Nonlocal model for the turbulent fluxes due to thermal convection in rectilinear shearing flow

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Abstract. We revisit a phenomenological description of turbulent thermal convection along the lines proposed originally by Gough (1965, 1977) in which eddies grow solely by extracting potential energy from the unstably stratified mean state and are subsequently destroyed by internal shear instability. This work is part of an ongoing investigation for finding a procedure to calculate the turbulent fluxes of heat and momentum in the presence of a shearing background flow in stars. In order to test and calibrate the formalism it is prudent first to compare its predictions with existing results from what we hope are more reliable investigations, such as experiments or numerical simulations. Here we compare the functional forms of the mean temperature profile and the distortion of an imposed horizontal shearing flow that is induced by the Reynolds stress implied by our theory with that of direct numerical simulations of Rayleigh-Bénard convection in air. In this contribution we present our latest results from a nonlocal generalization of our earlier local model (Smolec, Houdek & Gough 2011).

Introduction

Convection models based on the mixing-length approach still represent the main method for computing the turbulent fluxes in stars with convectively unstable regions. In such regions the pulsational stability of the star is affected not only by the radiative heat flux but also by the modulation of the convective heat flux and by direct mechanical coupling of the pulsation with the convective motion via the Reynolds stresses. Time-dependent formulations of the mixing-length approach for radial pulsation have been proposed by, for example, Gough (1965, 1977a) and Unno (1967). In a first step towards a generalization to nonradially pulsating stars, Gough & Houdek (2001) and Smolec, Houdek & Gough (2011) adopted Gough's (1977a) formulation, incorporating into it a treatment of the influence of a shearing background flow. In this generalized framework of the mixing-length formalism, in which turbulent convective eddies grow according to linearized theory and are subsequently broken up by internal shear instability, there is a consequent reduction in the mean amplitude of the eddy motion, and a corresponding reduction in the heat flux.

In this contribution we compare an extended, nonlocal version of Gough & Houdek's convection model with the results of Domaradzki & Metcalfe's (1988) Direct Numerical Simulations (DNS) of Rayleigh-Bénard convection in air in the presence of a strongly shearing background flow (see Fig. 1). Viscous terms, normally omitted in models of stellar convection, must therefore be retained in the model equations, which significantly adds to the complexity of the problem.

Turbulent fluxes in the presence of a shear

In Cartesian co-ordinates $(x,\ y,\ z)$ dimensionless equations (using d and the thermal diffusion time across d as units of space and time) describing the dynamics in a statistically stationary flow of a viscous Boussinesq liquid confined between two horizontal planes separated by d are (e.g. Chandrasekhar 1961):

$$\nabla \cdot \hat{\boldsymbol{u}} = 0 \,, \tag{1a}$$

$$[\partial_t + \hat{\boldsymbol{u}} \cdot \nabla]\hat{\boldsymbol{u}} = -\nabla p' + \sigma R T' \mathbf{e}_{\boldsymbol{z}} + \sigma \nabla^2 \hat{\boldsymbol{u}} \,, \tag{1b}$$

$$[\partial_t + \hat{\boldsymbol{u}} \cdot \nabla]T' - \beta \hat{\boldsymbol{u}} \cdot \mathbf{e}_{\boldsymbol{z}} = \nabla^2 T' \,, \tag{1c}$$

where $\hat{\boldsymbol{u}}$ is the total velocity field, which can be decomposed into a mean flow $\boldsymbol{U} = (U_1,\ U_2,\ 0.)$ and into the turbulent velocity fluctuations $\boldsymbol{u} = (u,\ v,\ w)$, and T' and p' are the Eulerian temperature and pressure fluctuations, respectively; $\sigma = \nu/\kappa$ is the Prandtl number (with ν and κ being the kinematic viscosity and thermal diffusivity, respectively), R is the Rayleigh number, $\beta \equiv -\mathrm{d}\overline{T}/\mathrm{d}z$ ($\overline{T}$ being the horizontal-mean temperature) is the mean temperature gradient, and $\mathbf{e}_z$ is the unit vector in the vertical direction. Radiative transfer is treated in the diffusion approximation. In accordance with a local formulation, β and the shear $\boldsymbol{E} = (\mathrm{d}\boldsymbol{U}/\mathrm{d}z) = (E_1,\ E_2,\ 0)$ are at first regarded as being constant over the vertical extent of an eddy.

