COSMIC MAGNETIC FIELDS

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MHD waves

UNIFORM MAGNETIC FIELD: EULER EQUATION

- $\dfrac{\partial \vec{\mathrm{v}}}{\partial t} + \left(\vec{\mathrm{v}} \cdot \vec{\nabla}\right)\vec{\mathrm{v}} = \dfrac{\vec{f}}{\rho}$

  $\vec{f} = -\vec{\nabla} P + \dfrac{\vec{j} \times \vec{B}}{c}$

- Consider a static background $\vec{\mathrm{v}}_0 = 0$ with uniform magnetic field $\vec{B}_0$. Since $\vec{j}_0 = 0$, there is no background Lorentz force density $\vec{f}_{\mathrm{L},0}$, and with uniform background pressure P_0 we have $\vec{f}_0 = 0$.

- $\dfrac{\partial \vec{\mathrm{v}}_1}{\partial t} + \left(\vec{\mathrm{v}}_1 \cdot \vec{\nabla}\right)\vec{\mathrm{v}}_0 + \left(\vec{\mathrm{v}}_0 \cdot \vec{\nabla}\right)\vec{\mathrm{v}}_1 = \dfrac{\vec{f}_1}{\rho_0} - \dfrac{\vec{f}_0}{\rho_0^2}\rho_1$

  $\rho_0 \dfrac{\partial \vec{\mathrm{v}}_1}{\partial t} = \vec{f}_1 = -\vec{\nabla} P_1 + \vec{f}_{\mathrm{L},1}$

PERTURBED LORENTZ FORCE

- $\vec{f}_{\mathrm{L}} = \dfrac{\vec{j} \times \vec{B}}{c}$

  $\vec{f}_{\mathrm{L},1} = \dfrac{\vec{j}_1 \times \vec{B}_0}{c}$

- $\vec{j}_1 = \dfrac{c}{4\pi}\left(\vec{\nabla} \times \vec{B}_1\right)$

- induction equation:

  $$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{\mathrm{v}} \times \vec{B}\right) = \left(\vec{B} \cdot \vec{\nabla}\right)\vec{\mathrm{v}} - \left(\vec{\mathrm{v}} \cdot \vec{\nabla}\right)\vec{B} - \vec{B}\left(\vec{\nabla} \cdot \vec{\mathrm{v}}\right)$$

  $$\frac{\partial \vec{B}_1}{\partial t} = \vec{\nabla} \times \left(\vec{\mathrm{v}}_1 \times \vec{B}_0\right) = \left(\vec{B}_0 \cdot \vec{\nabla}\right)\vec{\mathrm{v}}_1 - \vec{B}_0\left(\vec{\nabla} \cdot \vec{\mathrm{v}}_1\right)$$

PERTURBED LORENTZ FORCE

- Adopt oscillatory velocity perturbation $\overrightarrow{\mathrm{v}}_1 \propto \exp\left(i\omega t + i\overrightarrow{k}\cdot\vec{r}\right)$

- $$\frac{\partial \overrightarrow{B}_1}{\partial t} = \overrightarrow{\nabla} \times \left(\overrightarrow{\mathrm{v}}_1 \times \overrightarrow{B}_0\right)$$
  $$i\omega \overrightarrow{B}_1 = i\overrightarrow{k} \times \left(\overrightarrow{\mathrm{v}}_1 \times \overrightarrow{B}_0\right)$$
  $$\overrightarrow{B}_1 = \frac{1}{\omega}\left[\overrightarrow{k} \times \left(\overrightarrow{\mathrm{v}}_1 \times \overrightarrow{B}_0\right)\right] = \frac{1}{\omega}\left[\left(\overrightarrow{k}\cdot\overrightarrow{B}_0\right)\overrightarrow{\mathrm{v}}_1 - \left(\overrightarrow{k}\cdot\overrightarrow{\mathrm{v}}_1\right)\overrightarrow{B}_0\right]$$

- $$\vec{j}_1 = \frac{c}{4\pi}\left(\overrightarrow{\nabla} \times \overrightarrow{B}_1\right)$$
  $$\vec{j}_1 = \frac{c}{4\pi\omega}\left\{i\overrightarrow{k} \times \left[\overrightarrow{k} \times \left(\overrightarrow{\mathrm{v}}_1 \times \overrightarrow{B}_0\right)\right]\right\}$$

PERTURBED LORENTZ FORCE

- $\vec{f}_{\mathrm{L},1} = \dfrac{\vec{j}_1 \times \vec{B}_0}{c}$

  $\vec{j}_1 = \dfrac{c}{4\pi\omega}\left\{ i\vec{k} \times \left[\vec{k} \times \left(\vec{\mathrm{v}}_1 \times \vec{B}_0\right)\right]\right\}$

  $\vec{f}_{\mathrm{L},1} = \dfrac{i}{4\pi\omega}\left\{ \vec{k} \times \left[\vec{k} \times \left(\vec{\mathrm{v}}_1 \times \vec{B}_0\right)\right]\right\} \times \vec{B}_0$ a quadrupole cross product!

