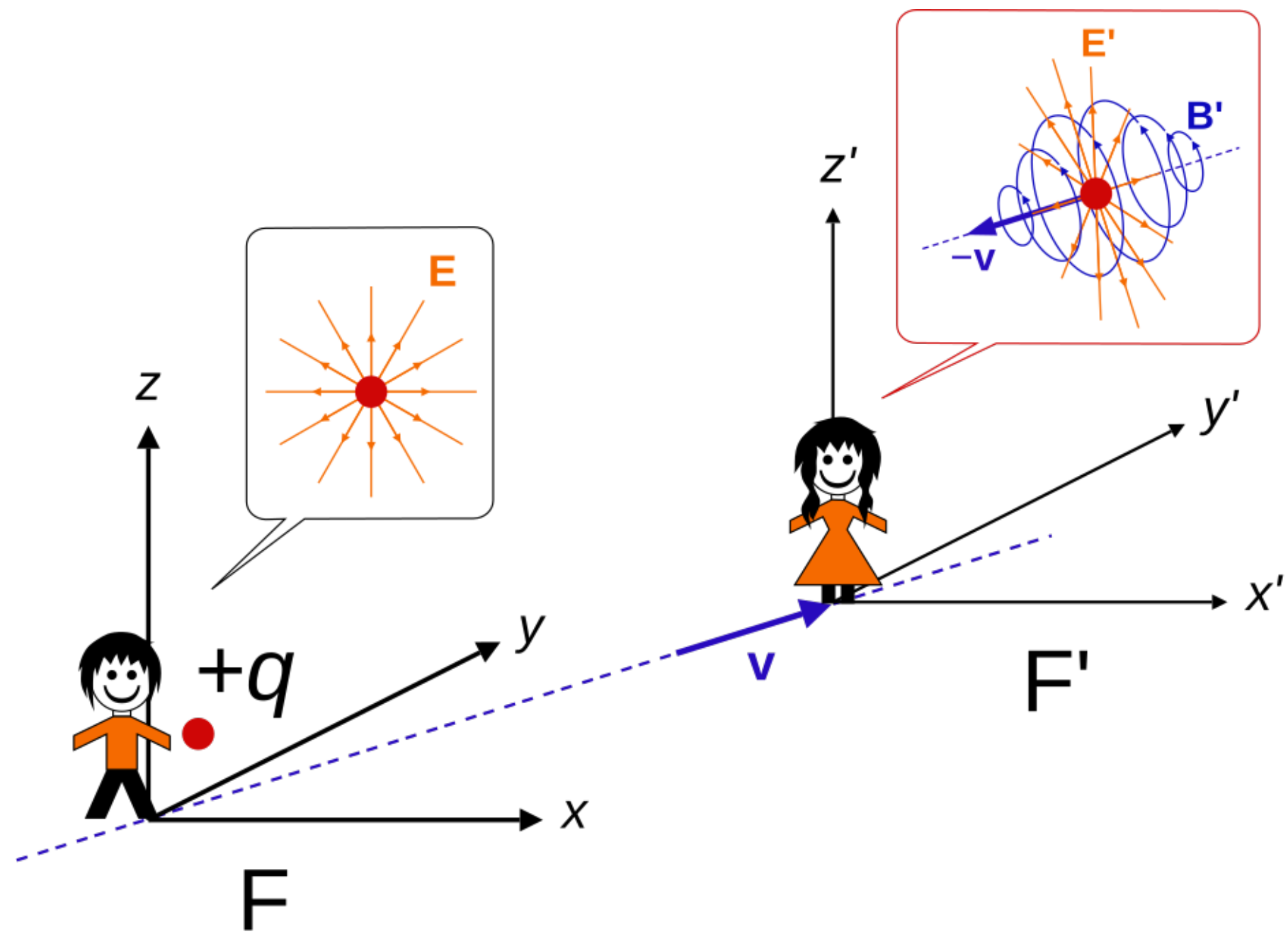
Force in Frame  $S$

$$F_S = m \frac{d^2x}{dt^2}$$

Force in Frame  $S'$

$$F_{S'} = m \frac{d^2x'}{dt'^2} = F_S$$

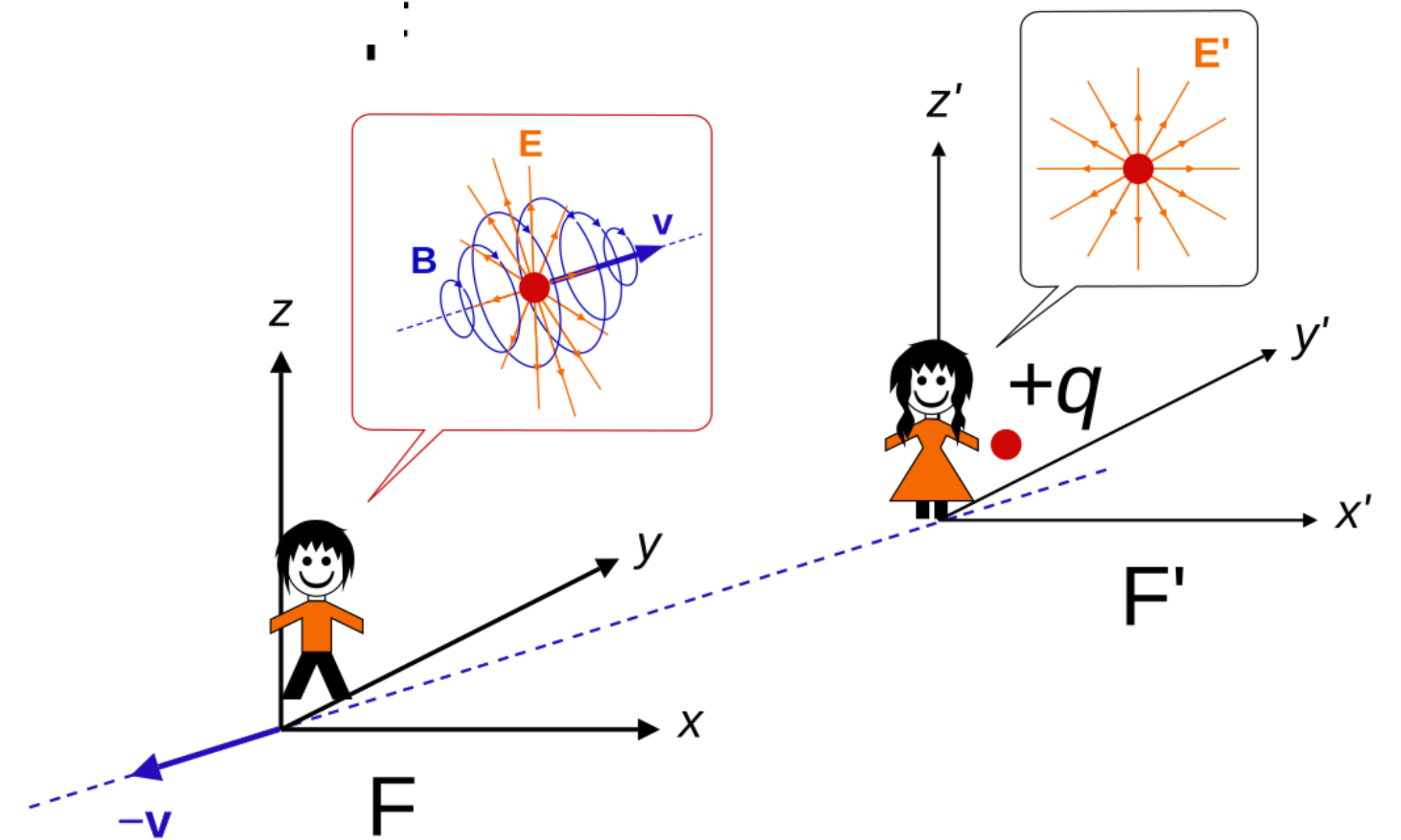
# Electrodynamics and Galilean relativity