The pressure fluctuations in Eq. (1b) can be eliminated by taking the curl and double curl of Eq. (1b). The resulting linearized equations can then be expanded into normal modes of the form:

$$w(x,y,z,t) = W(z) f(x,y) \mathrm{e}^{q(t-t_0)} \,, \tag{2a}$$

$$T'(x,y,z,t) = \Theta(z) f(x,\ y) \mathrm{e}^{q(t-t_0)} \,, \tag{2b}$$

$$\omega_3(x,y,z,t) = \Omega(z) f(x,\ y) \mathrm{e}^{q(t-t_0)} \,, \tag{2c}$$

where ω_3 denotes the vertical component of the fluctuating vorticity, $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$, and q is the linear growth rate with which the convective eddy fluctuations, created at the time t_0, grow with time, t. The planform, for rolls, $f(x) = \exp(\mathrm{i}ax)$, describes the horizontal structure of a component of the convective flow (a is a wavenumber in the x-direction). The resulting separated equations for the eigenvalue q and the eigenfunctions W, Θ and Ω

are:

$$\left[q + \mathrm{i}azE_1 - \sigma(\mathrm{D}^2 - a^2)\right](\mathrm{D}^2 - a^2)W + a^2\sigma R\Theta = 0\,, \tag{3a}$$

$$\left[q + \mathrm{i}azE_1 - \sigma(\mathrm{D}^2 - a^2)\right]\Omega + \mathrm{i}aE_2W = 0\,, \tag{3b}$$

$$\left[q + \mathrm{i}azE_1 - (\mathrm{D}^2 - a^2)\right]\Theta - \beta W = 0\,, \tag{3c}$$

in which $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}z$. For small amplitudes of the shear $E = |\boldsymbol{E}|$, Eqs (3) can be solved with linear perturbation theory about $\boldsymbol{E} = \boldsymbol{0}$. We take it to second order in $\boldsymbol{E}$. We adopt free-free boundary conditions, which permits an analytical treatment. The resulting expressions for the eigenfunctions of the fluctuating temperature and turbulent velocity field are then used to compute the turbulent fluxes in the manner of Gough (1977a).

- The expression for the vertical component of the convective heat flux is

$$F_\mathrm{c} = \frac{4}{\hat{C}\Phi\eta^2\sigma R\ell^4}\hat{q}^2\left[\left(\hat{q} + 2\sigma\right)\left(1 - \frac{E^2}{\Phi k^2}\hat{I}_1\right) - E^2\left(2/k^4\right)\hat{I}_2\right], \tag{4}$$

  in which $\hat{C}$ is an uncertain constant which accounts for imperfect fluctuation correlations, $k^2 = a^2 + (\pi/\ell)^2$ is the total wavenumber, $\Phi = k^2/a^2$ is an eddy shape parameter, ℓ the mixing length, $\hat{q} = 2q/k^2$, and $\hat{I}_1$ & $\hat{I}_2$ are (known) spatial integrals involving the computed eigenfunctions.

- The expression for the Reynolds stresses $\overline{uw}$, needed for solving the mean momentum equation, is

$$\overline{uw} = \frac{8q^2}{\hat{C}\Phi k^2}E\left(\hat{I}_4 - \frac{1}{2\tilde{q}}\right), \tag{5}$$

  where $\hat{I}_4$ is a spatial integral involving the velocity eigenfunctions, and an overbar denotes horizontal average.

The mean equations and the nonlocal formulation

The equations for the mean temperature, $\overline{T}$, and velocity, U, across the fluid layer are:

$$\begin{aligned} \frac{\partial \overline{T}}{\partial z} &= -N + \langle F_{\rm c} \rangle \,, && \text{(6a)} \\ \frac{\partial U}{\partial z} &= \frac{1}{\sigma}\langle \overline{uw} \rangle + S \,, && \text{(6b)} \end{aligned}$$

in which $\langle F_{\rm c} \rangle$ and $\langle \overline{uw} \rangle$ are nonlocal expressions for the turbulent fluxes, defined by Eqs (8) below. These equations express the conservation of the heat and momentum fluxes in terms of N and S: the Nusselt number N (dimensionless heat flux) and the total stress, S, which are eigenvalues of the system. The convective heat flux distorts the temperature gradient, and the Reynolds stress $\overline{uw}$ distorts the shear, and consequently $\overline{T}$ and the x-component U of the mean flow are no longer linear functions of z.

