Modelling turbulent fluxes due to thermal convection in rectilinear shearing flow

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Abstract. We revisit a phenomenological description of turbulent thermal convection along the lines proposed originally by Gough (1965, 1977a) in which eddies grow solely by extracting energy from the unstably stratified mean state and are subsequently destroyed by internal shear instability. This work is part of an ongoing investigation for finding a procedure to calculate the turbulent fluxes of heat and momentum in the presence of a shearing background flow in stars. In order to test and calibrate the formalism it is preferable first to compare its predictions with existing results of hopefully more reliable investigations, such as experiments or numerical simulations. Gough & Houdek (2001) have already indicated how shear in a plane Couette flow influences both the growth and annihilation rates of the convective eddies, thereby reducing the Nusselt number at fixed Rayleigh number, in agreement with experiments by Ingersoll (1966) and numerical simulations by Domaradzki & Metcalfe (1988). Here we compare the functional form of the temperature profile implied by our theory with that of the numerical simulations, and also the distortion of the shear produced by the Reynolds stress.

Introduction

Convection models based on the mixing-length approach still represent the main method for computing the turbulent fluxes in stars with convectively unstable regions. In such regions the pulsational stability of the star is affected not only by the radiative heat flux but also by the modulation of the convective heat flux and by direct mechanical coupling of the pulsation with the convective motion via the Reynolds stresses. Time-dependent formulations of the mixing-length approach for radial pulsation have been proposed for example, by Gough (1965, 1977a) and Unno (1967). In a first step towards a generalization to nonradially pulsating stars, Gough & Houdek (2001) adopted Gough's (1977a) formulation, incorporating into it a treatment of the influence of a shearing background flow. In this generalized framework of the mixing-length formalism, in which turbulent convective eddies grow according to linearized theory and are subsequently broken up by internal shear instability, there is a consequent reduction in the mean amplitude of the eddy motion, and a corresponding reduction in the heat flux. In its simplest form, the generalization of the formalism requires the explicit introduction of no new parameters. Consequently, the reduction in the heat flux predicted can be compared with experiment to provide an additional calibration of the formalism.

In this contribution we compare an extended version of Gough & Houdek's convection model with the results of Domaradzki & Metcalfe's (1988) Direct Numerical Simulations (DNS) of Rayleigh-Bénard convection in air in the presence of a strongly shearing background flow (see Fig. 1). Viscous terms must therefore be retained in the model equations, which significantly adds to the complexity of the problem.

Turbulent fluxes in the presence of a shear

In Cartesian co-ordinates $(x,\ y,\ z)$ dimensionless equations (using d and the thermal diffusion time across d as units of space and time) describing the dynamics in a statistically stationary flow of a viscous Boussinesq fluid confined between two horizontal planes separated by d are (e.g. Spiegel & Veronis 1960):

$$\nabla \cdot \hat{\boldsymbol{u}} = 0, \tag{1a}$$

$$[\partial_t + \hat{\boldsymbol{u}} \cdot \nabla]\hat{\boldsymbol{u}} = -\nabla p' + \sigma R T' \mathbf{e}_{\boldsymbol{z}} + \sigma \nabla^2 \hat{\boldsymbol{u}}, \tag{1b}$$

$$[\partial_t + \hat{\boldsymbol{u}} \cdot \nabla]T' - \beta \hat{\boldsymbol{u}} \cdot \mathbf{e}_{\boldsymbol{z}} = \nabla^2 T'. \tag{1c}$$

where $\hat{\boldsymbol{u}}$ is the total velocity field, which can be decomposed into a mean flow $\boldsymbol{U} = (U_1,\ U_2,\ 0.)$ and into the turbulent velocity fluctuations $\boldsymbol{u} = (u,\ v,\ w)$, and T' and p' are the Eulerian temperature and pressure fluctuations, respectively; $\sigma = \nu/\kappa$ is the Prandtl number (with ν and κ being the kinematic viscosity and thermal diffusivity, respectively), R is the Rayleigh number, $\beta \equiv -\mathrm{d}\overline{T}/\mathrm{d}z - g/c_p$ ($\overline{T}$ being the mean temperature, c_p the specific heat at constant pressure and g the acceleration due to gravity) is the superadiabatic lapse rate, and $\mathbf{e}_z$ is the unit vector in the vertical direction. Radiative transfer is treated in the diffusion approximation.