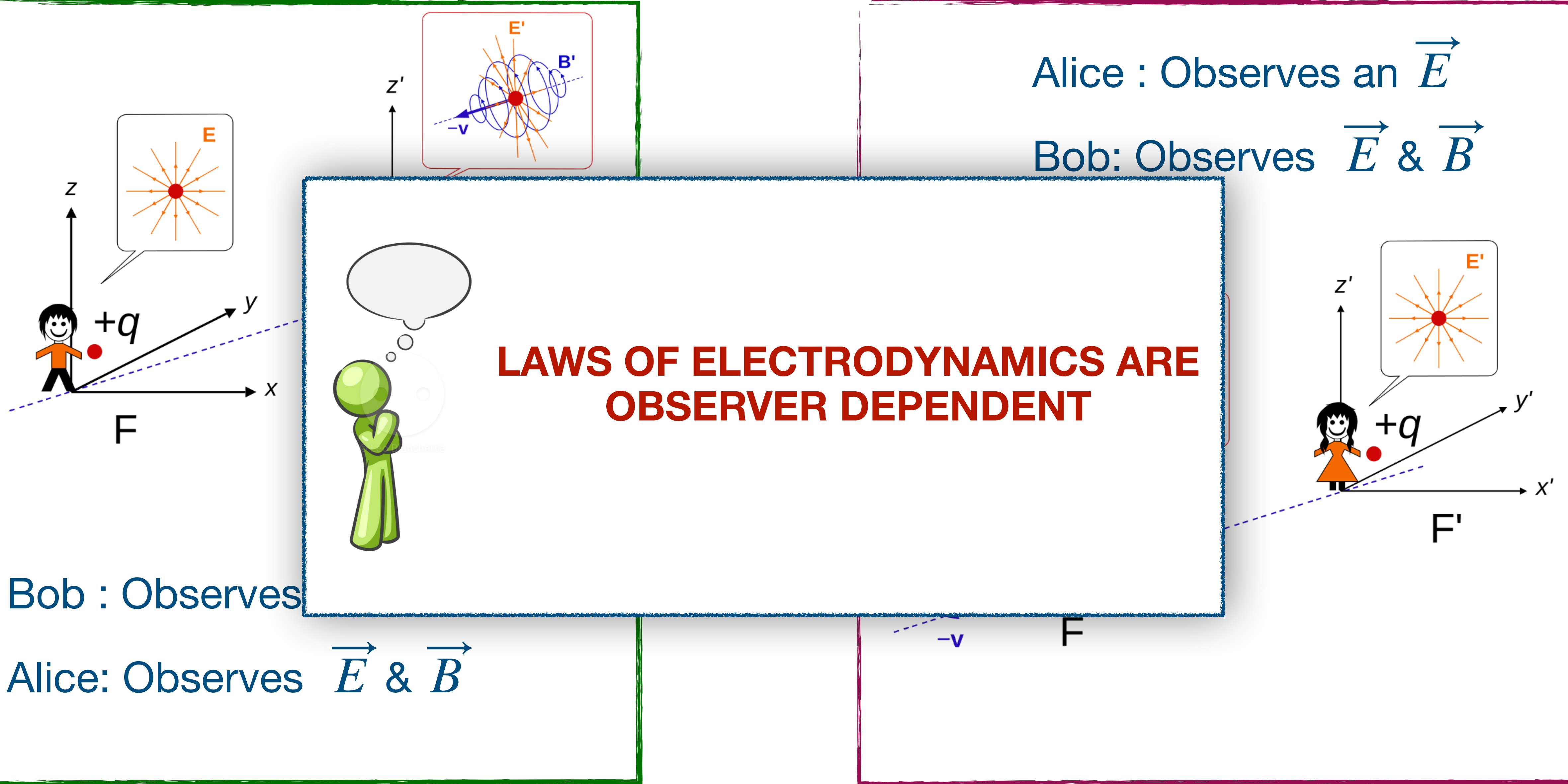
Bob : Observes an  $\vec{E}$

Alice: Observes  $\vec{E}$  &  $\vec{B}$

Alice : Observes an  $\vec{E}$   
 Bob: Observes  $\vec{E}$  &  $\vec{B}$



# Electrodynamics and Galilean relativity

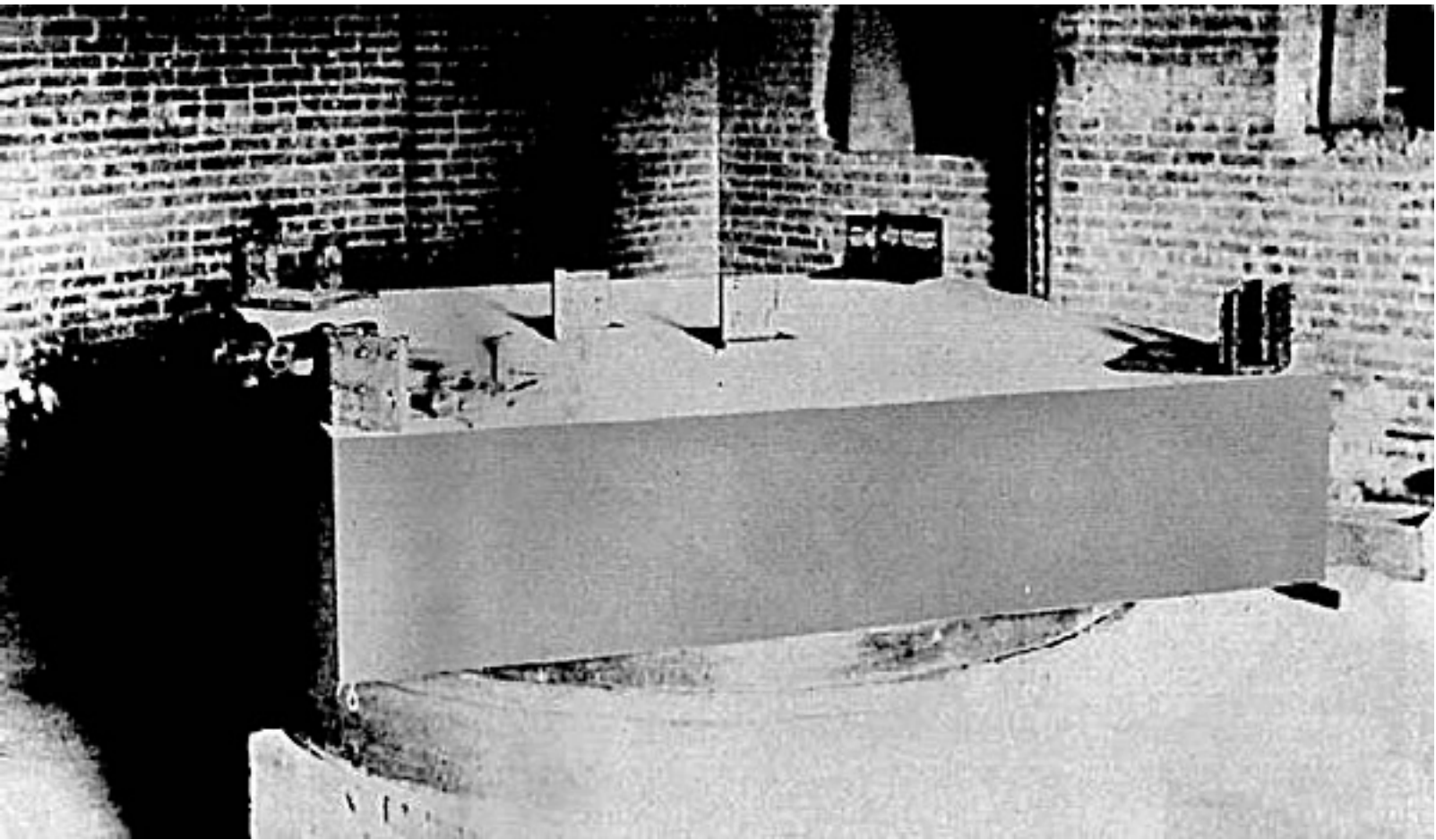


# What is wrong with Electrodynamics ?

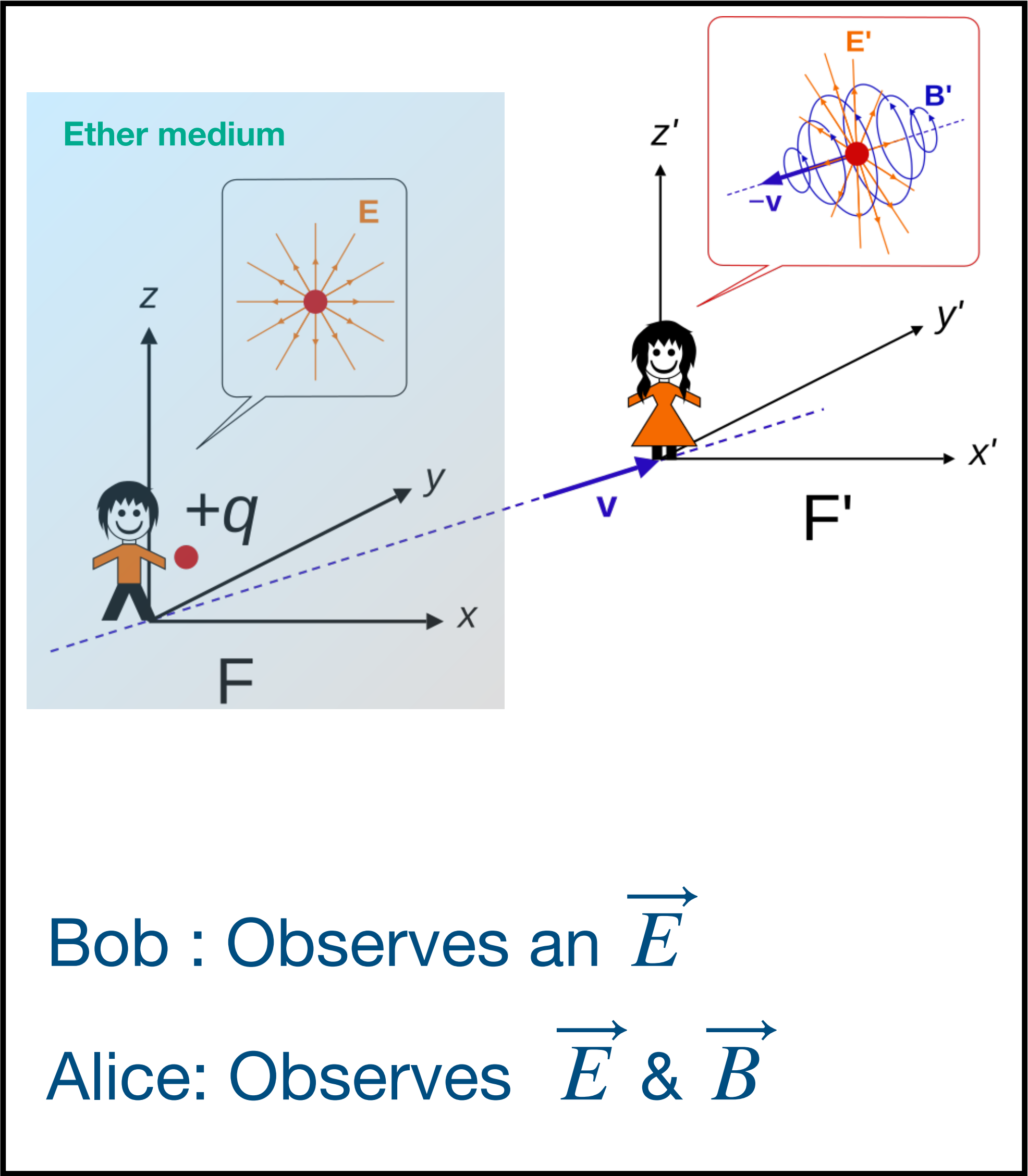
Laws of Electromagnetism should be universal like the laws of mechanics.

Speed of light  $c = \sqrt{\frac{1}{\mu_0 \epsilon_0}}$

It was believed that the incorporating the Ether drag can solve this problem



Michelson and Morley's [interferometric](#) setup



# The dawn of special relativity

## Postulates

Light travels with the same speed in all inertial frames of reference

The laws of physics have the same form for all observers.

