COSMIC MAGNETIC FIELDS

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Magnetohydrodynamics (MHD)

LORENTZ TRANSFORMATION

- Lorentz transformation preserves the fields components parallel to the boost vector $\overrightarrow{\mathrm{v}} = \overrightarrow{\beta} c$:
  $\overrightarrow{E}'_{\parallel} = \overrightarrow{E}_{\parallel}$ and $\overrightarrow{B}'_{\parallel} = \overrightarrow{B}_{\parallel}$.

- transformation of the perpendicular components:
  $$\overrightarrow{E}'_{\perp} = \Gamma\left(\overrightarrow{E}_{\perp} + \overrightarrow{\beta} \times \overrightarrow{B}\right),$$
  $$\overrightarrow{B}'_{\perp} = \Gamma\left(\overrightarrow{B}_{\perp} - \overrightarrow{\beta} \times \overrightarrow{E}\right),$$
  with $\Gamma = (1 - \beta^2)^{-1/2}$ the Lorentz factor.

- Maxwell's equations are Lorentz invariant!

NON-RELATIVISTIC TRANSFORMATION AND OHM'S LAW

- Consider the non-relativistic limit $\beta \ll 1$, hence $\Gamma \simeq 1 + \beta^2/2$.

- One can show that:
$\overrightarrow{E}' \simeq \overrightarrow{E} + \overrightarrow{\beta} \times \overrightarrow{B}$ and $\overrightarrow{B}' = \overrightarrow{B} - \overrightarrow{\beta} \times \overrightarrow{E}$ (symmetric!)

- Consider a basic Ohm's law in the co-moving frame $\vec{j}' = \sigma \overrightarrow{E}'$ with scalar electric conductivity σ.

- The $\overrightarrow{E}$ vs. $\overrightarrow{B}$ symmetry is broken for sufficiently high conductivity ($\sigma \gg c^2/\mathrm{v}L$):
$E' \ll E$, hence $\overrightarrow{E} \simeq \overrightarrow{B} \times \overrightarrow{\beta}$, hence $E \sim \beta B \ll B$, hence $\overrightarrow{B}' \simeq \overrightarrow{B}$.

- One can also show that $\vec{j}' \simeq \vec{j}$, hence a co-moving frame Ohm's law $\vec{j}' = \sigma \overrightarrow{E}'$ transforms to $\vec{j} \simeq \sigma \left(\overrightarrow{E} + \overrightarrow{\beta} \times \overrightarrow{B} \right)$ in general reference frame.

MAGNETIC DIFFUSIVITY

- Ohm's law in general reference frame: $\vec{j} = \sigma\left(\vec{E} + \vec{\beta} \times \vec{B}\right)$.

- Introducing the resistive (Faraday) time scale $\tau_\sigma = 1/(4\pi\sigma)$, and electric field variability time scale $\tau_{\rm E} = E/|\partial E/\partial t|$, the Ampère-Maxwell equation becomes:
$$\vec{\nabla} \times \vec{B} \simeq \frac{\vec{E} + \vec{\beta} \times \vec{B}}{c\tau_\sigma} + \frac{\vec{E}}{c\tau_{\rm E}}.$$

- $$\left(1 + \frac{\tau_\sigma}{\tau_{\rm E}}\right)\vec{E} \simeq \vec{B} \times \vec{\beta} + \frac{\eta}{c}\left(\vec{\nabla} \times \vec{B}\right),$$
with the magnetic diffusivity $\eta = \tau_\sigma c^2 = c^2/(4\pi\sigma)$.

- The displacement current can be neglected for $\tau_E \gg \tau_\sigma$. From the Spitzer resistivity, the standard magnetic diffusivity is $\eta \simeq 10^4\, T_6^{-3/2}\ {\rm cm^2\, s^{-1}}$ with $T_6 = T/(10^6\ {\rm K})$. This implies microscopic resistive scales: $c\tau_\sigma \simeq 3\, T_6^{-3/2}$ nm, hence it is really safe to neglect the displacement current for all astrophysical objects.