In our nonlocal formalism the eddies of vertical extent ℓ, centred at height z_0, sample the temperature gradient over the spatial region $(z_0 - \ell/2,\ z_0 + \ell/2)$. Furthermore, the nonlocal expressions $\langle F_{\rm c} \rangle$ and $\langle \overline{uw} \rangle$ are determined from the contributions of all eddies centred between $z_0 - \ell/2$ and $z_0 + \ell/2$, e.g.,

$$\langle F_{\rm c} \rangle = \int_{z_0-\ell/2}^{z_0+\ell/2} F_{\rm c}(z)\mathcal{K}(z - z_0)\mathrm{d}z \,. \tag{7}$$

The averaging kernel, $\mathcal{K}$, depends on the structure of the eddies. Here we follow Gough (1977b) and approximate the kernel $\mathcal{K}$ by $1/2\mu\exp(-\mu|z - z_0|)$, considered to extend over $(-\infty;\ \infty)$, which leads to the following set of second-order differential equations:

$$\frac{1}{\mathsf{b}^2}\frac{\partial^2\langle\beta\rangle}{\partial z^2} = \langle\beta\rangle - \beta\,, \tag{8a}$$

$$\frac{1}{\mathsf{a}^2}\frac{\partial^2\langle F_{\rm c}\rangle}{\partial z^2} = \langle F_{\rm c}\rangle - F_{\rm c}\,, \tag{8b}$$

$$\frac{1}{\mathsf{c}^2}\frac{\partial^2\langle\overline{uw}\rangle}{\partial z^2} = \langle\overline{uw}\rangle - \frac{1}{2+\gamma}\Big[\overline{uw}(z-\zeta) + \gamma\overline{uw}(z) + \overline{uw}(z+\zeta)\Big]\,, \tag{8c}$$

where a, b and c are potentially different values of μ, and γ is a constant of order unity. Unlike the expressions for $\langle\beta\rangle$ and $\langle F_{\rm c}\rangle$, which are dominated by contributions from the central region of an eddy, the expression for $\langle\overline{uw}\rangle$ includes two additional contributions off-centred by the distance ζ, where the horizontal velocity fluctuation, u, is large. The parameters a, b and c control the degree of nonlocality: $(\mathsf{a},\ \mathsf{b},\ \mathsf{c}) \to \infty$ reproduces local behaviour.

For a given velocity difference, ΔU, (see **Fig. 1**) unique solutions for velocity and temperature profiles are obtained and compared with the DNS data of Domaradzki & Metcalfe (1988).

Results

Domaradzki & Metcalfe (1988) used Direct Numerical Simulations of Rayleigh-Bénard convection in a shearing flow (see **Fig. 1**) in air ($\sigma = 0.71$). They adopted $R = 10^5$, and a dimensionless velocity difference $\Delta U = 700$ between the upper, horizontally moving, plate and the stationary lower plate. In our model computations we set the mixing length ℓ to be α times the distance to the nearest boundary. The values of α and $\hat{C}$ were chosen to make the modelled N (for different Rayleigh numbers) agree with the experimental determination by Rossby (1969) for water and mercury. We chose the value 5/3 for the eddy shape parameter Φ; it maximizes the heat flux at constant ℓ (with $\boldsymbol{E} = \mathbf{0}$). The remaining nonlocal parameters a, b and c, as well as the separation parameter ζ and parameter γ, are obtained from calibrating the model to the DNS data for $\Delta U = 0$.

In **Fig. 2** we illustrate the effect of varying the nonlocal parameter, a, on the temperature gradients, β and $\langle\beta\rangle$. Depending on the value of the parameter a, β can become negative in some layers of the model, producing a temperature inversion (see right panel of **Fig. 2**). Such inversions have been seen in experiments (e.g. Deardorff & Willis 1967).

In the left panels of **Fig. 3** and **Fig. 4** the normalized mean temperature and mean vertical velocity profiles are plotted for four values of ΔU: 100, 200, 300 and 400. The profiles are in reasonable agreement with the DNS data, but for rather smaller values of ΔU than the value adopted in the DNS: $\Delta U = 700$. However, different values of ΔU match the DNS temperature profile ($\Delta U \approx 300 - 400$) and velocity profile ($\Delta U \approx 200 - 300$).