On  
The Electrodynamics of Moving Bodies

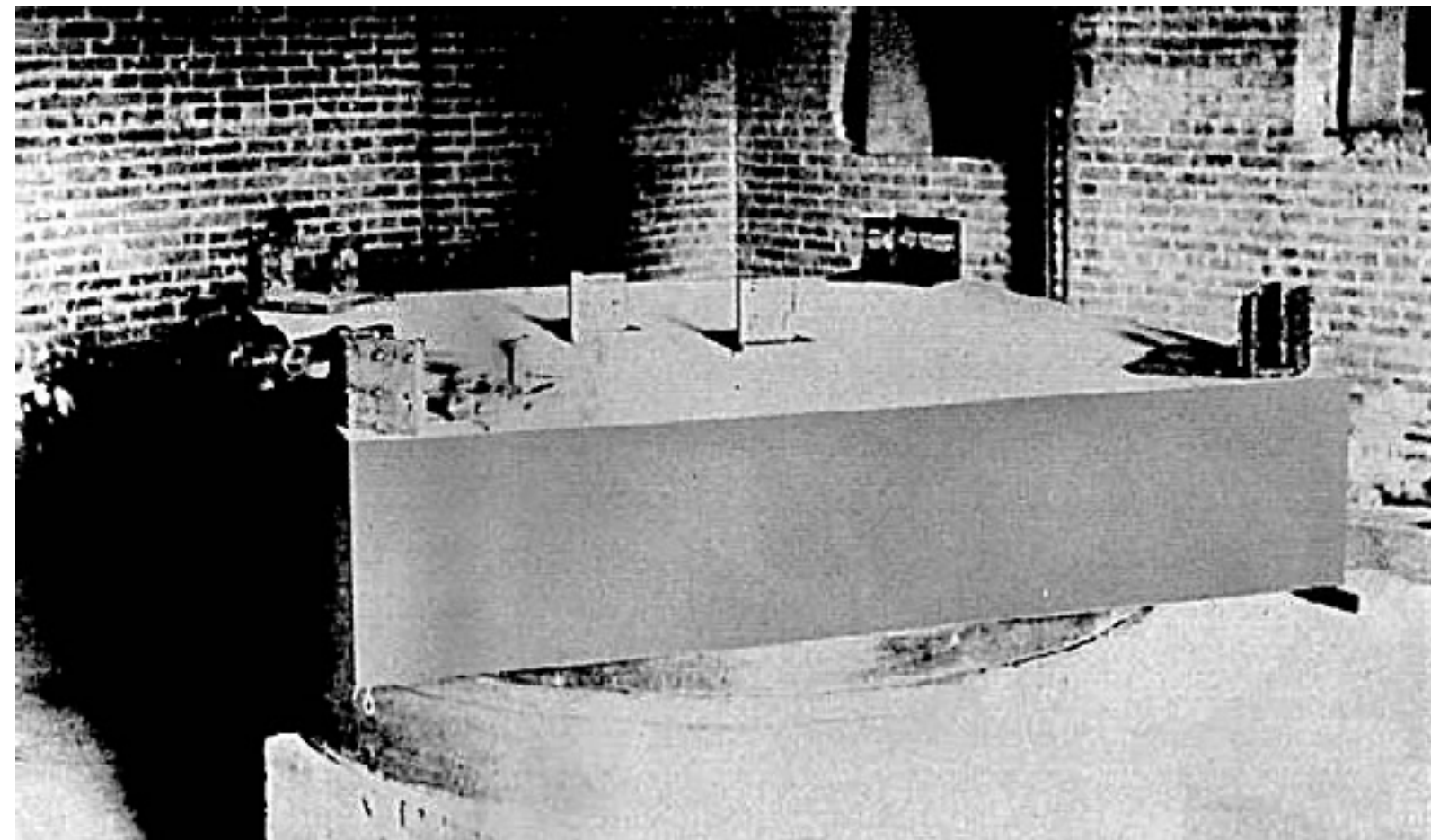
BY  
A. EINSTEIN.

**Length contraction:** Length of a moving body contracts

**Time dilation :** Time runs more slowly in a moving body

## Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

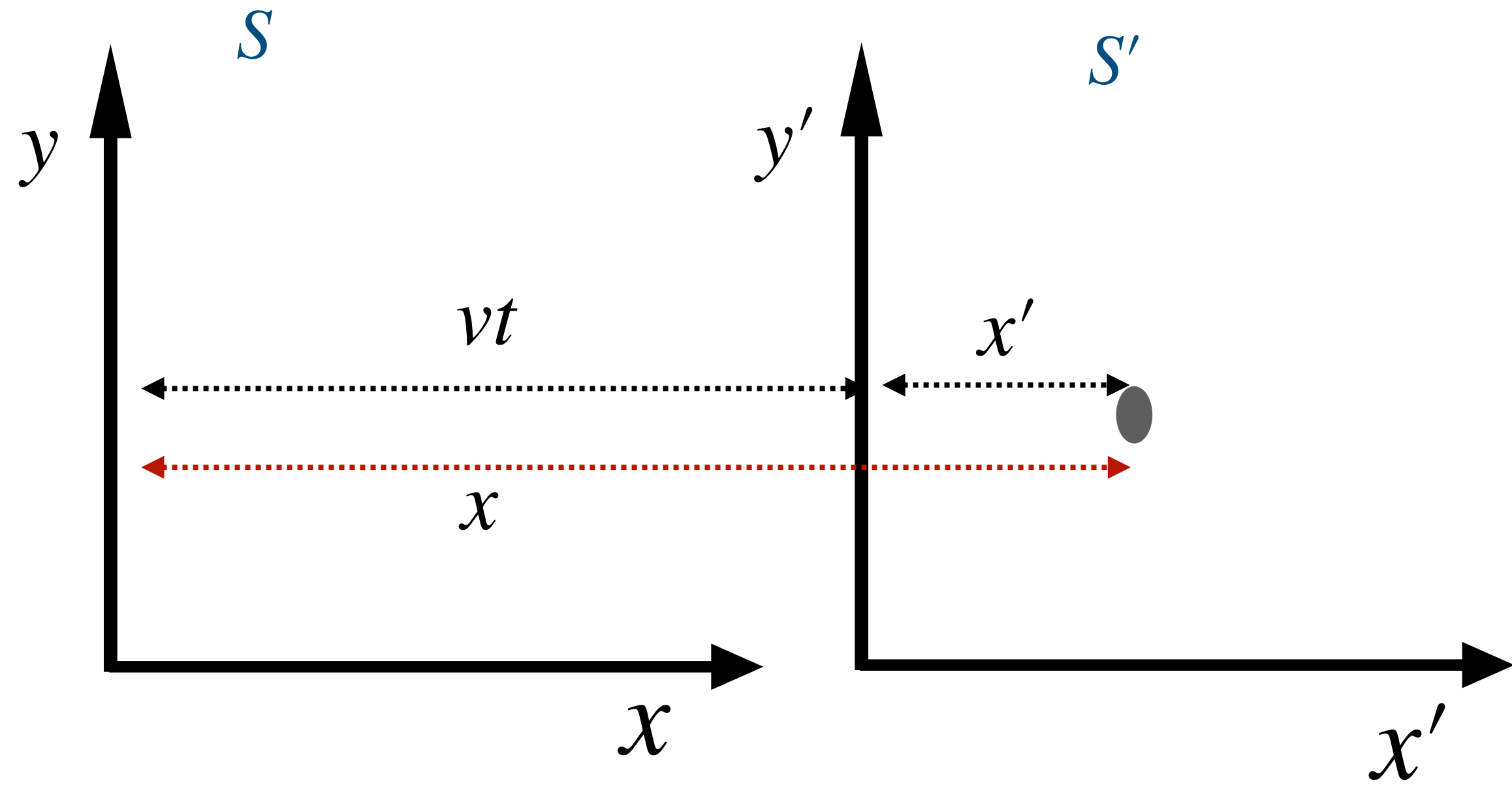


Michelson and Morley's [interferometric](#) setup



Hendrik Antoon Lorentz

# Lorentz Transformation



$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma \left( t - \frac{vx}{c^2} \right)$$

**Lorentz factor**

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

For  $v \ll c$ , **Lorentz transformations** reduces to **Galelian transformation**

Time Dilation and Length contractions are real and well studied experimental facts

- Hafele-Keating Experiment
- Muon decay
- Particle accelerators
- Atomic clocks at high altitudes
- GPS

# Notations

*Sometime we deal with spatial variables....*

*Sometimes we deal with time or just both...*

**NOTATIONS!!**

## **Greek letters**

$\mu, \nu, \alpha, \beta, \gamma, \delta \dots = t, x, y, z$

## **Latin letters**

$i, j, k, l, m, n \dots = x, y, z$

$A^\mu$  - Spacetime vector

$A^i$  - Spatial vector

# Minkowski spacetime



Herman Minkowski

Source : Wikipedia

*The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, time by itself, are doomed to fade away into mere shadows and only a kind of union of the two will preserve an independent reality.*

- Herman Minkowski

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$

$$[x'^{\mu}] = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

$$[x^{\nu}] = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$$

Transformation matrix for a boost along  $x$ -direction.

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{where, } \beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

# Einstein summation convention

Repeated indices are summed over.

$$A^\mu B_\mu = \sum_{\mu=0}^3 A^\mu B_\mu$$

$$A^\mu B_\mu = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3$$

Transformation matrix for a boost along  $x$ -direction.

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$x'^\mu = \Lambda^\mu{}_0 x^0 + \Lambda^\mu{}_1 x^1 + \Lambda^\mu{}_2 x^2 + \Lambda^\mu{}_3 x^3$$