- reducing the inner two cross products:

  $\vec{k} \times \left(\vec{\mathrm{v}}_1 \times \vec{B}_0\right) = \left(\vec{k} \cdot \vec{B}_0\right)\vec{\mathrm{v}}_1 - \left(\vec{k} \cdot \vec{\mathrm{v}}_1\right)\vec{B}_0$

  $\vec{f}_{\mathrm{L},1} = \dfrac{i}{4\pi\omega}\left\{ \left(\vec{k} \cdot \vec{B}_0\right)\left[\left(\vec{k} \times \vec{\mathrm{v}}_1\right) \times \vec{B}_0\right] - \left(\vec{k} \cdot \vec{\mathrm{v}}_1\right)\left[\left(\vec{k} \times \vec{B}_0\right) \times \vec{B}_0\right]\right\}$

- reducing the remaining double cross products:

  $\left(\vec{k} \times \vec{\mathrm{v}}_1\right) \times \vec{B}_0 = \left(\vec{B}_0 \cdot \vec{k}\right)\vec{\mathrm{v}}_1 - \left(\vec{B}_0 \cdot \vec{\mathrm{v}}_1\right)\vec{k}$

  $\left(\vec{k} \times \vec{B}_0\right) \times \vec{B}_0 = \left(\vec{B}_0 \cdot \vec{k}\right)\vec{B}_0 - \left(B_0^2\right)\vec{k}$

  $\vec{f}_{\mathrm{L},1} = \dfrac{i}{4\pi\omega}\left\{ \left(\vec{k} \cdot \vec{B}_0\right)^2 \vec{\mathrm{v}}_1 - \left(\vec{k} \cdot \vec{B}_0\right)\left(\vec{k} \cdot \vec{\mathrm{v}}_1\right)\vec{B}_0 + \left[B_0^2\left(\vec{k} \cdot \vec{\mathrm{v}}_1\right) - \left(\vec{k} \cdot \vec{B}_0\right)\left(\vec{B}_0 \cdot \vec{\mathrm{v}}_1\right)\right]\vec{k}\right\}$

PERTURBED LORENTZ FORCE

- $\vec{f}_{\mathrm{L},1} = \dfrac{i}{4\pi\omega}\left\{\left(\vec{k}\cdot\vec{B}_0\right)^2\vec{\mathrm{v}}_1 - \left(\vec{k}\cdot\vec{B}_0\right)\left(\vec{k}\cdot\vec{\mathrm{v}}_1\right)\vec{B}_0 + \left[B_0^2\left(\vec{k}\cdot\vec{\mathrm{v}}_1\right) - \left(\vec{k}\cdot\vec{B}_0\right)\left(\vec{B}_0\cdot\vec{\mathrm{v}}_1\right)\right]\vec{k}\right\}$

- dot products with $\vec{B}_0$, $\vec{k}$ and $\vec{k}\times\vec{B}_0$:
  $\vec{f}_{\mathrm{L},1}\cdot\vec{B}_0 = 0$
  $\vec{f}_{\mathrm{L},1}\cdot\vec{k} = \dfrac{ik^2}{4\pi\omega}\left[B_0^2\left(\vec{k}\cdot\vec{\mathrm{v}}_1\right) - \left(\vec{k}\cdot\vec{B}_0\right)\left(\vec{B}_0\cdot\vec{\mathrm{v}}_1\right)\right]$
  $\vec{f}_{\mathrm{L},1}\cdot\left(\vec{k}\times\vec{B}_0\right) = \dfrac{i}{4\pi\omega}\left(\vec{k}\cdot\vec{B}_0\right)^2\left[\vec{\mathrm{v}}_1\cdot\left(\vec{k}\times\vec{B}_0\right)\right]$

LINEARIZED EULER EQUATION ALONG $\vec{k} \times \vec{B}_0$

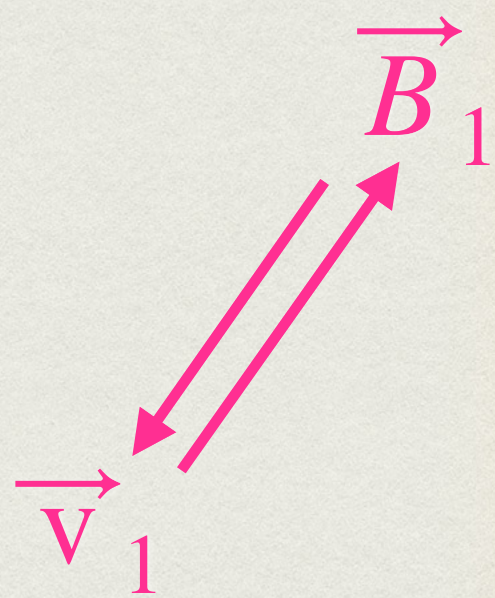
- $i\omega\rho_0 \vec{\mathrm{v}}_1 = -i\vec{k}P_1 + \vec{f}_{\mathrm{L},1}$
  $\vec{f}_{\mathrm{L},1} \cdot \left(\vec{k} \times \vec{B}_0\right) = \dfrac{i}{4\pi\omega}\left(\vec{k} \cdot \vec{B}_0\right)^2 \left[\vec{\mathrm{v}}_1 \cdot \left(\vec{k} \times \vec{B}_0\right)\right]$

- $i\omega\rho_0 \left[\vec{\mathrm{v}}_1 \cdot \left(\vec{k} \times \vec{B}_0\right)\right] = \dfrac{i}{4\pi\omega}\left(\vec{k} \cdot \vec{B}_0\right)^2 \left[\vec{\mathrm{v}}_1 \cdot \left(\vec{k} \times \vec{B}_0\right)\right]$
  note the lack of P_1 term since $\vec{k} \cdot \left(\vec{k} \times \vec{B}_0\right) = 0$.