INDUCTION EQUATION

- Neglecting the displacement current:
$\vec{j} \simeq \frac{c}{4\pi}\left(\vec{\nabla} \times \vec{B}\right)$ and $\vec{E} \simeq \vec{B} \times \vec{\beta} + \frac{\eta}{c}\left(\vec{\nabla} \times \vec{B}\right).$

- The Maxwell-Faraday equation becomes
the induction equation:
$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{\mathrm{v}} \times \vec{B} - \eta \vec{\nabla} \times \vec{B}\right)$$
This is the basic equation of
resistive magnetohydrodynamics (MHD).

MAGNETIC DIFFUSION

- The induction equation: $\frac{\partial \overrightarrow{B}}{\partial t} = \overrightarrow{\nabla} \times \left(\overrightarrow{\mathrm{v}} \times \overrightarrow{B} - \eta \overrightarrow{\nabla} \times \overrightarrow{B} \right)$

- for $\overrightarrow{\mathrm{v}} = 0$ and $\eta = \text{const}$:
  $\frac{\partial \overrightarrow{B}}{\partial t} = \eta \nabla^2 \overrightarrow{B}$: the diffusion equation (Fick's second law)
  with diffusion coefficient η

- Example: a Gaussian field
  $B_x(t, y) = B_0(t) \exp\left[-y^2/2\sigma_y^2(t)\right]$
  yields a solution $\sigma_y(t) = \sqrt{2\eta t}$ and $B_0(t) \propto 1/\sqrt{t}$

DIFFUSIVE TIME SCALES

- Consider a finite system with length scale L.
- The diffusive (dissipative) time scale $\tau_\eta = L^2/\eta$ or $\tau_\eta = L^2/(\pi^2\eta)$

	L [cm]	T [K]	η [cm²/s]	τ_η [yr]
Earth	3.5×10^{8}	3×10^{3}	10^{4}	5×10^{4}
Sun	1.5×10^{10}	10^{7}	2×10^{3}	2×10^{10}
Galaxy	6×10^{20}	10^{3}	3×10^{8}	4×10^{25}

MAGNETIC REYNOLDS NUMBER

- The diffusive time scale $\tau_\eta = L^2/\eta$.
- For a typical velocity v, this can be compared with the dynamical time scale $\tau_\mathrm{v} = L/\mathrm{v}$.
- The magnetic Reynolds number:

$$R_\mathrm{m} \equiv \frac{\text{induction}}{\text{diffusion}} = \frac{\tau_\eta}{\tau_\mathrm{v}} = \frac{\mathrm{v}L}{\eta}.$$

MAGNETIC PRANDTL NUMBER

- Considering a kinematic viscosity ν, the viscous time scale is $\tau_\nu = L^2/\nu$.

- The Reynolds number:

$$R \equiv \frac{\text{advection}}{\text{viscosity}} = \frac{\tau_\nu}{\tau_\mathrm{v}} = \frac{\mathrm{v}L}{\nu}$$

- the magnetic Prandtl number:

$$P_\mathrm{m} = \frac{R_\mathrm{m}}{R} \equiv \frac{\text{viscosity}}{\text{diffusion}} = \frac{\tau_\eta}{\tau_\nu} = \frac{\nu}{\eta}$$

MAGNETIC REYNOLDS AND PRANDTL NUMBERS

Table 1

Summary of some important parameters in various astrophysical settings. The values given should be understood as rough indications only. In particular, the applicability of Eq. (3.17) is questionable in some cases and has therefore not been used for protostellar discs (see text). We have assumed $\ln \Lambda = 20$ in computing R_m and P_m. CZ means convection zone, CV discs and similar refer to cataclysmic variables and discs around other compact objects such as black holes and neutron stars. AGNs are active galactic nuclei. Numbers in parenthesis indicate significant uncertainty due to other effects