$$x' = \gamma(x - vt)$$

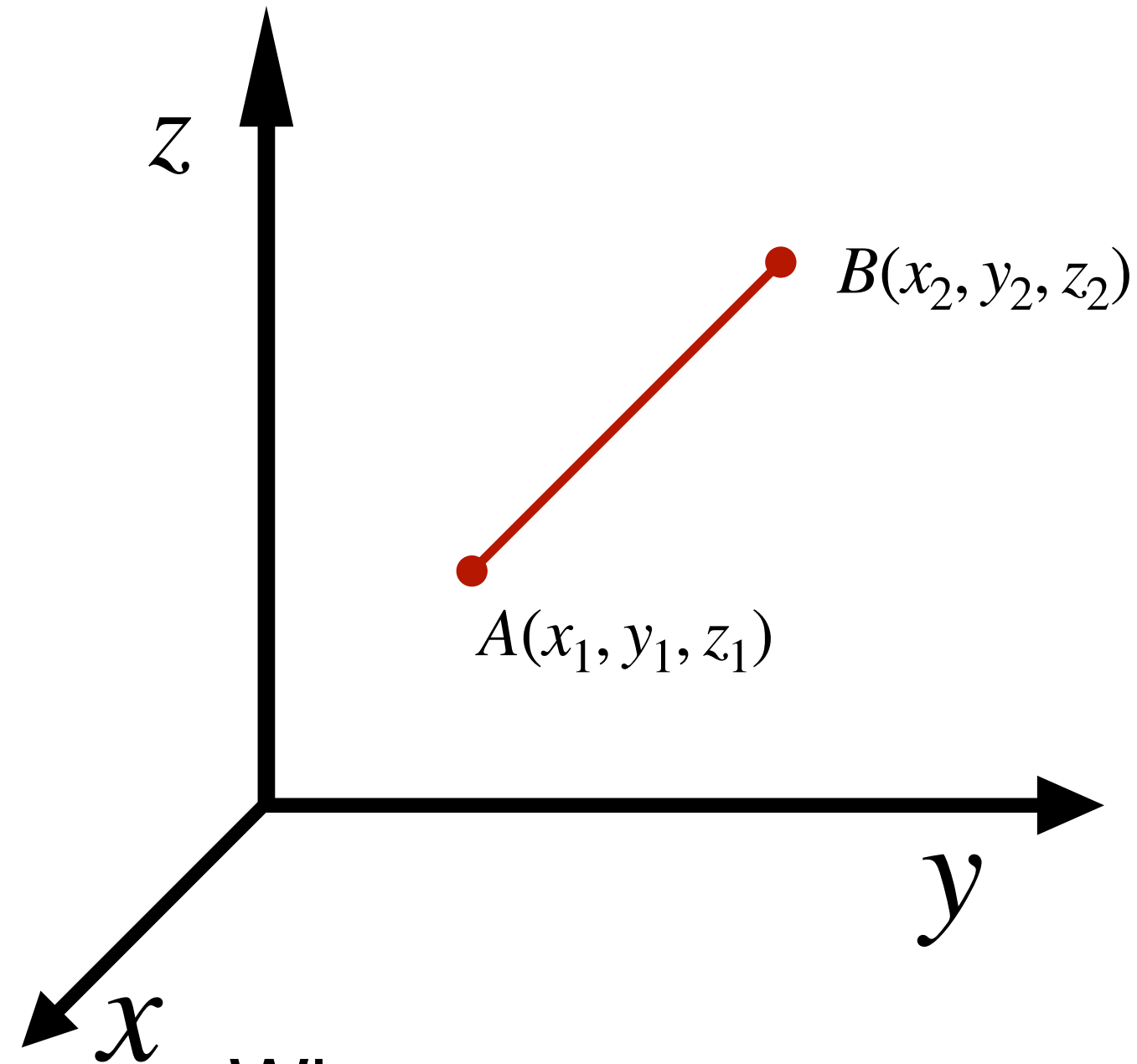
$$y' = y$$

$$z' = z$$

$$t' = \gamma \left( t - \frac{vx}{c^2} \right)$$

Good exercise to try and recover the transformation equations

# Measuring distances and the metric



Where

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$ds^2 = \delta_{ij} dx^i dx^j$$

Distance between two points

$$\begin{aligned} ds &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{dx^2 + dy^2 + dz^2} \end{aligned}$$

$$dx = x_2 - x_1$$

$$dy = y_2 - y_1$$

$$dz = z_2 - z_1$$

Kronecker delta

$$\delta_{ij} \equiv \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j. \end{cases}$$

Metric tensor for a flat 3D Cartesian coordinate system

# Minkowski spacetime

Flat Cartesian system

$$ds^2 = \delta_{ij} dx^i dx^j$$

A general spatial coordinate

$$ds^2 = g_{ij} dx^i dx^j$$

Spacetime in Special Relativity

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

## Minkowski metric tensor

$$\eta_{\mu\nu} = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Signature = +2

Einstein Summation in action

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu \\ &= \eta_{\mu 0} dx^\mu dx^0 + \eta_{\mu 1} dx^\mu dx^1 + \eta_{\mu 2} dx^\mu dx^2 + \eta_{\mu 3} dx^\mu dx^3 \\ &= - (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \\ &= - c^2 dt^2 + dx^2 + dy^2 + dz^2 \end{aligned}$$

## Possible cases

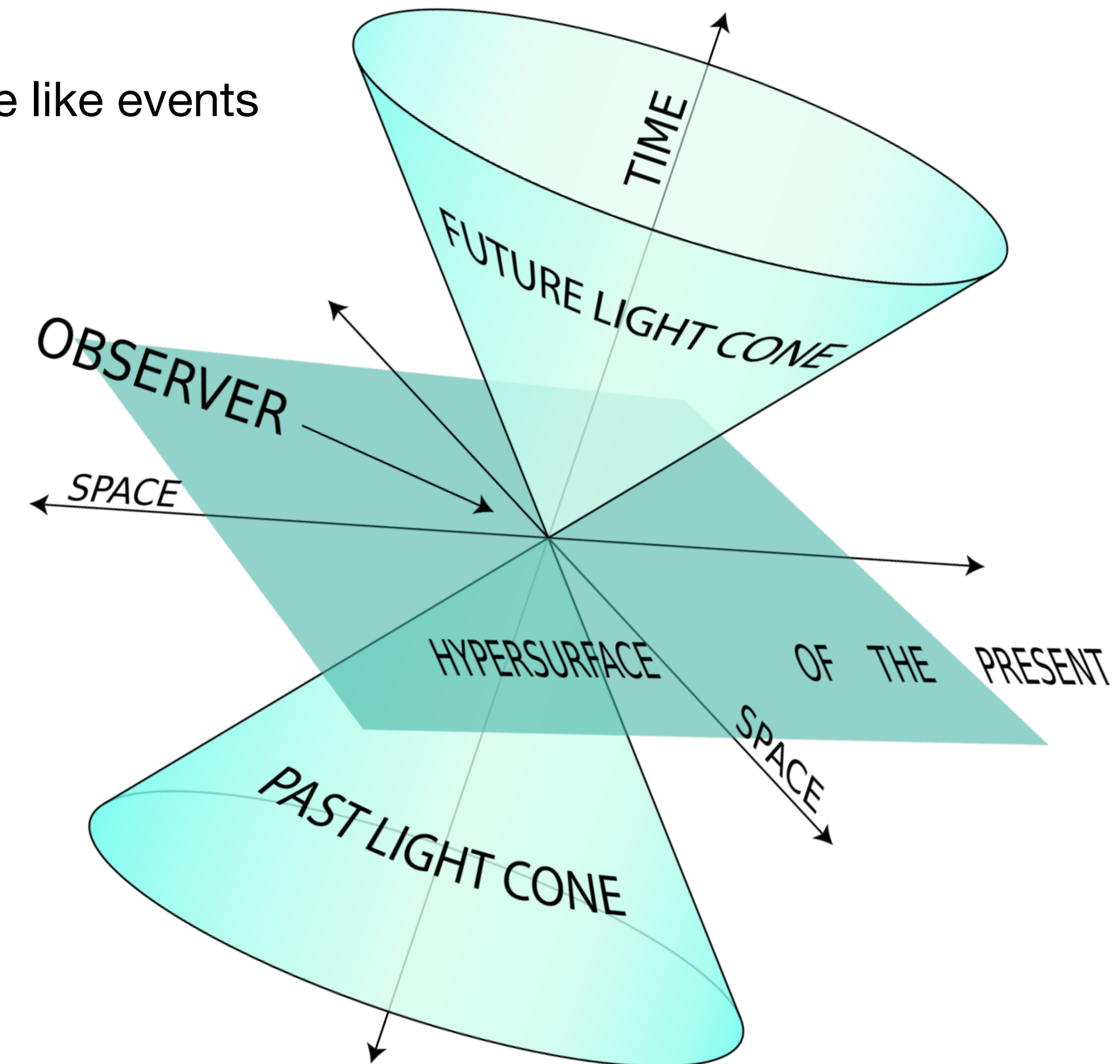
$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

One can attach a light cone to any event in spacetime.

- $ds^2 > 0$  Space like events
- $ds^2 = 0$  Light like events
- $ds^2 < 0$  Time like events