As reported before by Gough & Houdek (2001) and Smolec, Houdek & Gough (2011), the convective heat flux (hence the Nusselt number N) is reduced with increasing shear – see right panel of **Fig. 3** – in agreement with the DNS data and the experiments by Ingersoll (1966). The magnitude of the Reynolds stress increases with shear (**Fig. 4**).

Conclusions

- The heat flux is reduced by the shear, in agreement with simulations and experimental data.
- Agreement of the functional forms of the mean velocity and temperature profiles between our model results and the DNS data is obtained only for lesser shear (smaller ΔU) in the model than in the numerical simulations; and simultaneous agreement with both profiles is not possible with a single value of ΔU.
- There is a need for further numerical simulations to enable us to make comparisons over a much broader range of Rayleigh and Prandtl numbers.

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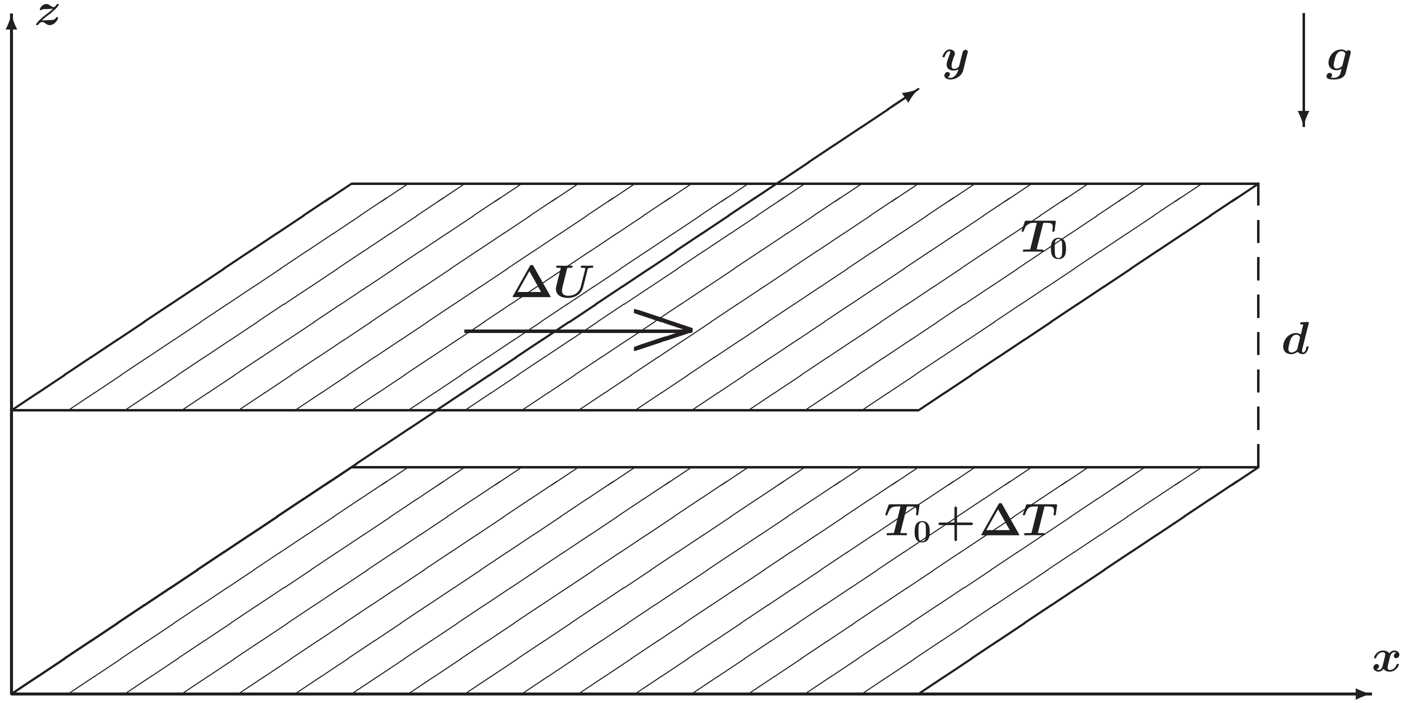

Figure 1. Rayleigh-Bénard convection. We consider a plane-parallel layer of fluid of infinite horizontal extent confined between rigid perfectly conducting boundaries at fixed temperatures. The boundaries are separated by a distance d, the lower being hotter than the upper by ΔT. In the presence of a shear, the upper boundary moves horizontally with constant velocity, ΔU.