- $\left[\omega^2 - \dfrac{1}{4\pi\rho_0}\left(\vec{k} \cdot \vec{B}_0\right)^2\right] \left[\vec{\mathrm{v}}_1 \cdot \left(\vec{k} \times \vec{B}_0\right)\right] = 0$

- if $\vec{\mathrm{v}}_1$ has a component in the $(\vec{B}_0, \vec{k})$ plane:
  $\omega^2 = \dfrac{1}{4\pi\rho_0}\left(\vec{k} \cdot \vec{B}_0\right)^2$ the Alfvén dispersion relation.

- introducing the wave vector inclination angle $\vec{k} \cdot \vec{B}_0 = kB_0 \cos\theta$
  and the background magnetization $\sigma_0 = \dfrac{B_0^2}{4\pi\rho_0 c^2}$, the propagation speed is
  $\mathrm{v}_\mathrm{A}^2 \equiv \dfrac{\omega^2}{k^2} = \sigma_0 c^2 \cos^2\theta \equiv c_{\mathrm{A},0}^2 \cos^2\theta,$ where $c_{\mathrm{A},0} = c\sqrt{\sigma_0} = \dfrac{B_0}{\sqrt{4\pi\rho_0}}$ is the Alfvén speed.



RELATIVISTIC ALFVÉN SPEED

- For relativistic plasmas we would use a momentum equation:
$$\left[\frac{\omega^2}{c^2}(1+\sigma_0)w_0 - \frac{1}{4\pi}\left(\vec{k}\cdot\vec{B}_0\right)^2\right]\left[\vec{\mathrm{v}}_1\cdot\left(\vec{k}\times\vec{B}_0\right)\right] = 0$$
with the relativistic enthalpy density $w_0 = \rho_0 c^2 + \dfrac{\kappa}{\kappa-1}P_0$
and relativistic magnetization $\sigma_0 = \dfrac{B_0^2}{4\pi w_0}$.

- This yields $\mathrm{v}_\mathrm{A}^2 \equiv \dfrac{\omega^2}{k^2} = \dfrac{\sigma_0}{1+\sigma_0}c^2\cos^2\theta \equiv c_{\mathrm{A},0}^2\cos^2\theta$
with the relativistic Alfvén speed $c_{\mathrm{A},0} \equiv c\sqrt{\dfrac{\sigma_0}{1+\sigma_0}}$.

- For ultra-relativistic magnetizations $\sigma_0 \gg 1$, hence $B_0^2/4\pi \gg w_0 > \rho_0 c^2$, we have $c_{\mathrm{A},0} \simeq c$, with the corresponding Alfvén Lorentz factor $\Gamma_{\mathrm{A},0} = \sqrt{1+\sigma_0}$.

- In the limit of non-relativistic magnetization $\sigma_0 \ll 1$ and pressure $P_0 \ll \rho_0 c^2$, we recover $\sigma_0 \simeq \dfrac{B_0^2}{4\pi\rho_0 c^2}$ and
$c_{\mathrm{A},0} \simeq c\sqrt{\sigma_0} \simeq \dfrac{B_0}{\sqrt{4\pi\rho_0}}$.