In accordance with a local formulation in the Boussinesq approximation, in which β is regarded as constant, we assume for the rectilinear shearing flow (i.e., plane Couette flow) a constant shear $\boldsymbol{E}$

$$\boldsymbol{E} = \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}z} = (E_1,\ E_2,\ 0). \tag{2}$$

The pressure fluctuations in Eq. (1b) can be eliminated by taking the curl and double curl. The resulting linearized equations can then be expanded into normal modes of the form,

$$w(x, y, z, t) = W(z) f(x, y) \mathrm{e}^{q(t-t_0)}\,, \tag{3a}$$

$$T'(x, y, z, t) = \Theta(z) f(x,\ y) \mathrm{e}^{q(t-t_0)}\,, \tag{3b}$$

$$\omega_3(x, y, z, t) = \Omega(z) f(x,\ y) \mathrm{e}^{q(t-t_0)}\,, \tag{3c}$$

where ω_3 denotes the vertical component of the fluctuating vorticity, $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$, and q is the linear growth rate with which the convective fluctuations of an eddy, created at the time t_0, grow with time, t. For convenience we adopt a complex notation, writing the planform, for rolls, as $f(x) = \cos ax + \mathrm{i} \sin ax$, to describe the horizontal structure of the flow. Here a is a wavenumber in the x direction. The resulting separable equations for the eigenvalue q and the eigenfunctions W, Θ and Ω are:

$$\left[q + \mathrm{i}azE_1 - \sigma(\mathrm{D}^2 - a^2)\right](\mathrm{D}^2 - a^2)W + a^2\sigma R\Theta = 0\,, \tag{4a}$$

$$\left[q + \mathrm{i}azE_1 - \sigma(\mathrm{D}^2 - a^2)\right]\Omega + \mathrm{i}aE_2 W = 0\,, \tag{4b}$$

$$\left[q + \mathrm{i}azE_1 - (\mathrm{D}^2 - a^2)\right]\Theta - \beta W = 0\,, \tag{4c}$$

in which $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}z$. For small amplitudes of the shear $\boldsymbol{E}$, Eqs (4) can be solved with linear perturbation theory about $\boldsymbol{E} = \mathbf{0}$. We take it to second order in $\boldsymbol{E}$. We adopt free-free boundary conditions, which permits an analytical treatment. The resulting expressions for the eigenfunctions of the fluctuating temperature and turbulent velocity field are then used to compute the turbulent fluxes in the manner of Gough (1977a). Note, that in a local formulation for a Boussinesq fluid the shear is assumed to be constant over the vertical extent of an eddy, as also is the superadiabatic lapse rate β.

- The expression for the vertical component of the convective heat flux is

$$F_{\mathrm{c}} = \frac{k^6\Phi}{2\hat{C}\left[\Phi k^2 + E^2\hat{I}_1\right]\sigma R}\hat{q}^2\left[(\hat{q} + 2\sigma) - E^2\hat{I}_2\right]\,, \tag{5}$$

  in which $\hat{C}$ is a constant, $k^2 = a^2 + (\pi/\ell)^2$ is the total wavenumber, $\Phi = k^2/a^2$ is an eddy shape parameter, ℓ the mixing length, $\hat{q} = 2q/k^2$, and $\hat{I}_1$ & $\hat{I}_2$ are (known) spatial integrals involving the computed eigenfunctions.