	T [K]	ρ [g cm^{-3}]	P_m	u_rms [cm s^{-1}]	L [cm]	R_m
Solar CZ (upper part)	10^4	10^{-6}	10^{-7}	10^6	10^8	10^6
Solar CZ (lower part)	10^6	10^{-1}	10^{-4}	10^4	10^{10}	10^9
Protostellar discs	10^3	10^{-10}	10^{-8}	10^5	10^{12}	10
CV discs and similar	10^4	10^{-7}	10^{-6}	10^5	10^7	10^4
AGN discs	10^7	10^{-5}	10^4	10^5	10^9	10^{11}
Galaxy	10^4	10^{-24}	(10^{11})	10^6	10^{20}	(10^{18})
Galaxy clusters	10^8	10^{-26}	(10^{29})	10^8	10^{23}	(10^{29})

IDEAL MHD

- Most astrophysical systems satisfy $R_\mathrm{m} \gg 1$ or $L \gg \eta/\mathrm{v}$ or $\sigma \gg c^2/\mathrm{v}L$, hence magnetic diffusion can be neglected.

- In the ideal MHD limit: $\overrightarrow{E} \simeq \overrightarrow{B} \times \overrightarrow{\beta}$ and the induction equation becomes:
$$\frac{\partial \overrightarrow{B}}{\partial t} \simeq \overrightarrow{\nabla} \times \left(\overrightarrow{\mathrm{v}} \times \overrightarrow{B} \right).$$

MICROSCOPIC CONDITIONS FOR IDEAL MHD

- Consider dimensionless parameters $y = R/L$ and $x = \sqrt{m_i/m_e}\,(\langle \mathrm{v} \rangle\, \tau_{\mathrm{coll}}/L)$ with R the gyroradius, $\langle \mathrm{v} \rangle$ the mean thermal particle speed, and τ_{coll} the collisional time scale.

- Three conditions for ideal MHD:
  1. small gyroradius $y \ll 1$;
  2. large collisionality $x \ll 1$;
  3. small resistivity $y^2 \ll x$.

- For more details, see Freidberg (1982), Sec. II.G-H.

MAGNETIC FLUX FREEZING

- magnetic flux through surface S at time t:

$$\Phi(t) = \iint_S \vec{B}(t) \cdot \mathrm{d}\vec{S}$$

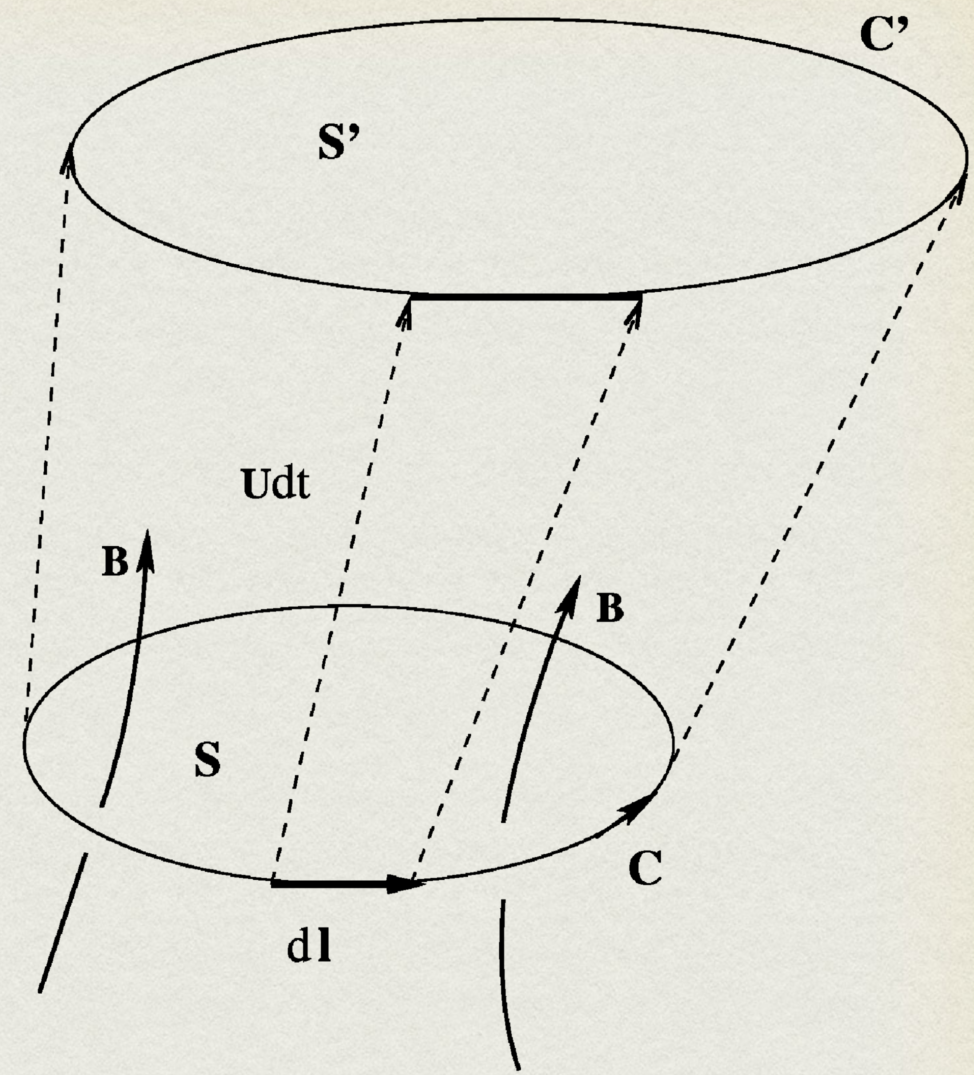
- due to flow velocity field $\vec{U}$ by time $t' = t + \mathrm{d}t$ we have $S(t) \to S'(t')$ sweeping volume $\mathrm{d}V$.
(not a magnetic flux tube!)