LINEARIZED EULER EQUATION ALONG $\overrightarrow{B}_0$

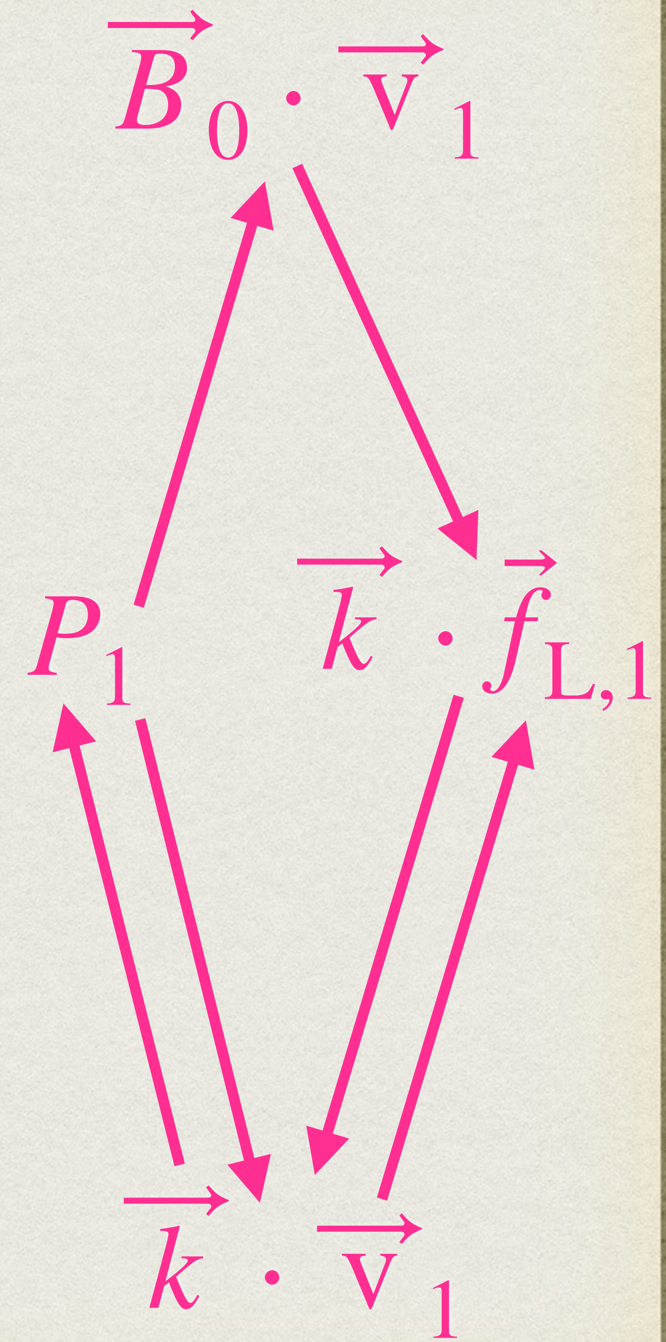
- $i\omega\rho_0 \overrightarrow{\mathrm{v}}_1 = -i\overrightarrow{k} P_1 + \vec{f}_{\mathrm{L},1}$
  $\vec{f}_{\mathrm{L},1} \cdot \overrightarrow{B}_0 = 0$

- $i\omega\rho_0 \left(\overrightarrow{\mathrm{v}}_1 \cdot \overrightarrow{B}_0\right) = -i\left(\overrightarrow{k} \cdot \overrightarrow{B}_0\right) P_1$

- the linearized pressure equation
  $P_1 = -\dfrac{\kappa P_0}{\omega}\left(\overrightarrow{k} \cdot \overrightarrow{\mathrm{v}}_1\right)$:
  $\omega^2 \left(\overrightarrow{B}_0 \cdot \overrightarrow{\mathrm{v}}_1\right) = \dfrac{\kappa P_0}{\rho_0}\left(\overrightarrow{k} \cdot \overrightarrow{B}_0\right)\left(\overrightarrow{k} \cdot \overrightarrow{\mathrm{v}}_1\right) = c_{\mathrm{s},0}^2\left(\overrightarrow{k} \cdot \overrightarrow{B}_0\right)\left(\overrightarrow{k} \cdot \overrightarrow{\mathrm{v}}_1\right)$

LINEARIZED EULER EQUATION ALONG $\vec{k}$

- $i\omega\rho_0 \vec{\mathrm{v}}_1 = -i\vec{k}P_1 + \vec{f}_{\mathrm{L},1}$
  $$\vec{f}_{\mathrm{L},1}\cdot\vec{k} = \frac{ik^2}{4\pi\omega}\left[B_0^2\left(\vec{k}\cdot\vec{\mathrm{v}}_1\right) - \left(\vec{k}\cdot\vec{B}_0\right)\left(\vec{B}_0\cdot\vec{\mathrm{v}}_1\right)\right]$$
- $$i\omega\rho_0\left(\vec{\mathrm{v}}_1\cdot\vec{k}\right) = -ik^2P_1 + \frac{ik^2}{4\pi\omega}\left[B_0^2\left(\vec{k}\cdot\vec{\mathrm{v}}_1\right) - \left(\vec{k}\cdot\vec{B}_0\right)\left(\vec{B}_0\cdot\vec{\mathrm{v}}_1\right)\right]$$
- substituting $P_1 = -\frac{\kappa P_0}{\omega}\left(\vec{k}\cdot\vec{\mathrm{v}}_1\right)$
  $$\omega^2\left(\vec{k}\cdot\vec{\mathrm{v}}_1\right) = k^2\frac{\kappa P_0}{\rho_0}\left(\vec{k}\cdot\vec{\mathrm{v}}_1\right) + \frac{k^2B_0^2}{4\pi\rho_0}\left(\vec{k}\cdot\vec{\mathrm{v}}_1\right) - \frac{k^2}{4\pi\rho_0}\left(\vec{k}\cdot\vec{B}_0\right)\left(\vec{B}_0\cdot\vec{\mathrm{v}}_1\right)$$
- substituting $\omega^2\left(\vec{B}_0\cdot\vec{\mathrm{v}}_1\right) = c_{\mathrm{s},0}^2\left(\vec{k}\cdot\vec{B}_0\right)\left(\vec{k}\cdot\vec{\mathrm{v}}_1\right)$
  $$\omega^2\left(\vec{k}\cdot\vec{\mathrm{v}}_1\right) = k^2\left[c_{\mathrm{s},0}^2 + \frac{B_0^2}{4\pi\rho_0} - \frac{c_{\mathrm{s},0}^2}{4\pi\rho_0\omega^2}\left(\vec{k}\cdot\vec{B}_0\right)^2\right]\left(\vec{k}\cdot\vec{\mathrm{v}}_1\right)$$
- if $\vec{\mathrm{v}}_1$ has a component along $\vec{k}$ (longitudinal):
  $$\frac{\omega^2}{k^2} = c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2 - \frac{k^2}{\omega^2}c_{\mathrm{s},0}^2c_{\mathrm{A},0}^2\cos^2\theta$$