- The expressions for the Reynolds stresses $\overline{\rho w^2}$ and $\overline{\rho uw}$ are

$$\overline{\rho w^2} = \frac{16q^2}{\hat{C}\left[\Phi k^2 + E^2 \hat{I}_1\right]}\left(\frac{1}{2} + E^2 \hat{I}_3\right), \tag{6a}$$

$$\overline{\rho uw} = \frac{16q^2}{\hat{C}\left[\Phi k^2 + E^2 \hat{I}_1\right]} E^2 \hat{I}_4\,, \tag{6b}$$

  where $\hat{I}_3$ and $\hat{I}_4$ are spatial integrals involving the velocity eigenfunctions, and a horizontal bar denotes horizontal average.

Mean equations

The sum of the radiative and convective fluxes is independent of z. In our dimensionless formulation it is equal to the Nusselt number (dimensionless heat flux) N of the layer:

$$\begin{aligned} N &= F_{\rm r}(z) + F_{\rm c}(z), \\ &= \beta + \beta_{\rm ad} + F_{\rm c}(z), \end{aligned} \tag{7}$$

where $\beta_{\rm ad} = g/c_p$. This is an equation for the mean temperature $\overline{T}$. It is solved simultaneously with the horizontal mean momentum equation

$$\frac{\rm d}{{\rm d}z}\overline{\rho wu} - \sigma \frac{{\rm d}^2 U}{{\rm d}z^2} = 0\,, \tag{8}$$

where $U = |\boldsymbol{U}|$, N being an eigenvalue. Note that the Reynolds stress $\overline{\rho wu}$ distorts the shear, and consequently the x-component U of the mean flow is no longer a linear function of z [see Eq. (2)].

Results

Domaradzki & Metcalfe (1988) used Direct Numerical Simulations of Rayleigh-Bénard convection in a shearing flow (see **Fig. 1**) in air ($\sigma = 0.71$). They assumed a Rayleigh number $R = 10^5$, and a dimensionless velocity difference $\Delta U = 700$ between the upper, horizontally moving, plate and the stationary lower plate. In our model computations we set the mixing length $\ell = \alpha$ times the distance to the nearest boundary. The values of α and $\hat{C}$ (see Eqs 5 & 6) were chosen to make the modelled N (for different Rayleigh numbers) agree with the experimental determination by Rossby (1969) for water and mercury. We chose the value 5/3 for the eddy shape parameter Φ; it maximizes the heat flux at constant ℓ (and $\boldsymbol{E} = \mathbf{0}$).

In **Fig. 2** the normalized mean vertical velocity profiles, $U/\Delta U$, are plotted for four values of ΔU: 50, 100, 150 and 200. Flow in the y direction is uninfluenced by the aligned rolls. The corresponding shear profiles, $E = |\boldsymbol{E}|$, are illustrated in **Fig. 3**, and the mean temperature profiles are displayed in **Fig. 4**. The mean velocity profiles (**Fig. 2**) are in reasonable agreement with the DNS data, but for smaller values of ΔU in our model computations. As expected, the shear (**Fig. 3**) is large and constant in the radiative boundary layers, where the velocity and temperature profiles vary linearly with height. The extent of the radiative zones increases with ΔU.

The modelled temperature profiles are in poorer agreement with the DNS data. Best agreement with the DNS data for the Nusselt number, N, which is equal to the slope of the temperature profile in the radiative zones (see Eq. 7), is obtained with $\Delta U = 150$. **Fig. 5** illustrates the corresponding heat flux (solid curve) together with the result obtained without shear ($\Delta U = 0$; dashed curve). As reported before by Gough & Houdek (2001), the convective heat flux (hence the Nusselt number N) is reduced with increasing shear, in agreement with the DNS data and the experiments by Ingersoll (1966).

Conclusions

- The heat flux is reduced by the shear, in agreement with simulations and experimental data.
- Agreement of the mean velocity and temperature profiles between model results and DNS data is obtained only for smaller shear values (ΔU values) in the model calculations.

Two factors may be responsible for the discrepancy in the values of ΔU:

- In order to treat the model equations analytically, we adopted free-free boundary conditions, whereas rigid boundary conditions were used in the simulations. This may account for up to a factor of about three in the ΔU differences between the model ($\Delta U\!=\!150$) and simulation ($\Delta U\!=\!700$) results.
- Nonlocal effects may also contribute to the remaining differences between the model results and DNS data (e.g. Tooth & Gough 1988). We therefore plan to extend our local model in the manner of Gough (1977b) to accommodate nonlocal behaviour. Then the convective fluxes of heat and momentum will no longer vanish in the (locally) stable boundary layers.