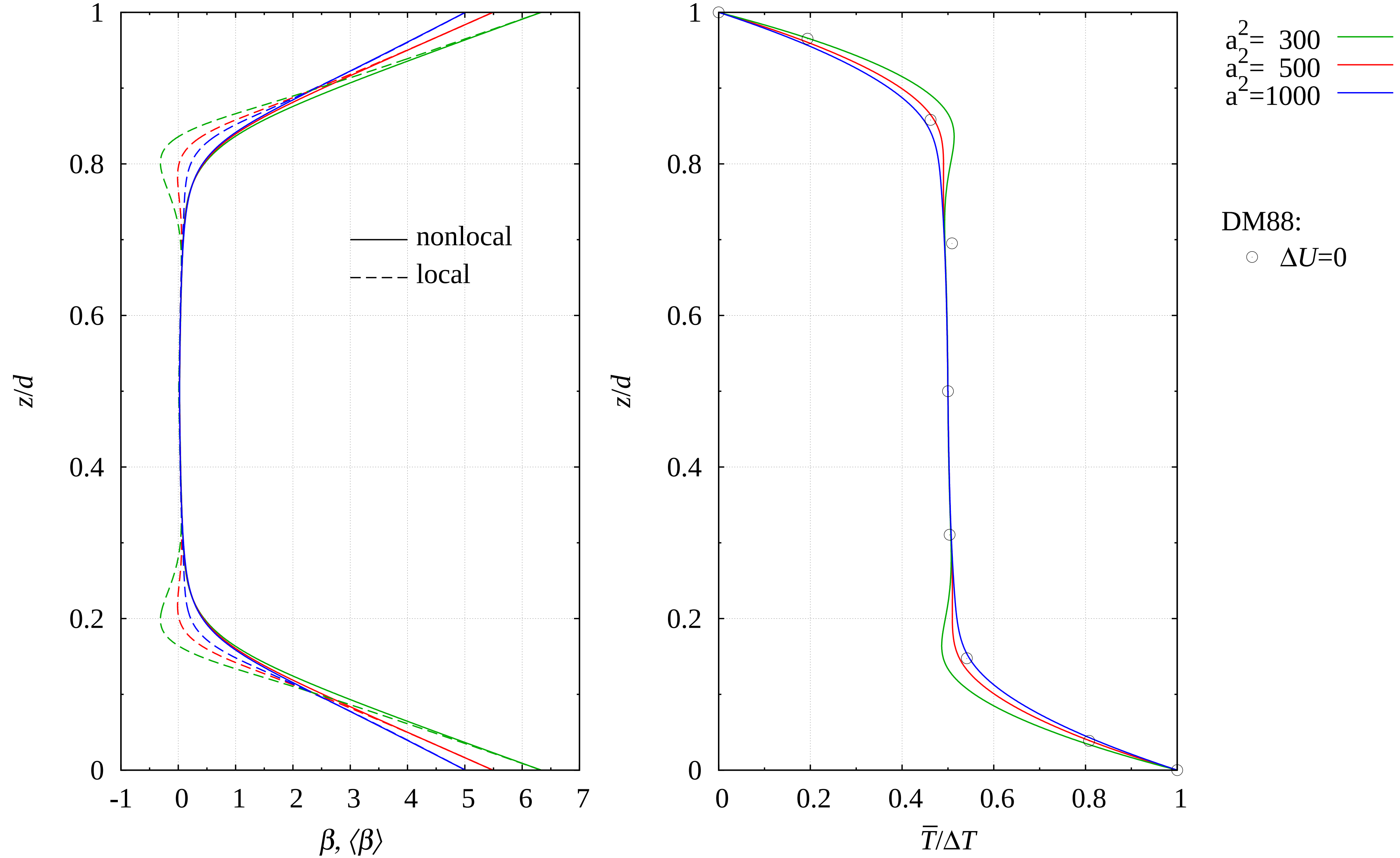

Figure 2. Effect of varying the nonlocal parameter a on the temperature gradient (left panel) and on the corresponding mean temperature profile of the nonlocal solution (right panel) for $\Delta U = 0$.