MAGNETOSONIC SPEEDS

- $$\frac{\omega^2}{k^2} = c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2 - \frac{k^2}{\omega^2} c_{\mathrm{s},0}^2 c_{\mathrm{A},0}^2 \cos^2\theta$$

- $$\frac{\omega^4}{k^4} - \left(c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2\right)\frac{\omega^2}{k^2} + c_{\mathrm{s},0}^2 c_{\mathrm{A},0}^2 \cos^2\theta = 0$$

  is the magnetosonic dispersion relation.

- this dispersion relation has two real solutions:

  $$\mathrm{v}_\pm^2 \equiv \frac{\omega^2}{k^2} = \frac{1}{2}\left[c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2 \pm \sqrt{\left(c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2\right)^2 - 4 c_{\mathrm{s},0}^2 c_{\mathrm{A},0}^2 \cos^2\theta}\right]$$

  corresponding to the slow ($\mathrm{v}_{\mathrm{SM}} \equiv \mathrm{v}_-$) and fast ($\mathrm{v}_{\mathrm{FM}} \equiv \mathrm{v}_+$) magnetosonic waves.

- for relativistic magnetizations:

  $$\frac{\omega^4}{k^4} - \left[\frac{1+\sigma_0\cos^2\theta}{1+\sigma_0} c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2\right]\frac{\omega^2}{k^2} + c_{\mathrm{s},0}^2 c_{\mathrm{A},0}^2 \cos^2\theta = 0$$

WAVES ALONG $\overrightarrow{B}_0$

- Consider the case of $\theta = 0$, hence $\overrightarrow{k} \parallel \overrightarrow{B}_0$.

- Note that $\overrightarrow{k} \times \overrightarrow{B}_0 = 0$. We can only use the magnetosonic dispersion relation:

$$\mathrm{v}_{\pm}^2 = \frac{1}{2}\left[c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2 \pm \sqrt{\left(c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2\right)^2 - 4c_{\mathrm{s},0}^2 c_{\mathrm{A},0}^2 \cos^2\theta}\right]$$

$$\mathrm{v}_{\pm}^2 = \frac{1}{2}\left[c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2 \pm \sqrt{\left(c_{\mathrm{s},0}^2 - c_{\mathrm{A},0}^2\right)^2}\right]$$

- the solutions reduce to $c_{\mathrm{s},0}^2$ and $c_{\mathrm{A},0}^2$,
  which correspond to the sound and Alfvén waves, respectively.

WAVES ALONG $\overrightarrow{B}_0$

- Consider the perturbed Lorentz force, splitting the perturbation velocity $\overrightarrow{\mathrm{v}}_1 = \overrightarrow{\mathrm{v}}_{1,\parallel} + \overrightarrow{\mathrm{v}}_{1,\perp}$ into components parallel and perpendicular to $\overrightarrow{k}$:
$$\overrightarrow{B}_1 = \frac{1}{\omega}\left[\left(\overrightarrow{k}\cdot\overrightarrow{B}_0\right)\overrightarrow{\mathrm{v}}_1 - \left(\overrightarrow{k}\cdot\overrightarrow{\mathrm{v}}_1\right)\overrightarrow{B}_0\right]$$
$$\overrightarrow{B}_1 = \frac{kB_0}{\omega}\overrightarrow{\mathrm{v}}_{1,\perp}$$

- $$\vec{f}_{\mathrm{L},1} = \frac{i}{4\pi\omega}\left\{\left(\overrightarrow{k}\cdot\overrightarrow{B}_0\right)^2\overrightarrow{\mathrm{v}}_1 - \left(\overrightarrow{k}\cdot\overrightarrow{B}_0\right)\left(\overrightarrow{k}\cdot\overrightarrow{\mathrm{v}}_1\right)\overrightarrow{B}_0 + \left[B_0^2\left(\overrightarrow{k}\cdot\overrightarrow{\mathrm{v}}_1\right) - \left(\overrightarrow{k}\cdot\overrightarrow{B}_0\right)\left(\overrightarrow{B}_0\cdot\overrightarrow{\mathrm{v}}_1\right)\right]\overrightarrow{k}\right\}$$
$$\vec{f}_{\mathrm{L},1} = \frac{ik^2B_0^2}{4\pi\omega}\overrightarrow{\mathrm{v}}_{1,\perp}$$