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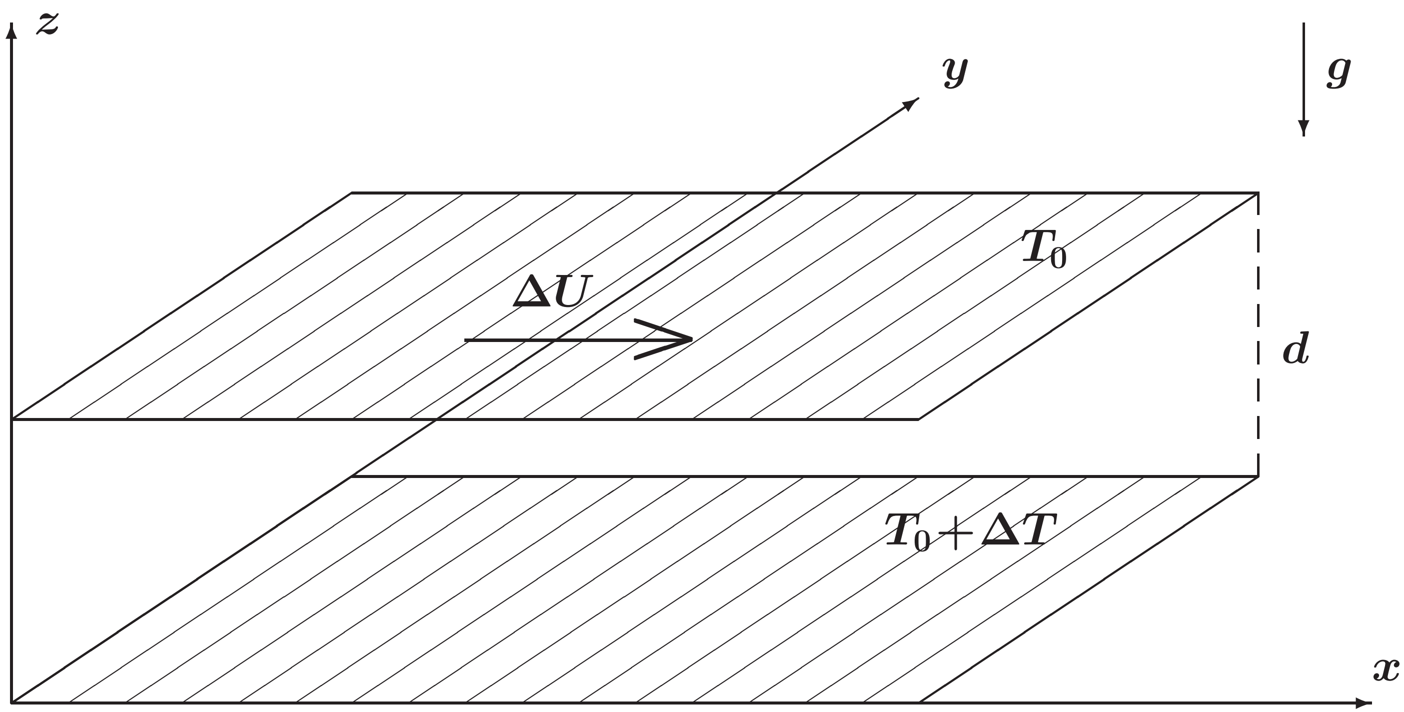

Fig. 1. Rayleigh-Bénard convection. We consider a plane-parallel layer of fluid of infinite horizontal extent confined between rigid perfectly conducting boundaries at fixed temperatures. The boundaries are separated by a distance d, the lower being hotter than the upper by ΔT. In the presence of a shear, the upper boundary moves horizontally with constant velocity, ΔU.