## Some remarks

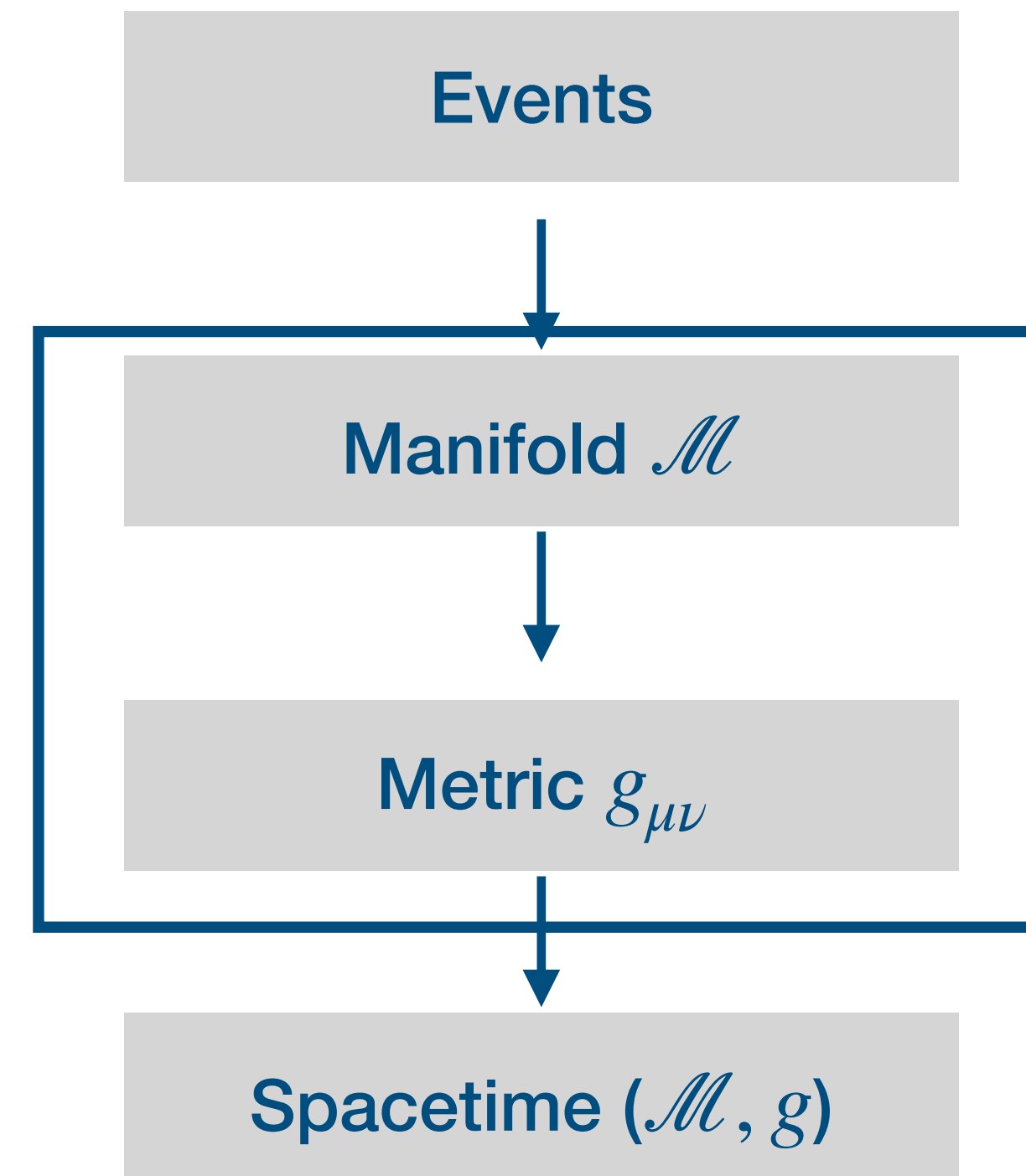
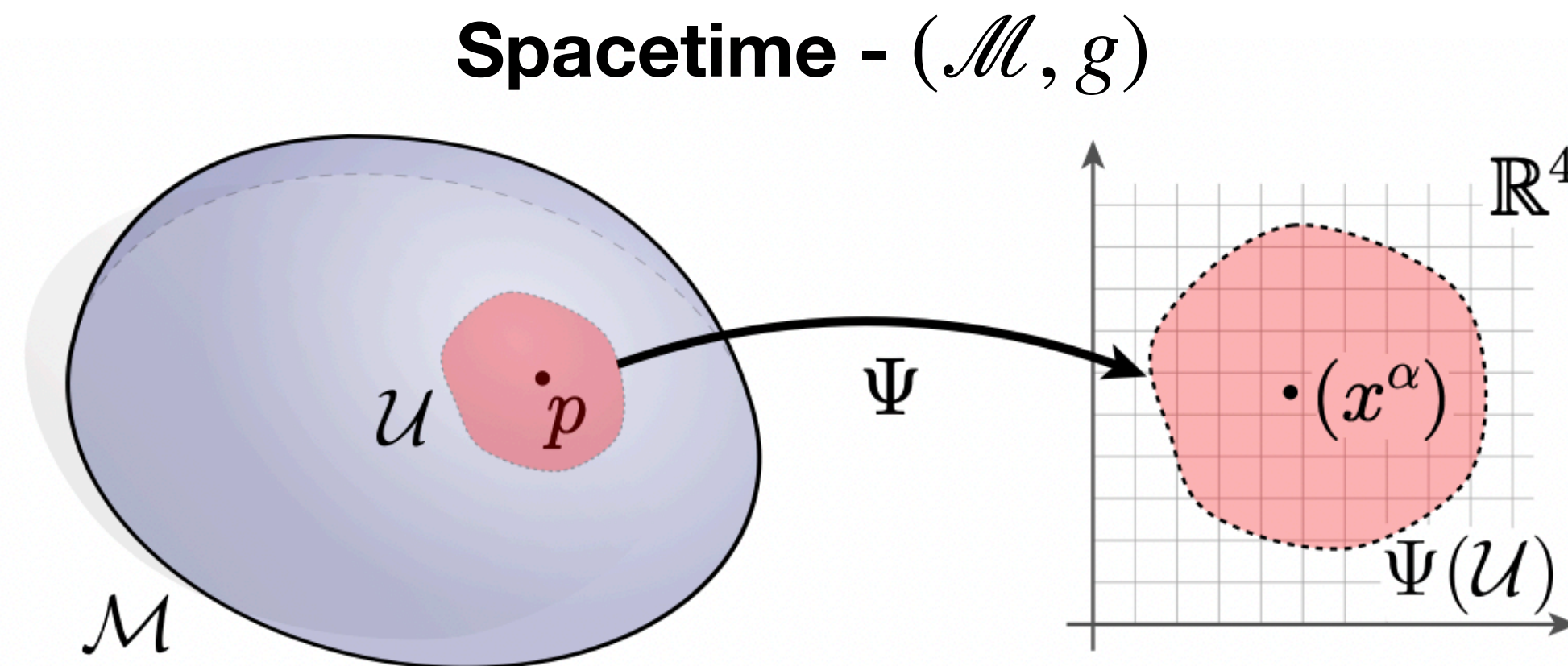
- Space and Time are completely independent absolute quantities in Newtonian mechanics.
- Using **Galelian relativity** in the laws of Electromagnetism lead to observer dependence ! That's not how physics works.
- Galelian transformation is a subclass of **Lorentz transformation** ( applicable for fast moving objects)
- We have seen that metric tensor is useful tool to measure distance between any two points in spacetime and **Einstein summation convention** simplifies our notation.



# Notion of manifolds $\mathcal{M}$ and spacetime $(\mathcal{M}, g)$

A **manifold** is a topological space that **locally resembles Euclidean space**.

Spacetime is modeled as a **4-dimensional differentiable manifold  $\mathcal{M}$**  with a Lorentzian  $g_{\mu\nu}$



## Why Lorentzian metric?

- Causal structure
- Distances
- Levi-Civita connection
- Covariant derivative
- Geodesics and Curvature tensor

## Why differentiable ?

- Scalar fields
- Vector fields
- Tensor fields
- To perform calculus in spacetime
- To define particle motions....

# Tensors

Tensors are mathematical objects that help us describe quantities in a **coordinate independent** way.

## Scalars

- Temperature
- Mass
- Electric charge

## Vectors

- Velocity
- Force
- Electric field

## Tensors

- Moment of Inertia
- Stress
- EM field  $F_{\mu\nu}$
- Curvature  $R^{\alpha}_{\nu\mu\beta}$

- **Scalars** (Rank 0 tensors)
- **Vectors** (Rank 1 tensors)
- **Tensors** (Rank higher than 1)

Equations written in tensor form are valid in **any frame of reference**,

These objects have certain transformation properties

Tensor type

$$\begin{pmatrix} m \\ n \end{pmatrix}$$

# Representing a vector (Rank 1 tensors)

$$\vec{A} = A^\alpha \vec{e}_\alpha \quad A^\alpha - \text{components}$$
$$\hat{e}_\alpha - \text{basis}$$

## Components of the same vector in another

$$A^{\bar{\alpha}} = \Lambda_{\beta}^{\bar{\alpha}} A^\beta$$

$$\text{Note : } \Lambda_{\beta}^{\bar{\alpha}} = \frac{\partial x^{\bar{\alpha}}}{\partial x^\beta}$$

A bar on top of the component is used to denote the components in a different frame.

### Things to note:

- $\beta$  here is a dummy index, which should be summed over
- $\alpha$  is a free index
- It is a set of four equations, one for each component.

## Example:

A vector in Euclidian geometry

$$\vec{A} = 2\hat{x} + 3\hat{y} - 6\hat{z}$$

$$\hat{e}_1 = \hat{x}; \quad \hat{e}_2 = \hat{y}; \quad \hat{e}_3 = \hat{z}$$

# Basis vectors & Transformation

$$\begin{aligned}\vec{A} &= A^\alpha \hat{e}_\alpha \\ &= A^0 \hat{e}_0 + A^1 \hat{e}_1 + A^2 \hat{e}_2 + A^3 \hat{e}_3\end{aligned}$$

Basis vectors in frame  $\mathcal{O}$

$$\vec{e}_0 \rightarrow (1,0,0,0)$$

$$\vec{e}_1 \rightarrow (0,1,0,0)$$

$$\vec{e}_2 \rightarrow (0,0,1,0)$$

$$\vec{e}_3 \rightarrow (0,0,0,1)$$

How are the basis transformed in another frame ?

Same vector in another frame  $\bar{\mathcal{O}}$        $\vec{A} = A^{\bar{\alpha}} \hat{e}_{\bar{\alpha}}$       Note :  $\Lambda_{\beta}^{\bar{\alpha}} = \frac{\partial x^{\bar{\alpha}}}{\partial x^{\beta}}$

$$A^\alpha \vec{e}_\alpha = A^{\bar{\alpha}} \vec{e}_{\bar{\alpha}}$$

$$A^\alpha \vec{e}_\alpha = \Lambda_{\beta}^{\bar{\alpha}} A^\beta \vec{e}_{\bar{\alpha}}$$

$$= \Lambda_{\alpha}^{\bar{\beta}} A^\alpha \vec{e}_{\bar{\beta}}$$

$$\vec{e}_\alpha = \Lambda_{\alpha}^{\bar{\beta}} \vec{e}_{\bar{\beta}}$$

(Relabelling the dummy index, changing  $\beta \rightarrow \alpha$  and  $\bar{\alpha} \rightarrow \bar{\beta}$ )

$$A^\alpha (\vec{e}_\alpha - \Lambda_{\alpha}^{\bar{\beta}} \vec{e}_{\bar{\beta}}) = 0$$

The components and basis vectors transform in two different ways

# Scalar product of two vectors

Consider two vectors  $\vec{A} = A^\alpha \vec{e}_\alpha$  and  $\vec{B} = B^\alpha \vec{e}_\alpha$

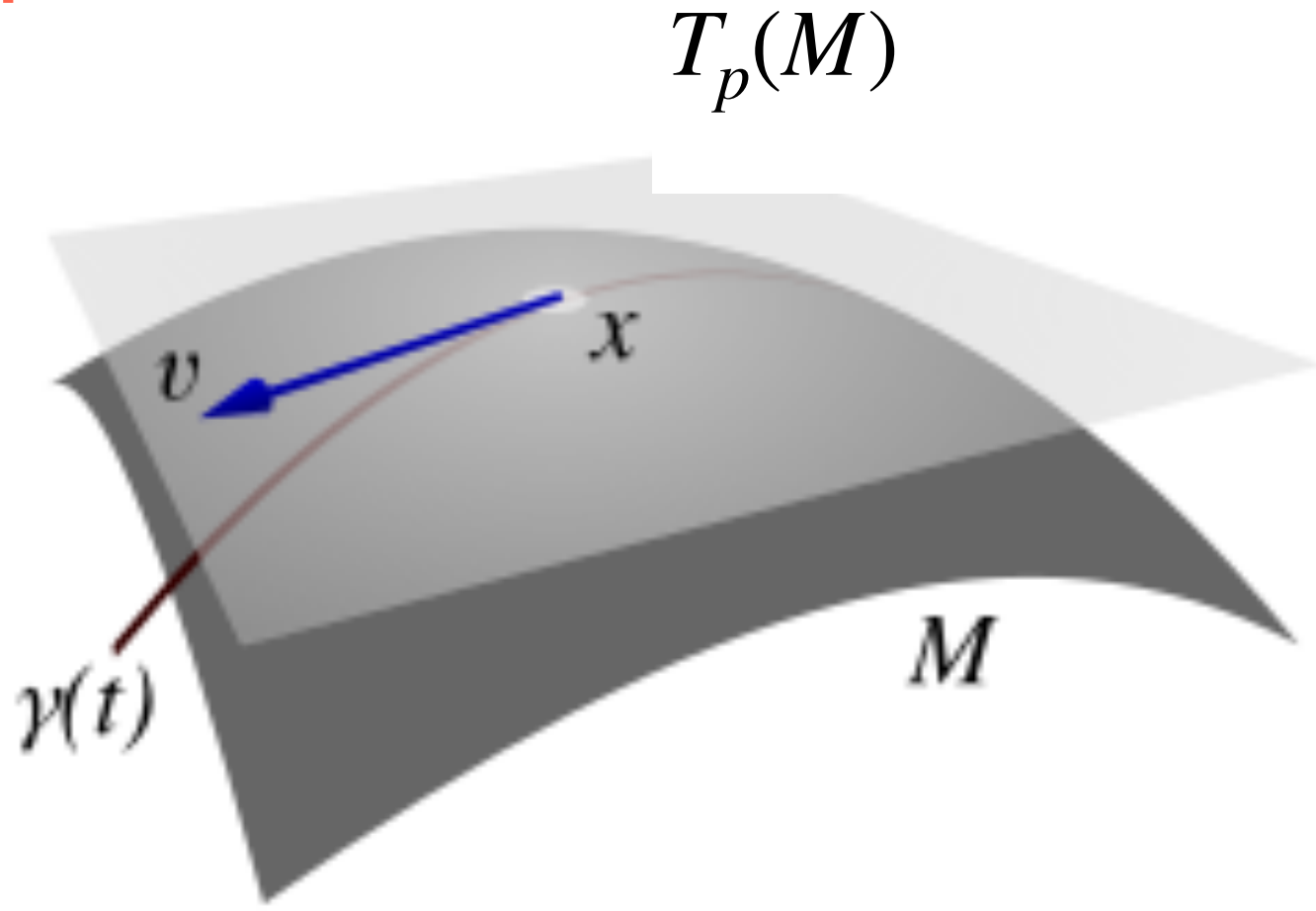
$$\begin{aligned} \vec{A} \cdot \vec{B} &= A^\alpha B^\beta (\vec{e}_\alpha \cdot \vec{e}_\beta) \\ &= A^\alpha B^\beta \eta_{\alpha\beta} \quad \text{Frame Invariant number} \\ &= -A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \end{aligned}$$