- $$i\omega\rho_0\overrightarrow{\mathrm{v}}_1 = -i\overrightarrow{k}P_1 + \vec{f}_{\mathrm{L},1}$$
$$P_1 = -\frac{\kappa P_0}{\omega}k\mathrm{v}_{1,\parallel}$$
$$\omega\rho_0\overrightarrow{\mathrm{v}}_1 = k^2\frac{\kappa P_0}{\omega}\overrightarrow{\mathrm{v}}_{1,\parallel} + \frac{k^2B_0^2}{4\pi\omega}\overrightarrow{\mathrm{v}}_{1,\perp}$$

- $$\left(\omega^2 - k^2c_{\mathrm{s},0}^2\right)\overrightarrow{\mathrm{v}}_{1,\parallel} + \left(\omega^2 - k^2c_{\mathrm{A},0}^2\right)\overrightarrow{\mathrm{v}}_{1,\perp} = 0$$

WAVES ALONG $\vec{B}_0$

- $\left(\omega^2 - k^2 c_{\mathrm{s},0}^2\right) \vec{\mathrm{v}}_{1,\parallel} + \left(\omega^2 - k^2 c_{\mathrm{A},0}^2\right) \vec{\mathrm{v}}_{1,\perp} = 0$

- For the sound wave: $\omega^2 = k^2 c_{\mathrm{s},0}^2$ and hence $\vec{\mathrm{v}}_{1,\perp} = 0$: velocity perturbations are longitudinal $\vec{\mathrm{v}}_1 \parallel \vec{k}$ and compressible $P_1 \neq 0$, there is no Lorentz force (and $\vec{B}_1 = 0$).

- For the Alfvén wave: $\omega^2 = k^2 c_{\mathrm{A},0}^2$ and hence $\vec{\mathrm{v}}_{1,\parallel} = 0$: velocity perturbations are transverse $\vec{\mathrm{v}}_1 \perp \vec{k}$ and incompressible $P_1 = 0$, there are transverse perturbations of magnetic field and Lorentz force density $\vec{B}_1 \parallel \vec{f}_{\mathrm{L},1} \parallel \vec{\mathrm{v}}_1$.

WAVES ACROSS $\overrightarrow{B}_0$

- Consider the case of $\theta = \pi/2$, hence $\overrightarrow{k} \perp \overrightarrow{B}_0$.

- The Alfvén dispersion relation reduces to $\omega^2 = \dfrac{1}{4\pi\rho_0}\left(\overrightarrow{k} \cdot \overrightarrow{B}_0\right)^2 = 0$.

- The magnetosonic dispersion relation:

$$\mathrm{v}_\pm^2 = \frac{1}{2}\left[c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2 \pm \sqrt{\left(c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2\right)^2 - 4c_{\mathrm{s},0}^2 c_{\mathrm{A},0}^2 \cos^2\theta}\right]$$

$$\mathrm{v}_\pm^2 = \frac{1}{2}\left[c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2 \pm \sqrt{\left(c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2\right)^2}\right] = \left\{0, c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2\right\}$$

- Let us thus define the fast magnetosonic speed $c_{\mathrm{FM},0}^2 \equiv c_{\mathrm{s},0}^2 + c_{\mathrm{A},0}^2$.

WAVES ACROSS $\overrightarrow{B}_0$

- Consider the perturbed magnetic field and Lorentz force, splitting the perturbation velocity $\overrightarrow{\mathrm{v}}_1 = \overrightarrow{\mathrm{v}}_{1,\parallel} + \overrightarrow{\mathrm{v}}_{1,\perp}$ into components parallel and perpendicular to $\overrightarrow{k}$:

  $$\overrightarrow{B}_1 = \frac{1}{\omega}\left[\left(\overrightarrow{k}\cdot\overrightarrow{B}_0\right)\overrightarrow{\mathrm{v}}_1 - \left(\overrightarrow{k}\cdot\overrightarrow{\mathrm{v}}_1\right)\overrightarrow{B}_0\right]$$

  $$\overrightarrow{B}_1 = -\frac{k}{\omega}\mathrm{v}_{1,\parallel}\overrightarrow{B}_0$$

- $$\vec{f}_{\mathrm{L},1} = \frac{i}{4\pi\omega}\left\{\left(\overrightarrow{k}\cdot\overrightarrow{B}_0\right)^2\overrightarrow{\mathrm{v}}_1 - \left(\overrightarrow{k}\cdot\overrightarrow{B}_0\right)\left(\overrightarrow{k}\cdot\overrightarrow{\mathrm{v}}_1\right)\overrightarrow{B}_0 + \left[B_0^2\left(\overrightarrow{k}\cdot\overrightarrow{\mathrm{v}}_1\right) - \left(\overrightarrow{k}\cdot\overrightarrow{B}_0\right)\left(\overrightarrow{B}_0\cdot\overrightarrow{\mathrm{v}}_1\right)\right]\overrightarrow{k}\right\}$$

  $$\vec{f}_{\mathrm{L},1} = \frac{ik^2B_0^2}{4\pi\omega}\overrightarrow{\mathrm{v}}_{1,\parallel}$$