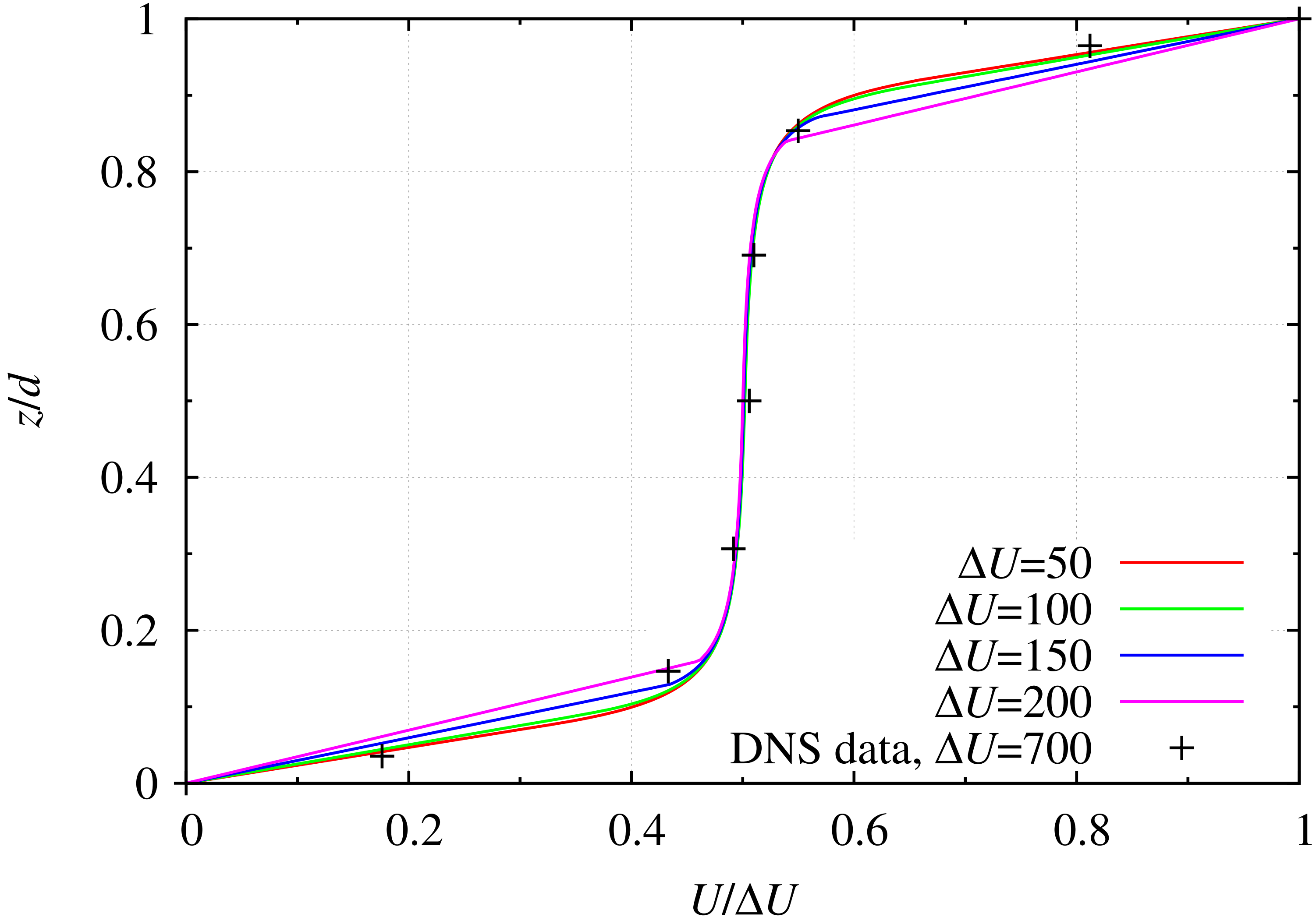

Figure 2. Mean velocity profiles, $U/\Delta U$, for different ΔU, compared with DNS data.