- magnetic flux through S' at t': $\Phi'(t') = \iint_{S'} \vec{B}(t') \cdot \mathrm{d}\vec{S}'$

- divergence theorem at time t':

$$\Phi'(t') - \Phi(t') + \Phi_{\mathrm{side}}(t') = \iiint_{\mathrm{d}V} \left(\vec{\nabla} \cdot \vec{B}(t')\right) \mathrm{d}V = 0$$

$$\Phi_{\mathrm{side}}(t') = \oint_C \vec{B}(t') \cdot \left(\mathrm{d}\vec{l} \times \vec{U}\,\mathrm{d}t\right)$$



Brandenburg & Subramanian (2005)

MAGNETIC FLUX FREEZING

- $\Phi'(t') - \Phi(t) = \Phi(t') - \Phi(t) - \oint_C \vec{B}(t') \cdot \left(\mathrm{d}\vec{l} \times \vec{U}\,\mathrm{d}t \right)$

- using the identity $\left(\mathrm{d}\vec{l} \times \vec{U} \right) \cdot \vec{B} = \left(\vec{U} \times \vec{B} \right) \cdot \mathrm{d}\vec{l}$:

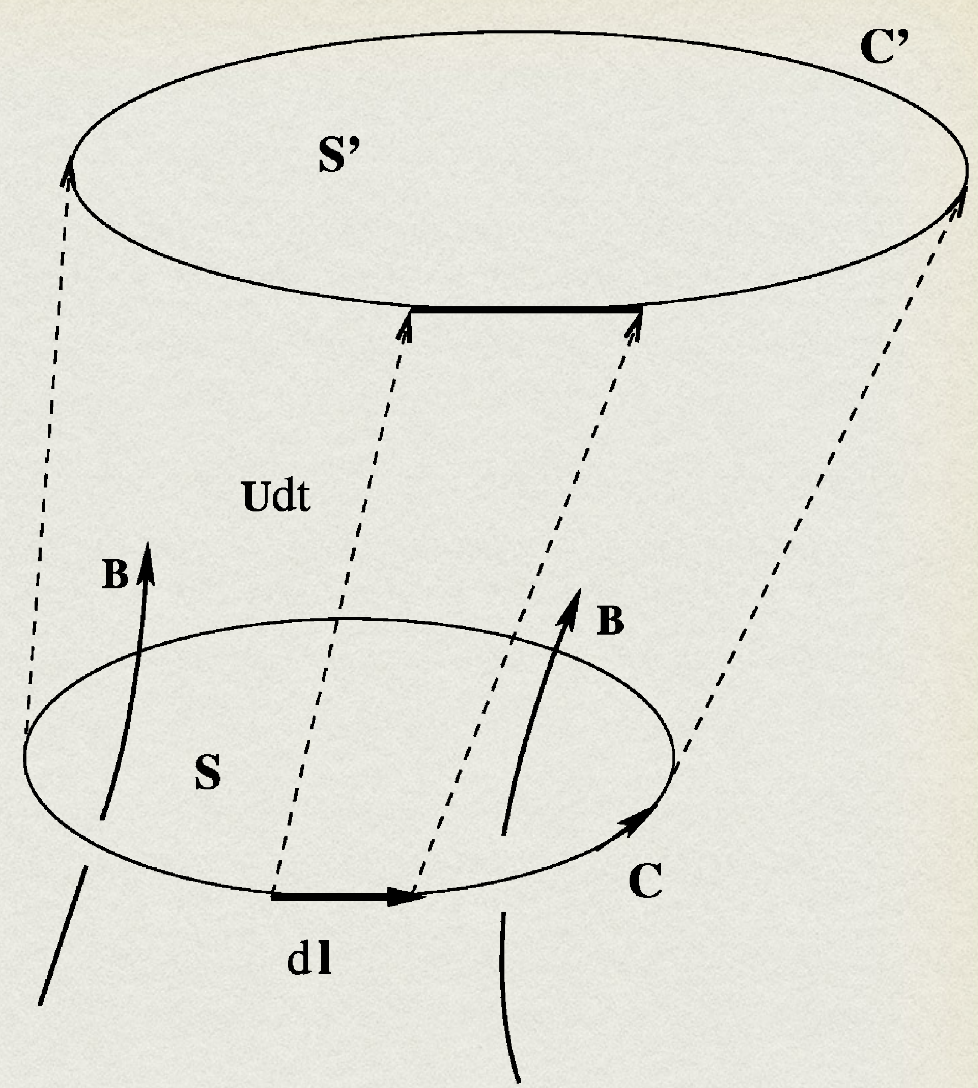
$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\partial \Phi}{\partial t} - \oint_C \left(\vec{U} \times \vec{B} \right) \cdot \mathrm{d}\vec{l}$$

- applying the Stokes' theorem:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \iint_S \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{S} - \iint_S \vec{\nabla} \times \left(\vec{U} \times \vec{B} \right) \cdot \mathrm{d}\vec{S}$$