- $$i\omega\rho_0\overrightarrow{\mathrm{v}}_1 = -i\overrightarrow{k}P_1 + \vec{f}_{\mathrm{L},1}$$

  $$P_1 = -\frac{\kappa P_0}{\omega}k\mathrm{v}_{1,\parallel}$$

  $$\omega\rho_0\overrightarrow{\mathrm{v}}_1 = k^2\frac{\kappa P_0}{\omega}\overrightarrow{\mathrm{v}}_{1,\parallel} + \frac{k^2B_0^2}{4\pi\omega}\overrightarrow{\mathrm{v}}_{1,\parallel}$$

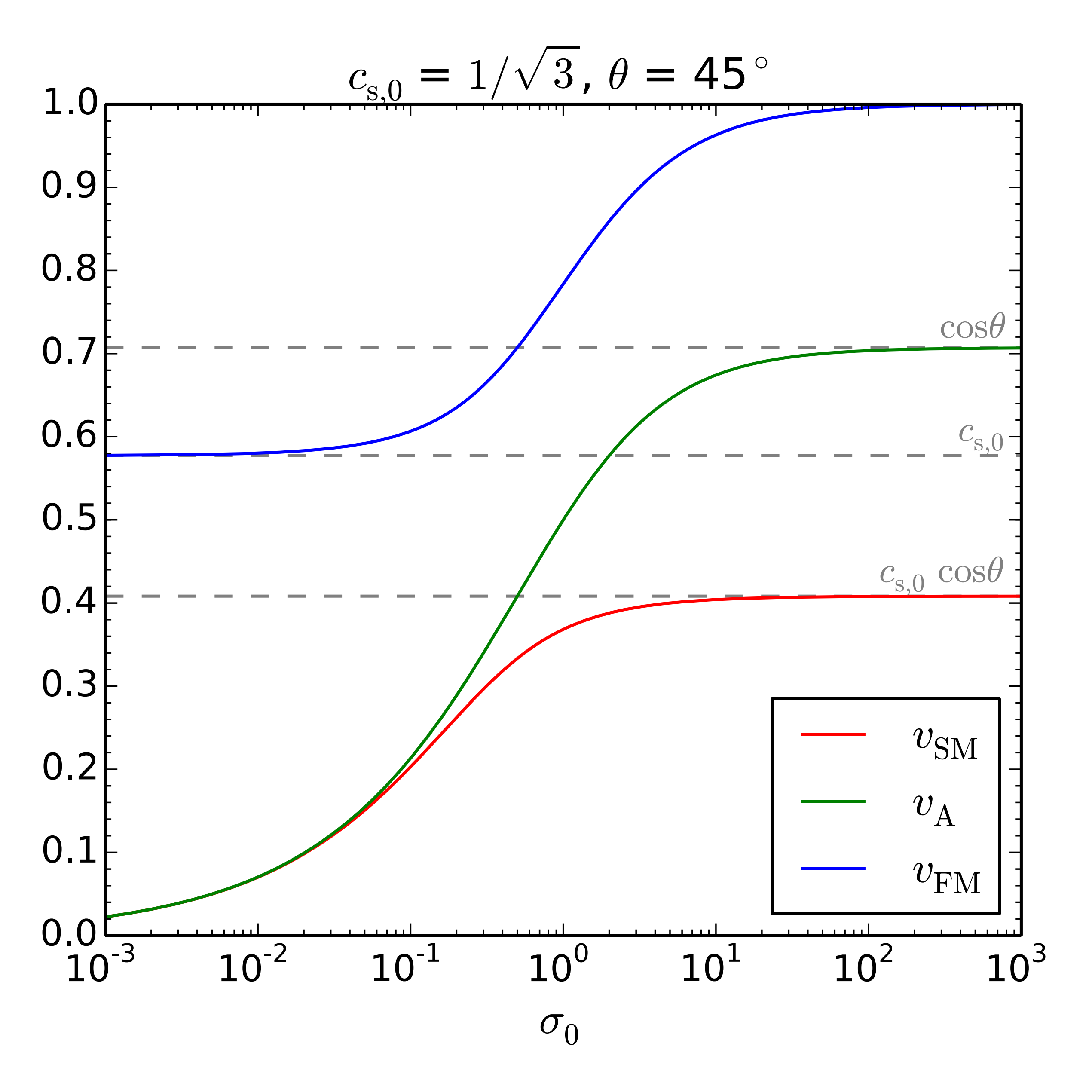
- $$\left(\omega^2 - k^2c_{\mathrm{FM},0}^2\right)\overrightarrow{\mathrm{v}}_{1,\parallel} + \omega^2\overrightarrow{\mathrm{v}}_{1,\perp} = 0$$