**Note**

**If  $\vec{A} \cdot \vec{B} = 0$  the vectors are orthogonal. But they need not be at right angles.**

The vectors live in a space called  $T_p(M)$  and through dot product we obtained a rank 0 tensor (scalar). To make this possible we require a third object the metric tensor.

## Tangent space



$$g(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta}$$

# Metric tensor - $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

## Properties

- Symmetry property

$$g_{\mu\nu} = g_{\nu\mu}$$

- Inverse metric

$$g^{\mu\alpha} g_{\alpha\nu} = \delta_\nu^\mu$$

- **Raising and lowering indices**

## Rank 1

$$V_\mu = g_{\mu\nu} V^\nu, \quad V^\mu = g^{\mu\nu} V_\nu$$

## Rank 2

$$T^\mu{}_\nu = g^{\mu\alpha} T_{\alpha\nu}, \quad T_{\mu\nu} = g_{\mu\alpha} T^\alpha{}_\nu$$

- $g_{\mu\nu}$  encodes the geometry of the spacetime
- It tells the distance between any points in spacetime
- Geodesics are determined by the metric
- The metric tells you how to measure angles and orthogonality in spacetime

**Inner products:**  $\langle A, B \rangle = g_{\mu\nu} A^\mu B^\nu$

In Schwarzschild coordinates, the line element is written as

$$ds^2 = - \left( 1 - \frac{2GM}{r} \right) dt^2 + \left( 1 - \frac{2GM}{r} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

Let us consider a contravariant vector

$$V^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \omega \end{pmatrix}$$

$$g_{\mu\nu} = \begin{pmatrix} - \left( 1 - \frac{2GM}{r} \right) & 0 & 0 & 0 \\ 0 & \left( 1 - \frac{2GM}{r} \right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

$$V_\mu = g_{\mu\nu} V^\nu \quad \longrightarrow \quad V_\mu = \begin{pmatrix} - \left( 1 - \frac{2GM}{r} \right) \\ 0 \\ 0 \\ r^2 \sin^2 \theta \cdot \omega \end{pmatrix}$$

## Some additional notes

- **Symmetric tensor**  $T_{\mu\nu} = T_{\nu\mu}$

Metric tensor is an example of a symmetric tensor

- **Antisymmetric tensor**  $T_{\mu\nu} = -T_{\nu\mu}$

The electromagnetic Faraday tensor  $F_{\mu\nu}$  is an example of Antisymmetric tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

- **Higher rank tensor**  $p + q$   $T_{\nu_1\nu_2\dots\nu_q}^{\mu_1\mu_2\dots\mu_p}$

- **Transformation of a rank 2 tensor**

$$T_{\bar{\mu}\bar{\nu}} = \Lambda_{\bar{\mu}}^{\alpha} \Lambda_{\bar{\nu}}^{\beta} T_{\alpha\beta} = \frac{\partial x^{\alpha}}{\partial x^{\bar{\mu}}} \frac{\partial x^{\beta}}{\partial x^{\bar{\nu}}} T_{\alpha\beta}$$

$$T^{\mu\nu} = \underbrace{\frac{1}{2}(T^{\mu\nu} + T^{\nu\mu})}_{\text{symmetric part}} + \underbrace{\frac{1}{2}(T^{\mu\nu} - T^{\nu\mu})}_{\text{antisymmetric part}}$$

$$T^{\mu\nu} = T^{(\mu\nu)} + T^{[\mu\nu]}$$

### Some properties

- A rank 2 tensor in  $n$  dimensions has  $n^2$  components
- A symmetric rank 2 tensor has  $\frac{n(n+1)}{2}$  components
- An anti-symmetric rank 2 tensor has  $\frac{n(n-1)}{2}$  components

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

## Length of a curve

$$l = \int \sqrt{g_{\alpha\beta} dx^\alpha dx^\beta}$$

## Four volume element

$$dV_4 = \sqrt{-g} d^4x$$

Takes into account the  
Jacobian of transformation

## Spatial volume

$$dV = \sqrt{\gamma} d^3x$$

## Example

### Spatial metric

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

### Determinant

$$g = \det(g_{\mu\nu}) = r^4 \sin^2 \theta$$

### Volume element

$$dV = r^2 \sin \theta dr d\theta d\phi$$

# Covariant derivatives

Consider a vector field  $\vec{V} = V^\alpha \vec{e}_\alpha$

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} \vec{e}_\alpha + V^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\beta}$$

Term that exists in Cartesian coord. Arises in curved spacetime

$$= \frac{\partial V^\alpha}{\partial x^\beta} \vec{e}_\alpha + V^\alpha \Gamma_{\alpha\beta}^\mu \vec{e}_\mu$$

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma_{\alpha\beta}^\mu \vec{e}_\mu$$

Relabelling dummy indices  $\alpha \rightarrow \mu, \mu \rightarrow \alpha$  we obtain

$$\nabla_\beta V^\alpha = \left( \frac{\partial V^\alpha}{\partial x^\beta} + \Gamma_{\mu\beta}^\alpha V^\mu \right) \vec{e}_\alpha$$

For flat spaces the covariant derivative reduces to partial derivatives as  $\Gamma_{\mu\nu}^\alpha = 0$

- Covariant derivative of a covariant vector

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma_{\mu\nu}^\lambda V_\lambda$$

- Covariant derivative of a tensor

$$\nabla_\lambda T^\mu_\nu = \partial_\lambda T^\mu_\nu + \Gamma_{\lambda\sigma}^\mu T^\sigma_\nu - \Gamma_{\lambda\nu}^\sigma T^\mu_\sigma$$

**Christoffel symbols describe how coordinates change in curved spacetime and appear in the geodesic equation**

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2}g^{\lambda\sigma} \left( \partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\nu} \right)$$

- Symmetry in lower indices  $\Gamma_{\mu\nu}^{\lambda} = \Gamma_{\nu\mu}^{\lambda}$
- Transformations under coordinate change **do not follow tensor rules.**
- In a flat the connection vanishes  $\Gamma_{\mu\nu}^{\lambda} \sim 0$

**The Levi-Civita connection is uniquely defined under the following assumptions**

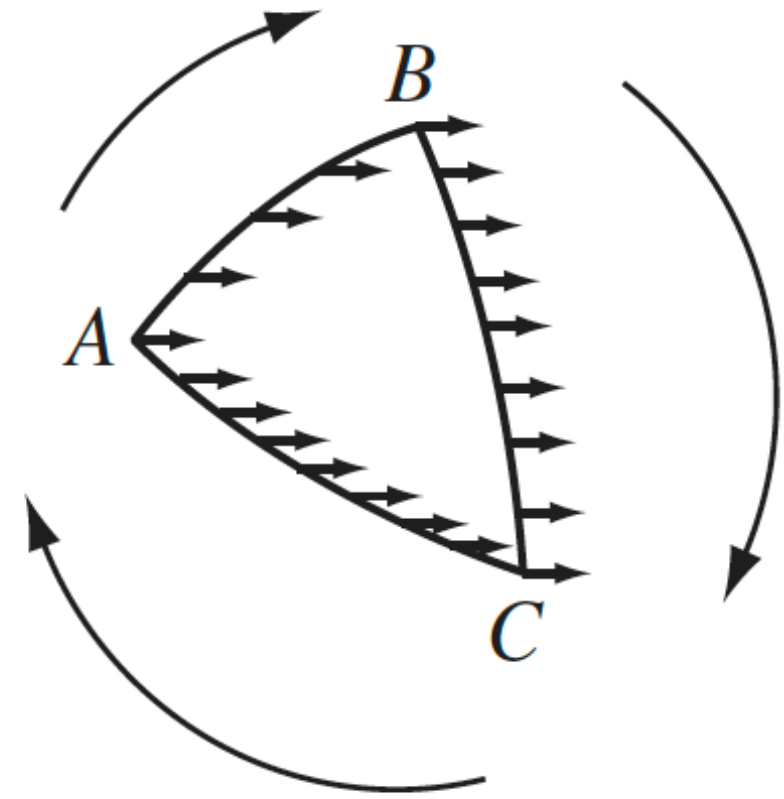
- Metric compatibility  $\nabla_{\lambda}g_{\mu\nu} = 0$
- Torsion free  $\Gamma_{\mu\nu}^{\lambda} = \Gamma_{\nu\mu}^{\lambda}$

## Note

In alternative theories the assumption of metric compatibility, torsion free and dependence of connection on the metric can be broken