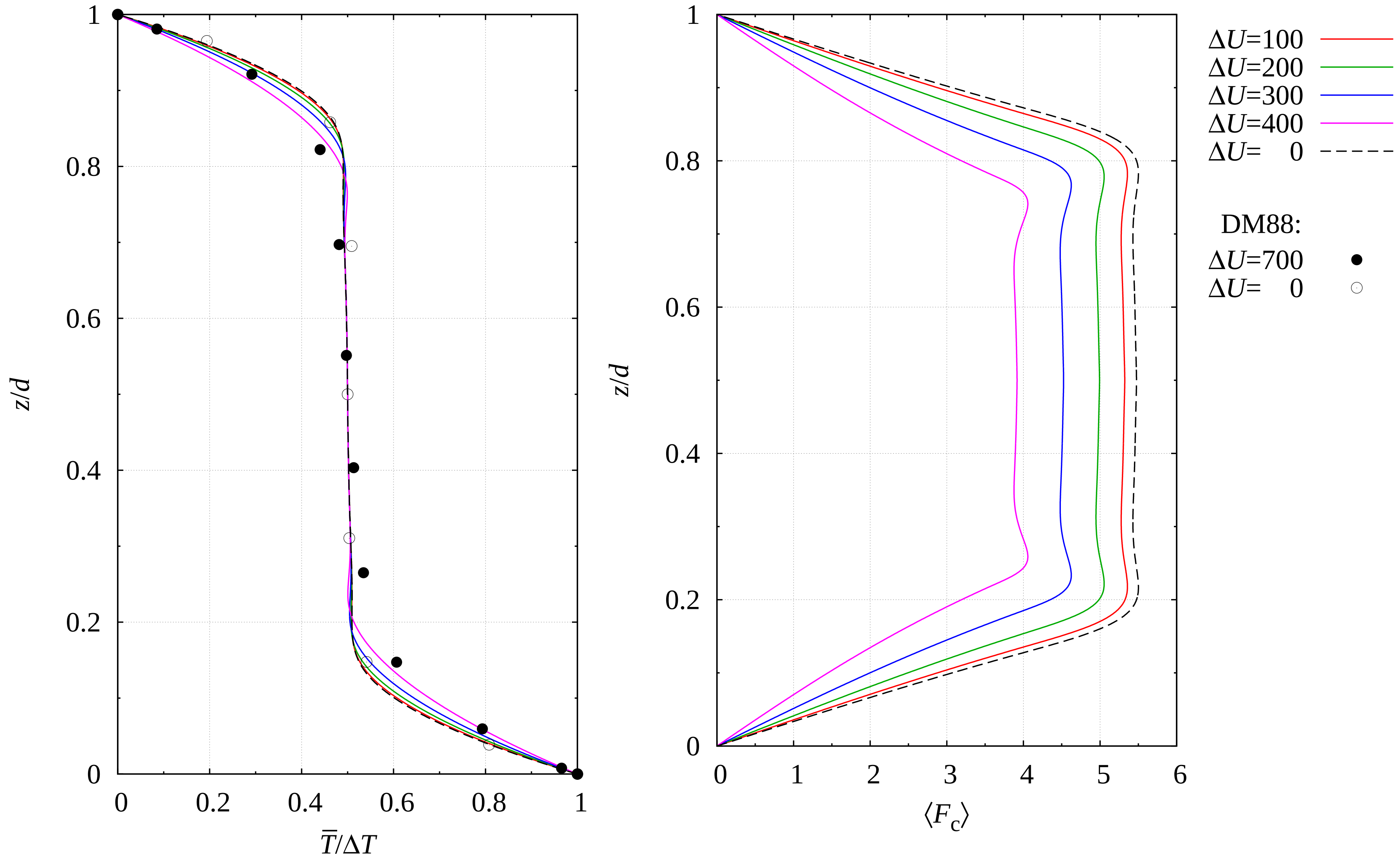

Figure 3. Mean temperature profiles $\overline{T}/\Delta T$ (left panel) and nonlocal convective flux $\langle F_c \rangle$ (right panel), for different values of ΔU. The temperature profiles are compared with DNS data (circles).