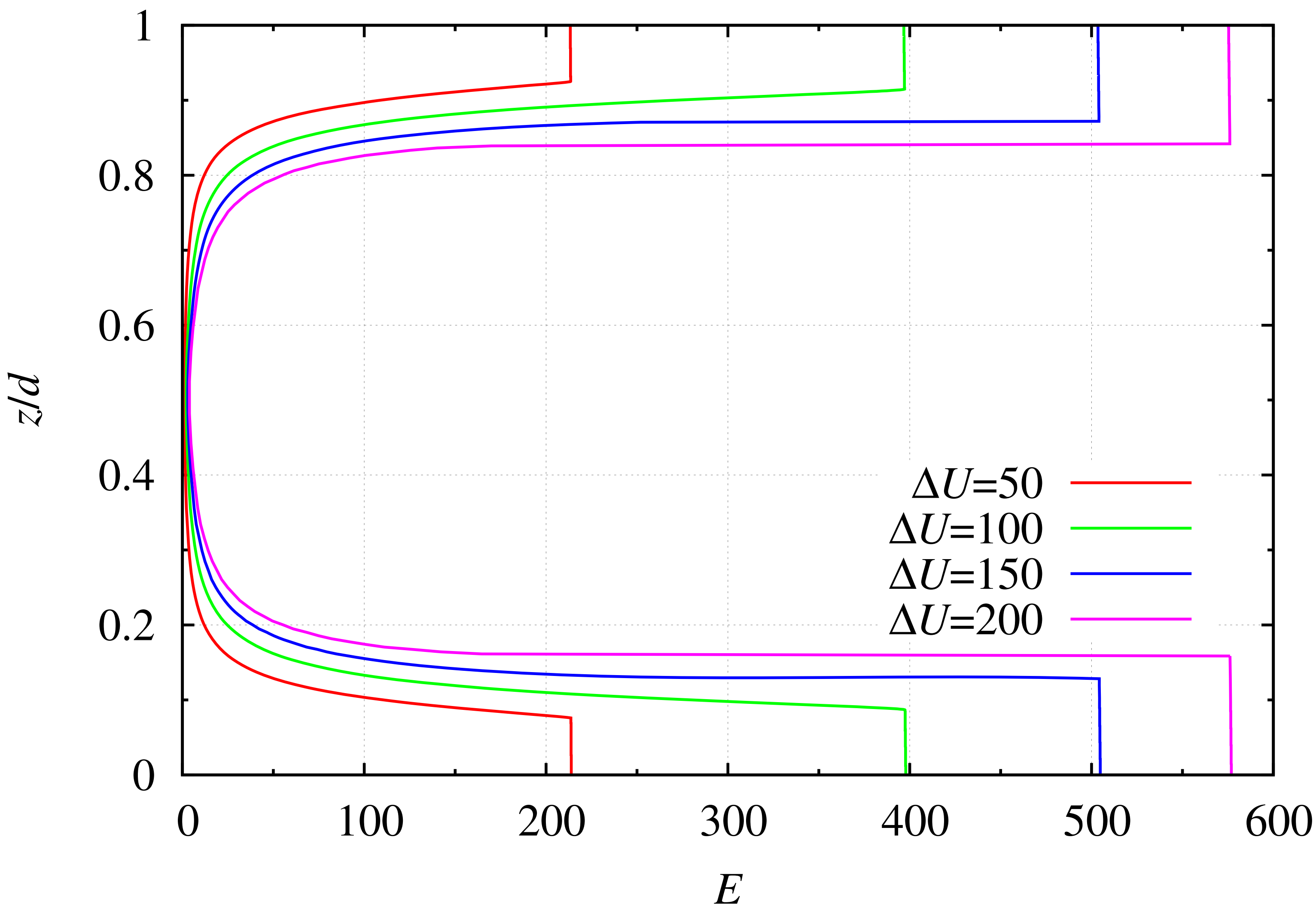

Figure 3. Dimensionless vertical shear profiles, $E(z)$, for different ΔU.