- from the induction equation the RHS is zero, hence $\mathrm{d}\Phi = 0$ (Alfvén's theorem).

- Magnetic flux is thus frozen into the flow.



Brandenburg & Subramanian (2005)

LORENTZ FORCE DENSITY IN NON-RELATIVISTIC MHD

- $\overrightarrow{f_{\rm L}} = \rho_{\rm e}\overrightarrow{E} + \frac{1}{c}\left(\vec{j} \times \overrightarrow{B}\right)$

- Electric charge density: $\rho_{\rm e} = \frac{1}{4\pi}\overrightarrow{\nabla} \cdot \overrightarrow{E} = \frac{1}{4\pi c}\overrightarrow{\nabla} \cdot \left(\overrightarrow{B} \times \overrightarrow{\rm v}\right)$

- Electric current density: $\vec{j} = \frac{c}{4\pi}\left(\overrightarrow{\nabla} \times \overrightarrow{B}\right)$

- $\overrightarrow{f_{\rm L}} = \frac{1}{4\pi}\overrightarrow{E}\left(\overrightarrow{\nabla} \cdot \overrightarrow{E}\right) + \frac{1}{4\pi}\left[\left(\overrightarrow{\nabla} \times \overrightarrow{B}\right) \times \overrightarrow{B}\right]$

- In non-relativistic MHD: $E \sim \beta B \ll B$, hence the electric term can be neglected.