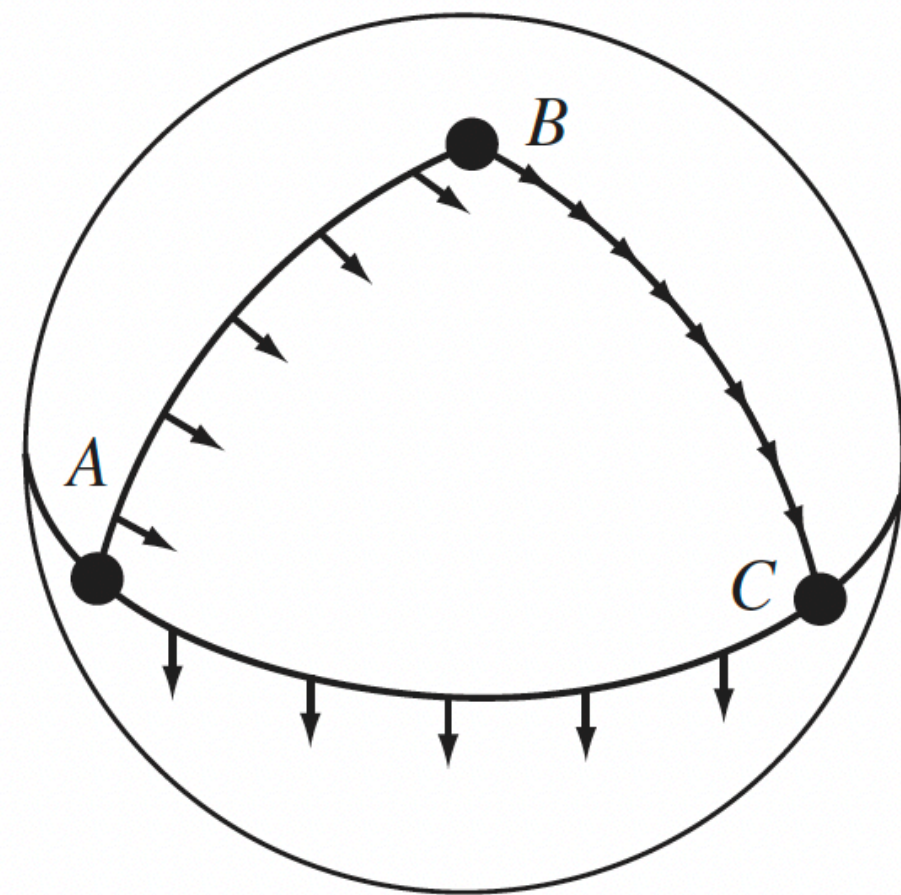
- Palatini  $f(R)$  theory
- Einstein Cartan theory
- Symmetric Teleparallel gravity

# Parallel transport



A 'triangle' made of curved lines in flat space.

It helps us define the notion of curvature !



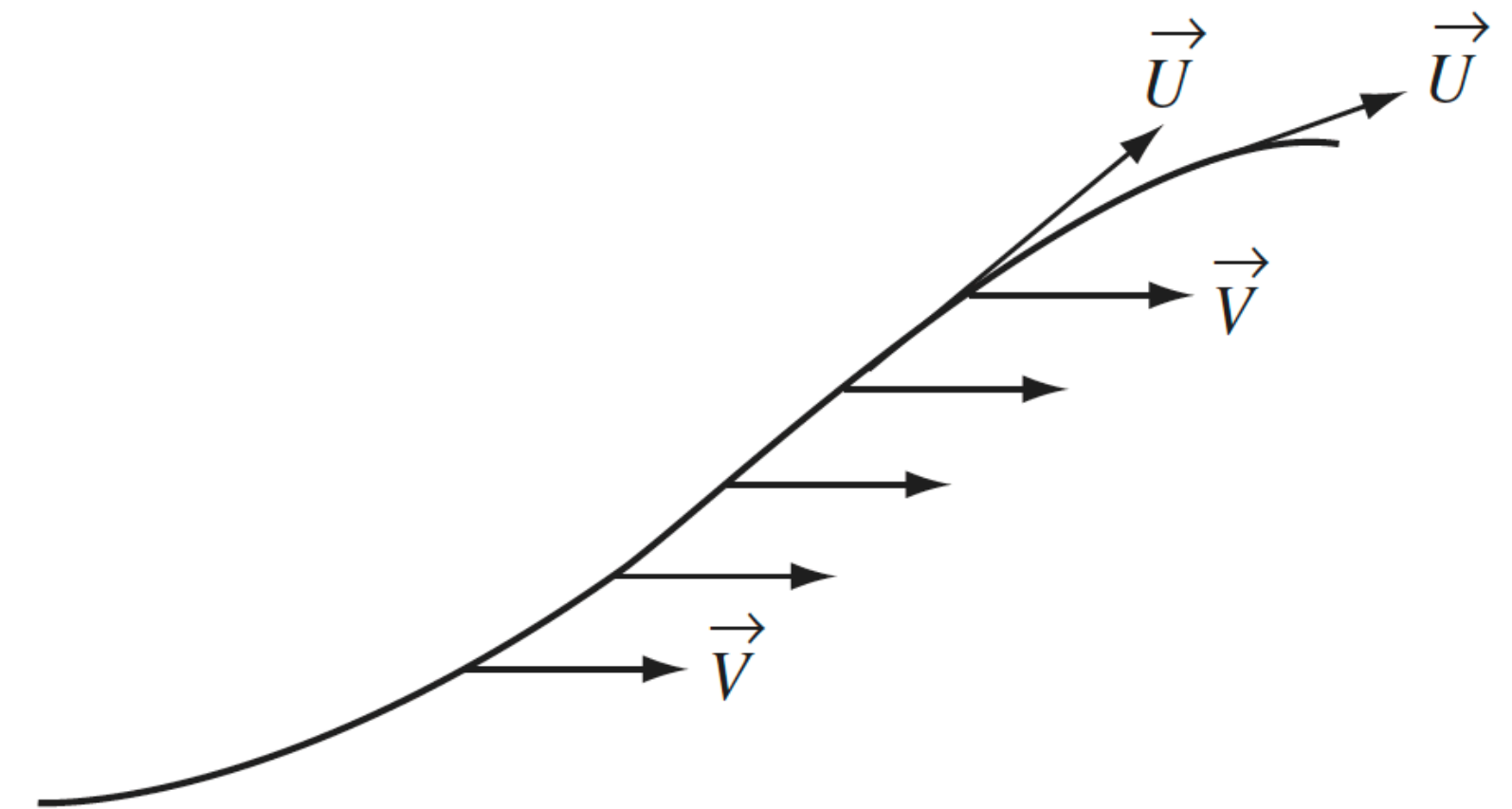
Parallel transport around a spherical triangle.

- It is not possible to define parallel vector fields in a curved spacetime
- Local parallelism can still be defined

$$\frac{dV^\alpha}{d\lambda} = 0 \text{ at } \mathcal{P}$$

$$\frac{dV^\alpha}{d\lambda} = U^\beta \partial_\beta V^\alpha = U^\beta \nabla_\beta V^\alpha$$

$$\nabla_{\vec{U}} \vec{V} = 0$$



## Remark :

Another requirement of metric compatibility  $\nabla_\lambda g_{\mu\nu} = 0$  is the necessity to preserve length and angles of vector during parallel transport

Geodesics are curves that parallel transport its own tangent vector, here  $U^\alpha = \frac{dx^\alpha}{d\lambda}$

$$\nabla_{\vec{U}} \vec{U} = 0$$

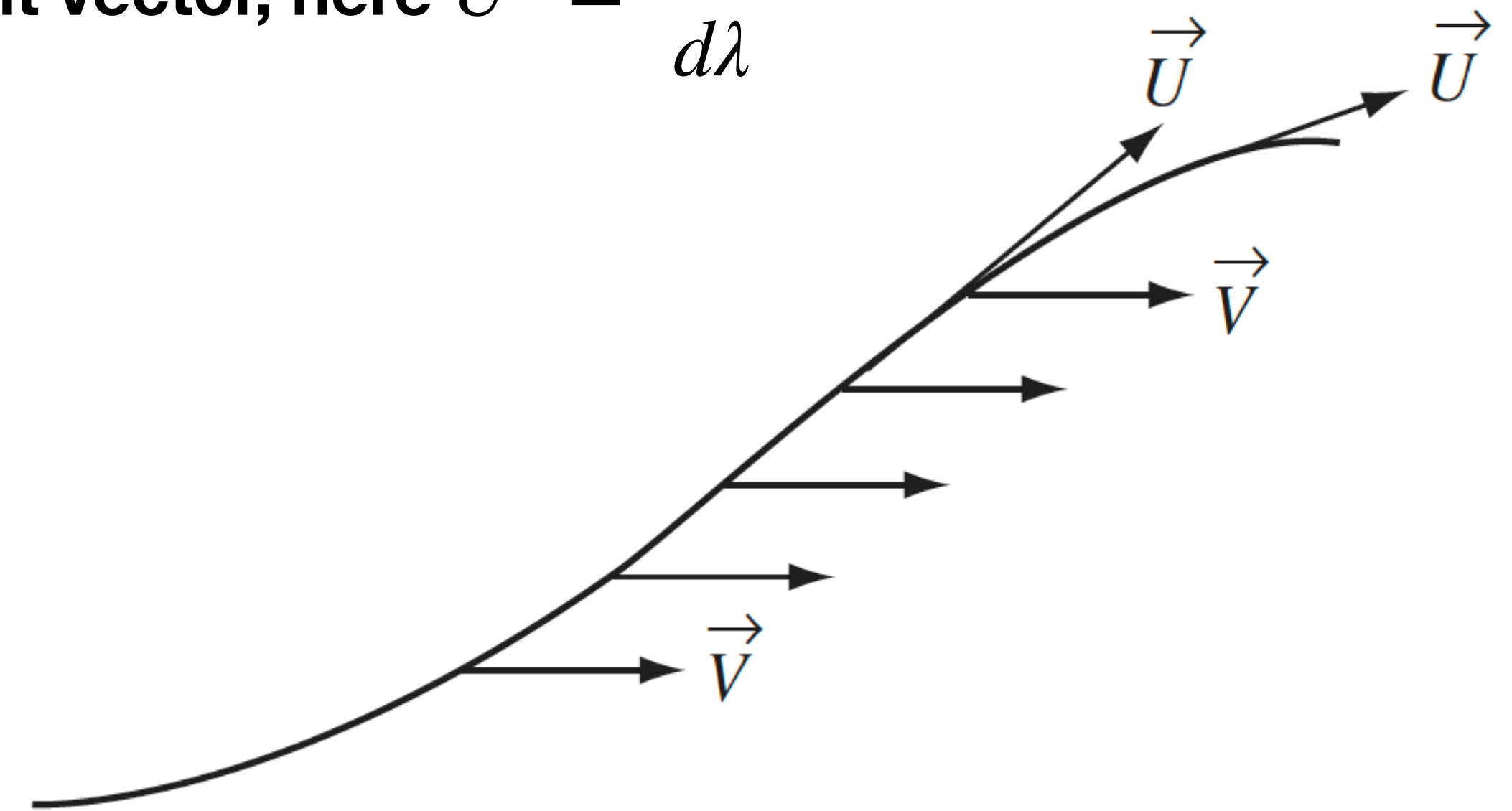
$$U^\beta \nabla_\beta U^\alpha = 0$$

Using the definition of covariant derivative

$$U^\beta \partial_\beta U^\alpha + \Gamma_{\mu\beta}^\alpha U^\mu U^\beta = 0$$

Geodesics satisfy the differential equation

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma_{\mu\beta}^\alpha \frac{dx^\mu}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$$