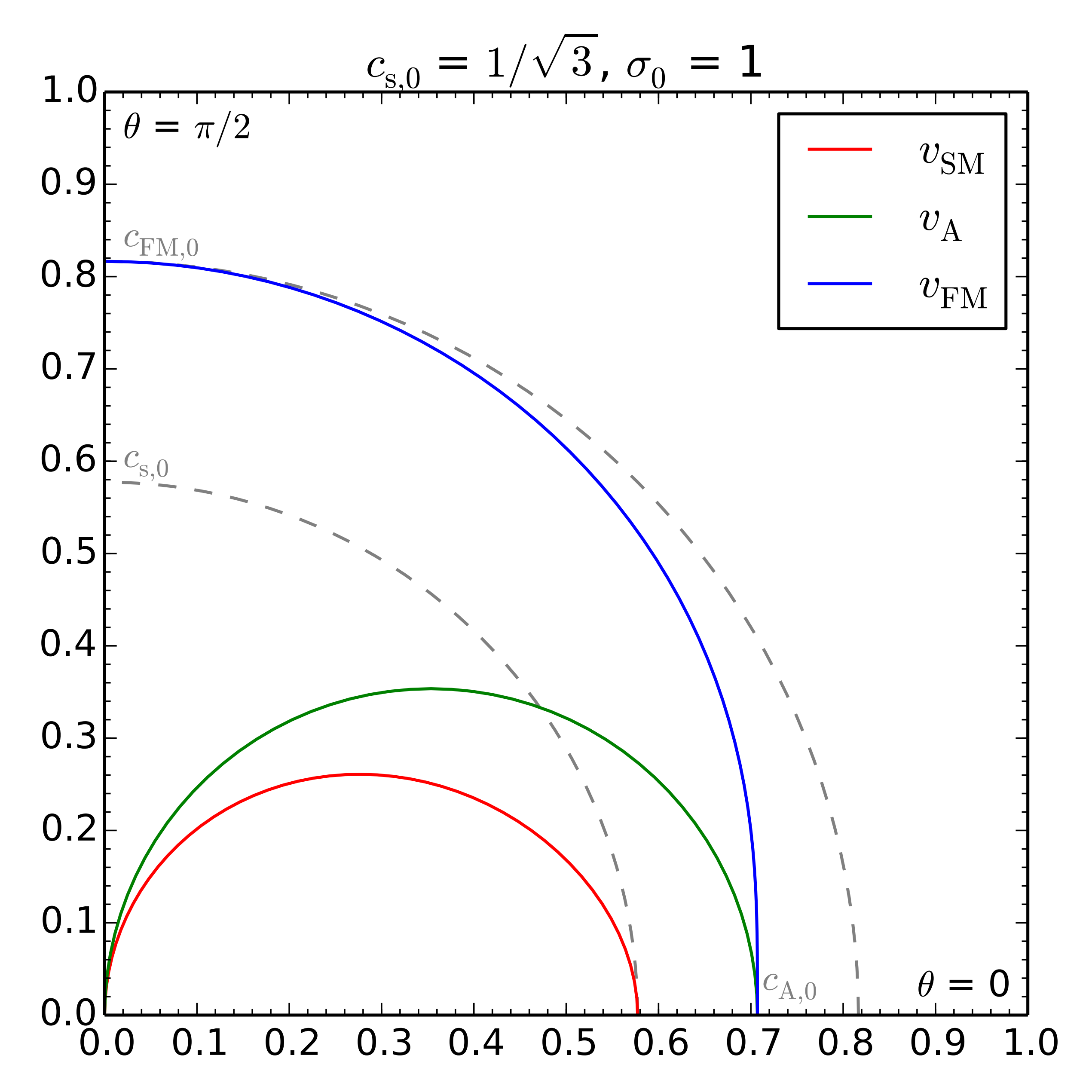
WAVES ACROSS $\overrightarrow{B}_0$

- $\left(\omega^2 - k^2 c_{\mathrm{FM},0}^2\right) \overrightarrow{\mathrm{v}}_{1,\parallel} + \omega^2 \overrightarrow{\mathrm{v}}_{1,\perp} = 0$

- For the fast magnetosonic wave: $\omega^2 = k^2 c_{\mathrm{FM},0}^2$ and hence $\overrightarrow{\mathrm{v}}_{1,\perp} = 0$: velocity perturbations are longitudinal $\overrightarrow{\mathrm{v}}_1 \parallel \overrightarrow{k}$ and compressible $P_1 \neq 0$, the perturbed magnetic field is $\overrightarrow{B}_1 \parallel \overrightarrow{B}_0$ (polarized), and the perturbed Lorentz force density is $\vec{f}_{\mathrm{L},1} \parallel \overrightarrow{k}$.

MHD WAVE SPEEDS





SUMMARY

- Linearization of ideal gas with uniform background magnetic field $\overrightarrow{B}_0$ yields 3 stable modes: Alfvén (intermediate) and slow/fast magnetosonic.
- Modes propagating along $\overrightarrow{B}_0$ reduce to the Alfvén and sound waves.
- Modes propagating across $\overrightarrow{B}_0$ reduce to the fast magnetosonic wave.
- Propagation speeds of Alfvén and fast magnetosonic waves can be ultra-relativistic for $\sigma_0 = \dfrac{B_0^2}{4\pi w_0} \gg 1$, with the Alfvén Lorentz factor $\Gamma_{\mathrm{A},0} = \sqrt{1 + \sigma_0}$.