PROBLEM 4: MAGNETIC HELICITY

- Magnetic helicity is defined for a system of volume V as the integral $\mathscr{H} = \int_V \left(\overrightarrow{A} \cdot \overrightarrow{B}\right) \mathrm{d}V$, where $\overrightarrow{A}$ is the magnetic vector potential.

- Calculate $\dfrac{\mathrm{d}\mathscr{H}}{\mathrm{d}t}$ in the regime of resistive MHD in terms of $\overrightarrow{B}$. Assume that potentials $\overrightarrow{A}, \phi$ vanish at the system boundaries.

This problem is worth 5 points. Solutions should be sent as 1-page PDF files to knalew@camk.edu.pl before the next lecture.

VARIATION OF $\overrightarrow{A} \cdot \overrightarrow{B}$

- $\mathscr{H} = \int_V \left(\overrightarrow{A} \cdot \overrightarrow{B}\right) \mathrm{d}V = \int_V \left[\overrightarrow{A} \cdot \left(\overrightarrow{\nabla} \times \overrightarrow{A}\right)\right] \mathrm{d}V$

- $\dfrac{\partial \overrightarrow{B}}{c\,\partial t} = -\overrightarrow{\nabla} \times \overrightarrow{E}$ Maxwell-Faraday equation

- $\dfrac{\partial \overrightarrow{A}}{c\,\partial t} = -\overrightarrow{E} - \overrightarrow{\nabla}\phi$ electric field in terms of potentials $\overrightarrow{A}, \phi$

- $\dfrac{\partial}{c\,\partial t}\left(\overrightarrow{A} \cdot \overrightarrow{B}\right) = \dfrac{\partial \overrightarrow{A}}{c\,\partial t} \cdot \overrightarrow{B} + \overrightarrow{A} \cdot \dfrac{\partial \overrightarrow{B}}{c\,\partial t} = -\overrightarrow{E} \cdot \overrightarrow{B} - \overrightarrow{B} \cdot \overrightarrow{\nabla}\phi - \overrightarrow{A} \cdot \left(\overrightarrow{\nabla} \times \overrightarrow{E}\right)$

- One can identify a divergence term:
  $\overrightarrow{\nabla} \cdot \left(\overrightarrow{A} \times \overrightarrow{E} - \phi\overrightarrow{B}\right) = \overrightarrow{E} \cdot \overrightarrow{B} - \overrightarrow{B} \cdot \overrightarrow{\nabla}\phi - \overrightarrow{A} \cdot \left(\overrightarrow{\nabla} \times \overrightarrow{E}\right)$

- $\dfrac{\partial}{c\,\partial t}\left(\overrightarrow{A} \cdot \overrightarrow{B}\right) = -2\overrightarrow{E} \cdot \overrightarrow{B} + \overrightarrow{\nabla} \cdot \left(\overrightarrow{A} \times \overrightarrow{E} - \phi\overrightarrow{B}\right)$

VARIATION OF $\mathscr{H}$

- $$\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}t} = \int_V \frac{\partial}{\partial t}\left(\vec{A}\cdot\vec{B}\right)\mathrm{d}V = c\int_V \left[-2\vec{E}\cdot\vec{B} + \vec{\nabla}\cdot\left(\vec{A}\times\vec{E} - \phi\vec{B}\right)\right]\mathrm{d}V$$

- Using the divergence theorem:
$$\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}t} = -2c\int_V \left(\vec{E}\cdot\vec{B}\right)\mathrm{d}V + c\oint_{\partial V}\left[\hat{n}\cdot\left(\vec{A}\times\vec{E} - \phi\vec{B}\right)\right]\mathrm{d}S$$
the surface integral vanishes since $\vec{A}, \phi$ vanish at the boundary.

- In resistive MHD: $\vec{E} = \vec{B}\times\vec{\beta} + \frac{\eta}{c}\left(\vec{\nabla}\times\vec{B}\right)$

- The solution is $\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}t} = -2\eta\int_V \left[\left(\vec{\nabla}\times\vec{B}\right)\cdot\vec{B}\right]\mathrm{d}V$

- One can also write $\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}t} = -\frac{8\pi\eta}{c}\mathscr{C}$, where $\mathscr{C} = \int_V \left(\vec{j}\cdot\vec{B}\right)\mathrm{d}V$ is the current helicity.