- $\overrightarrow{f_{\rm L}} \simeq \frac{1}{4\pi}\left[\left(\overrightarrow{\nabla} \times \overrightarrow{B}\right) \times \overrightarrow{B}\right] = \frac{1}{4\pi}\left(\overrightarrow{B} \cdot \overrightarrow{\nabla}\right)\overrightarrow{B} - \frac{1}{8\pi}\overrightarrow{\nabla}\left(B^2\right)$

LORENTZ FORCE DENSITY IN IDEAL MHD

- $\overrightarrow{f_\mathrm{L}} \simeq \frac{1}{4\pi}\left(\overrightarrow{B}\cdot\overrightarrow{\nabla}\right)\overrightarrow{B} - \frac{1}{8\pi}\overrightarrow{\nabla}\left(B^2\right)$

- The first term is the magnetic tension.
  Example: toroidal field in cylindrical coordinates $\overrightarrow{B} = B_\phi(r,z)\hat{\phi}$:
  $\left(\overrightarrow{B}\cdot\overrightarrow{\nabla}\right)\overrightarrow{B} = -\frac{B_\phi^2}{r}\hat{r}$ a radial component directed inwards.

- The second term is the magnetic pressure gradient $-\overrightarrow{\nabla}P_\mathrm{B}$.
  Example: $\overrightarrow{B} = B_x(y)\hat{x}$: $\overrightarrow{\nabla}\left(B^2\right) = \frac{\partial(B_x^2)}{\partial y}\hat{y}$.

- Uniform magnetic field is force-free.

PROBLEM 3: MAGNETIC BRAKING

- Consider a thin ring of radius R centered in cylindrical coordinates (r, ϕ, z) of conducting plasma rotating with angular velocity $\overrightarrow{\Omega} = \Omega\hat{z}$ and threaded by an axisymmetric magnetic field $\overrightarrow{B} = [B_r(r), B_\phi(r), 0]$.

- By considering how Ω changes due to the torque exerted by the Lorentz force, derive an expression for the magnetic braking time scale $t_{\rm L} \equiv \dfrac{\Omega}{|\mathrm{d}\Omega/\mathrm{d}t|}$.

This problem is worth 5 points. Solutions should be sent as 1-page PDF files to knalew@camk.edu.pl before the next lecture.

MHD STRESS TENSOR

$$T^{00}_{\rm EM} = u_{\rm EM} = \frac{E^2 + B^2}{8\pi} = \frac{(1+\beta^2)B^2 - (\overrightarrow{\beta}\cdot\overrightarrow{B})^2}{8\pi}$$

$$T^{0i}_{\rm EM} = \frac{S^i}{c} = \frac{1}{4\pi}\left(\overrightarrow{E}\times\overrightarrow{B}\right)^i = \frac{B^2\beta^i - (\overrightarrow{\beta}\cdot\overrightarrow{B})B^i}{4\pi}$$

$$T^{ij}_{\rm EM} = \frac{E^2 + B^2}{8\pi}\delta^{ij} - \frac{E^iE^j + B^iB^j}{4\pi} =$$

$$= \frac{B^2}{4\pi}\beta^i\beta^j + \frac{(1-\beta^2)B^2 + (\overrightarrow{\beta}\cdot\overrightarrow{B})^2}{8\pi}\delta^{ij} - (1-\beta^2)\frac{B^iB^j}{4\pi} - \frac{\overrightarrow{\beta}\cdot\overrightarrow{B}}{4\pi}\left(\beta^iB^j + \beta^jB^i\right)$$

SUMMARY

- A simple Ohm's law with scalar resistivity, non-relativistic Lorentz transformation and neglecting the displacement current are the basis for resistive MHD: $\vec{E} \simeq \vec{B} \times \vec{\beta} + \frac{\eta}{c}\left(\vec{\nabla} \times \vec{B}\right)$.

- In most astrophysical systems, magnetic diffusivity can be neglected, which leads to the ideal MHD limit $\vec{E} \simeq \vec{B} \times \vec{\beta}$.

- In ideal MHD, magnetic flux is frozen into the flow.

- In non-relativistic MHD, the Lorentz force density is $\vec{f_{\rm L}} \simeq \frac{1}{4\pi}\left(\vec{B} \cdot \vec{\nabla}\right)\vec{B} - \frac{1}{8\pi}\vec{\nabla}\left(B^2\right)$: (tension) - (pressure gradient).