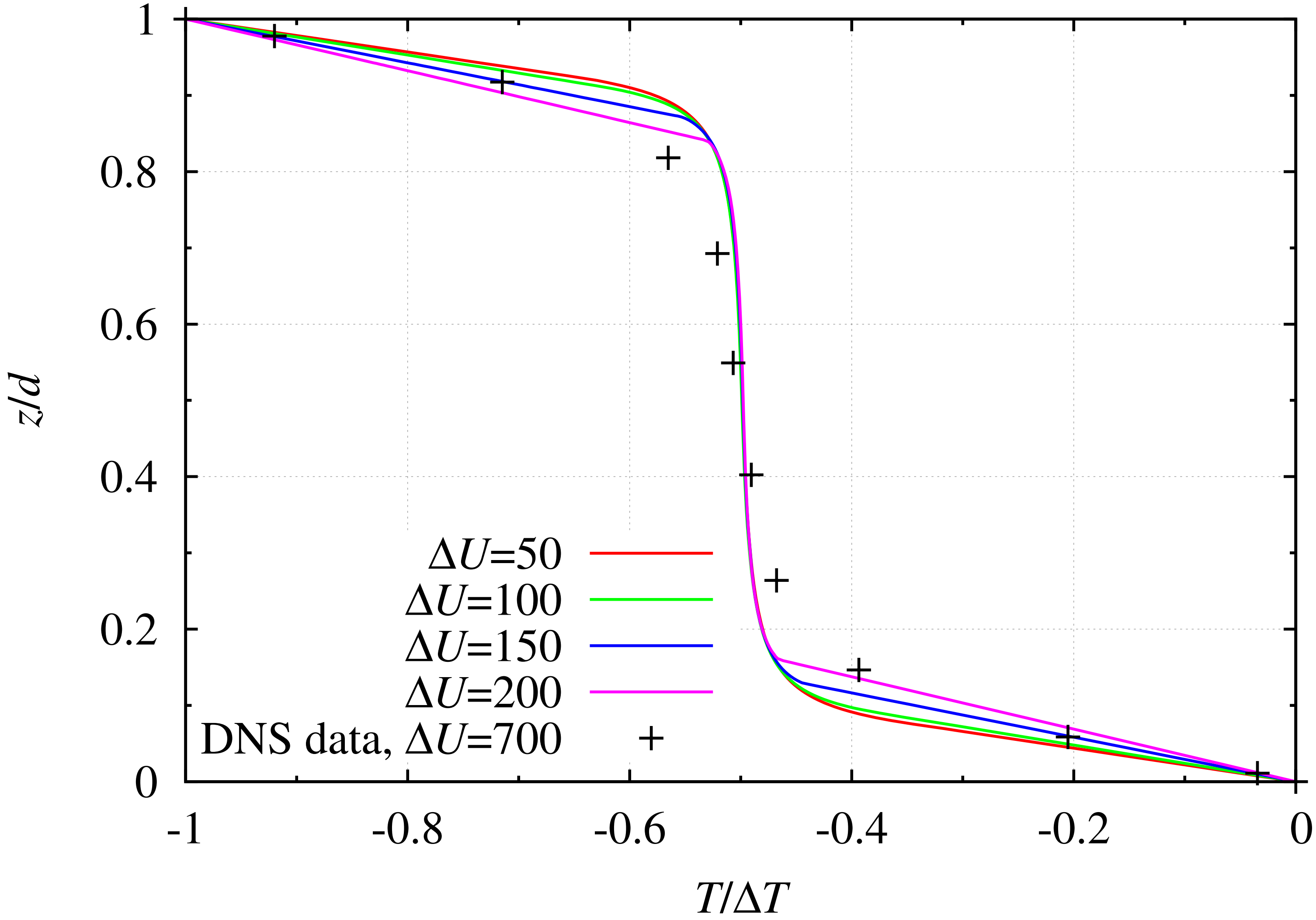

Figure 4. Mean temperature profiles, $\overline{T}/\Delta T$, for different ΔU, compared with DNS data.