## Note

- Geodesics are extension of straight lines in curved space
- In Euclidian space straight lines are the only curves that parallel transports its own tangent vector
- Describe free fall motion in curved spacetime

If we define a new parameter  $\phi = a\lambda + b$

$$\frac{d^2 x^\alpha}{d\phi^2} + \Gamma_{\mu\beta}^\alpha \frac{dx^\mu}{d\phi} \frac{dx^\beta}{d\phi} = 0$$

**Beware  $a$  and  $b$  must be constants....**

If the above equation is satisfied then  $\phi$  is an affine parameter

---

## Newton's equation

In the static, weak-field, slow-velocity regime EOM of Newtonian gravity is recovered

$$g_{00} = -(1 + 2\Phi) \quad \Gamma_{00}^i = \partial^i \Phi$$

$$\frac{d^2 x^i}{dt^2} = -\partial_i \Phi \quad \text{Try to obtain this !!}$$

**Geodesics are also curves of extremal length**

$$\delta \int ds = 0$$

where  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

**Try it !!**

Find the Euler- Lagrange equations by solving the equation.

A mathematical description of an intrinsic curvature of a manifold is given by

**Riemann Curvature tensor**

$$\begin{aligned}
 [\nabla_{\mu}, \nabla_{\nu}]V^{\rho} &= \nabla_{\mu}\nabla_{\nu}V^{\rho} - \nabla_{\nu}\nabla_{\mu}V^{\rho} \\
 &= \nabla_{\mu}[\partial_{\nu}V^{\rho} + \Gamma^{\rho}_{\nu\sigma}V^{\sigma}] - (\mu \leftrightarrow \nu) \\
 &= \partial_{\mu}[\partial_{\nu}V^{\rho} + \Gamma^{\rho}_{\nu\sigma}V^{\sigma}] - \Gamma^{\lambda}_{\mu\nu}[\partial_{\lambda}V^{\rho} + \Gamma^{\rho}_{\lambda\sigma}V^{\sigma}] + \Gamma^{\rho}_{\mu\lambda}[\partial_{\nu}V^{\lambda} + \Gamma^{\lambda}_{\nu\sigma}V^{\sigma}] - (\mu \leftrightarrow \nu) \\
 &= \partial_{\mu}\partial_{\nu}V^{\rho} + \underbrace{\partial_{\mu}(\Gamma^{\rho}_{\nu\sigma})V^{\sigma} + \Gamma^{\rho}_{\nu\sigma}\partial_{\mu}V^{\sigma}}_{\partial_{\mu}(\Gamma^{\rho}_{\nu\sigma}V^{\sigma})} - \Gamma^{\lambda}_{\mu\nu}\partial_{\lambda}V^{\rho} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\rho}_{\lambda\sigma}V^{\sigma} + \Gamma^{\rho}_{\mu\lambda}\partial_{\nu}V^{\lambda} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma}V^{\sigma} \\
 &\quad - \partial_{\nu}\partial_{\mu}V^{\rho} - \underbrace{\partial_{\nu}(\Gamma^{\rho}_{\mu\sigma})V^{\sigma} + \Gamma^{\rho}_{\mu\sigma}\partial_{\nu}V^{\sigma}}_{\partial_{\nu}(\Gamma^{\rho}_{\mu\sigma}V^{\sigma})} + \Gamma^{\lambda}_{\nu\mu}\partial_{\lambda}V^{\rho} + \Gamma^{\lambda}_{\nu\mu}\Gamma^{\rho}_{\lambda\sigma}V^{\sigma} - \Gamma^{\rho}_{\nu\lambda}\partial_{\mu}V^{\lambda} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}V^{\sigma} \\
 &= \underbrace{\left[\partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\right]}_{R^{\rho}_{\sigma\mu\nu}}V^{\sigma} \\
 &= R^{\rho}_{\sigma\mu\nu}V^{\sigma}
 \end{aligned}$$

$$R^{\rho}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}.$$

$R^{\alpha}_{\beta\mu\nu} = 0 \implies$  flat manifold

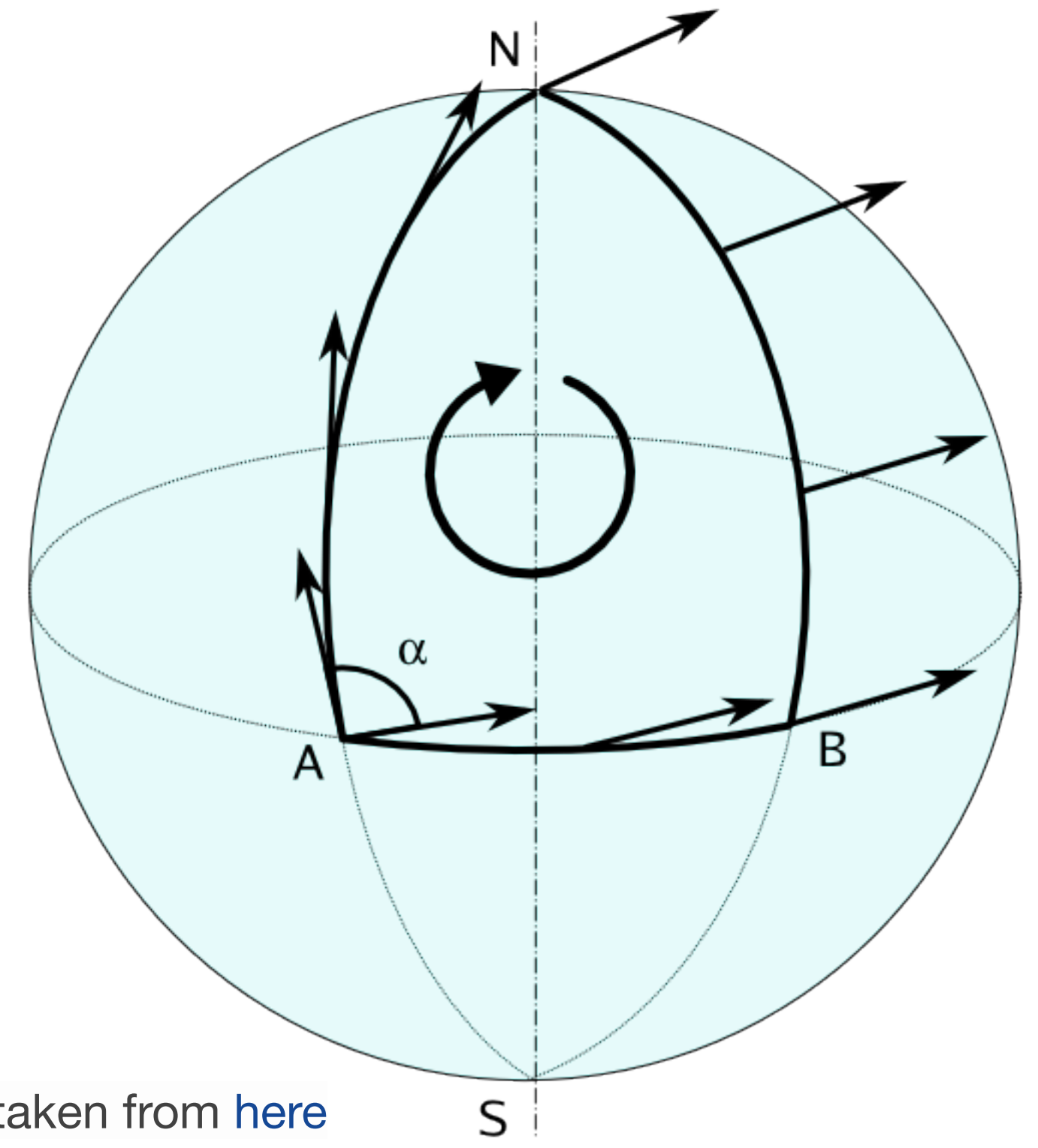


Image taken from [here](#)

## Definition

$$R^{\rho}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}.$$

## Properties

- Antisymmetry in the last two indices

$$R^{\rho}_{\sigma\mu\nu} = -R^{\rho}_{\sigma\nu\mu}$$

- Antisymmetry in the first two indices

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}.$$

- Symmetry under the exchange of index pairs

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$$

## Bianchi identities

- First identity

$$R_{\rho[\sigma\mu\nu]} = 0$$

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0.$$

- Second identity

$$\nabla_{\lambda}R^{\rho}_{\sigma\mu\nu} + \nabla_{\mu}R^{\rho}_{\sigma\nu\lambda} + \nabla_{\nu}R^{\rho}_{\sigma\lambda\mu} = 0.$$

## Ricci tensor

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$$

$$R_{\mu\nu} = \partial_{\rho}\Gamma^{\rho}_{\mu\nu} - \partial_{\nu}\Gamma^{\rho}_{\mu\rho} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\rho}.$$

## Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu}$$

## Einstein- Hilbert action

$$S = \frac{1}{16\pi G} \int R \sqrt{-g} d^4x + S_{\text{matter}}$$

with cosmological constant

$$S = \frac{1}{16\pi G} \int (R - 2\Lambda) \sqrt{-g} d^4x + S_{\text{matter}}$$

$$\delta S = 0 \quad \text{(Action principle)}$$

Varying the action w.r.t to the metric gives the EFE

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- The action is scalar under diffeomorphisms, it ensures physics is coordinate independent.
- It is symmetric and has 10 independent components
- The field equations are second order in derivatives
- Locally Lorentz invariant
- Reduces to Newtonian gravity in the weak field limit
- Energy momentum conservation  $\nabla_{\mu} T^{\mu\nu} = 0$
- And the Bianchi identity  $\nabla_{\mu} G^{\mu\nu} = 0$  independent of matter