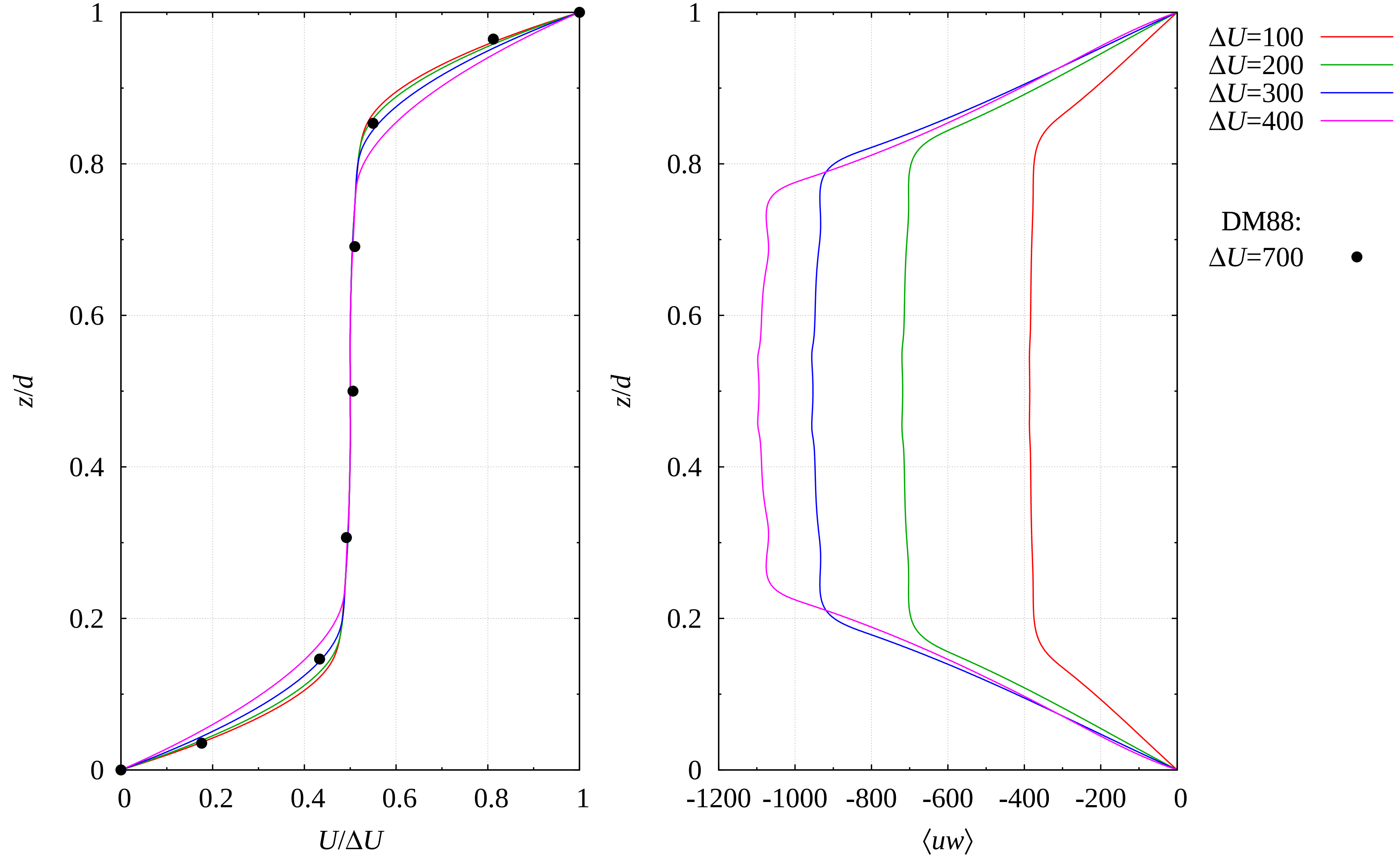

Figure 4. Mean velocity profiles $U/\Delta U$ (left panel) and nonlocal Reynolds stress $\langle\overline{uw}\rangle$ (right panel), for different values of ΔU. The velocity profiles are compared with DNS data (full circles).