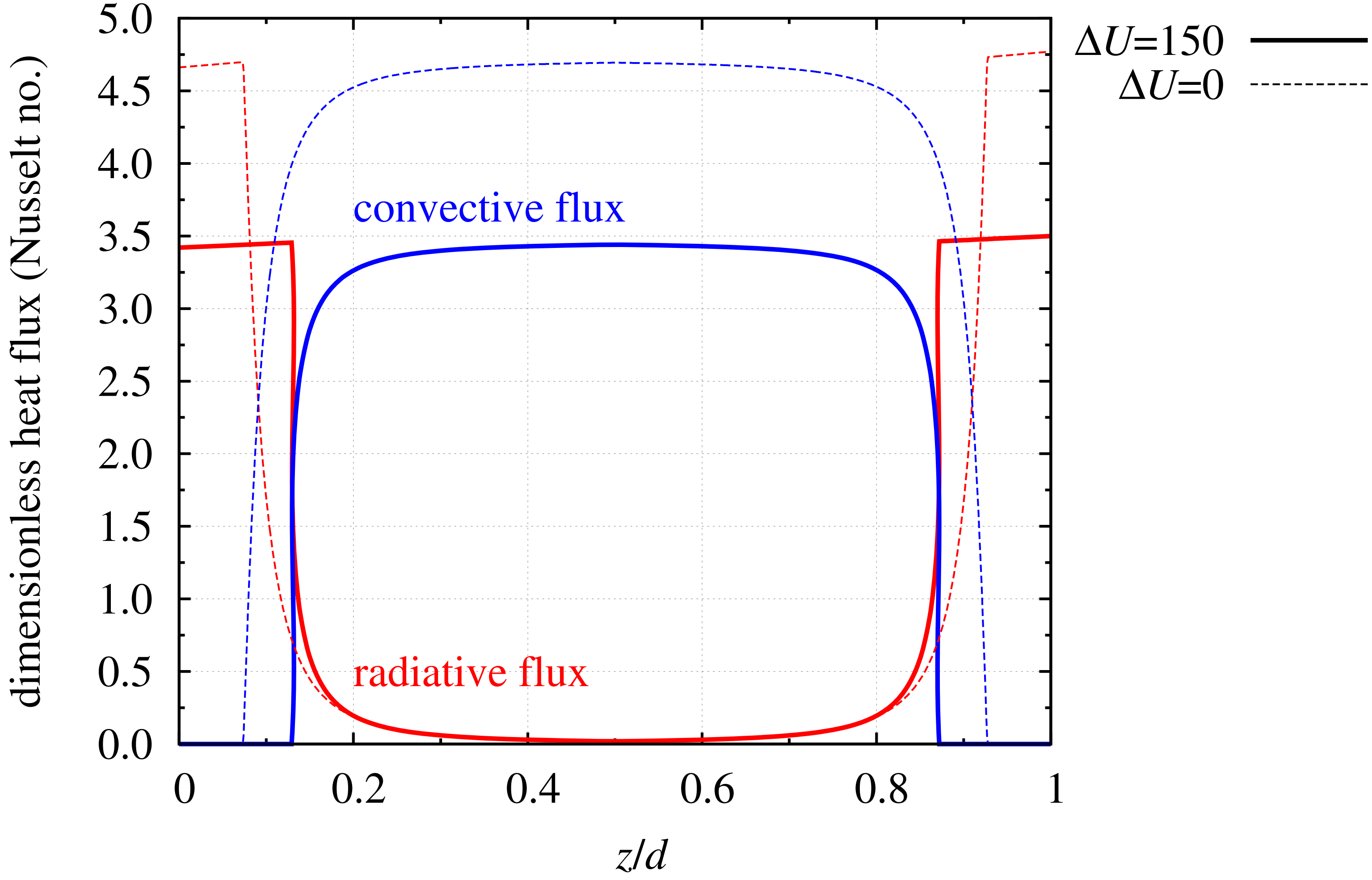

Figure 5. Radiative and convective fluxes with ($\Delta U = 150$; solid lines) and without ($\Delta U = 0$; thin dashed lines) shear.