COSMIC MAGNETIC FIELDS

KRZYSZTOF NALEWAJKO, CAMK PAN
KNALEW@CAMK.EDU.PL

Rayleigh-Taylor instability (HD)

HYDROSTATIC EQUILIBRIUM UNDER GRAVITY

- Consider the problem of a static background $\overrightarrow{\mathrm{v}}_0 = 0$ under uniform gravitational acceleration $\overrightarrow{g} = -g\hat{z}$.

- The gravitational force density: $\vec{f}_{\mathrm{g}} = \rho\overrightarrow{g}$.

- Hydrostatic equilibrium: $\rho_0\overrightarrow{g} = \overrightarrow{\nabla} P_0$ or $P_0'(z) \equiv \dfrac{\mathrm{d}P_0}{\mathrm{d}z} = -g\rho_0(z) < 0.$

LINEARIZED EULER EQUATION

- $\rho_0 \frac{\partial \overrightarrow{\mathrm{v}}_1}{\partial t} = -\overrightarrow{\nabla} P_1 + \overrightarrow{g} \rho_1$

- $\overrightarrow{\mathrm{v}}_1 \propto \exp\left(i\omega t + ik_x x + ik_y y + ik_z z\right)$:

  (however the z dependence of ρ_1, P_1 can be more complex)

  $i\omega\rho_0 \mathrm{v}_{1,x} = -ik_x P_1$

  $i\omega\rho_0 \mathrm{v}_{1,y} = -ik_y P_1$

  $i\omega\rho_0 \mathrm{v}_{1,z} = -P_1' - g\rho_1$

- eliminating horizontal velocities:

  $\mathrm{v}_{1,x} = -\frac{k_x P_1}{\omega\rho_0}$ and $\mathrm{v}_{1,y} = -\frac{k_y P_1}{\omega\rho_0}$

LINEARIZED CONTINUITY EQUATION

- linearized continuity equation: $\frac{\partial \rho_1}{\partial t} + \left(\vec{\mathrm{v}}_1 \cdot \vec{\nabla}\right)\rho_0 + \rho_0\left(\vec{\nabla} \cdot \vec{\mathrm{v}}_1\right) = 0$

- $i\omega\rho_1 + \mathrm{v}_{1,z}\rho_0' + \rho_0\left(ik_x\mathrm{v}_{1,x} + ik_y\mathrm{v}_{1,y} + ik_z\mathrm{v}_{1,z}\right) = 0$

- $ik_x\mathrm{v}_{1,x} + ik_y\mathrm{v}_{1,y} = -\frac{ik_{xy}^2}{\omega\rho_0}P_1$

- introducing density height scale $\rho_0' \equiv \frac{\rho_0}{\lambda_\rho}$ assuming that $\lambda_\rho = \mathrm{const}$,

  associated with free-fall time scale $t_\rho \equiv \sqrt{|\lambda_\rho/g|}$ and velocity $\mathrm{v}_\rho \equiv \sqrt{|g\lambda_\rho|}$:

  $i\omega\rho_1 + \frac{\rho_0}{\lambda_\rho}\mathrm{v}_{1,z} - \frac{ik_{xy}^2}{\omega}P_1 + ik_z\rho_0\mathrm{v}_{1,z} = 0$

- $\rho_1 = \left(\frac{i}{\lambda_\rho} - k_z\right)\frac{\rho_0}{\omega}\mathrm{v}_{1,z} + \frac{k_{xy}^2}{\omega^2}P_1$

LINEARIZED EQUATION OF STATE

- linearized adiabatic equation of state:
  $$\frac{\partial P_1}{\partial t} + \left(\overrightarrow{\mathrm{v}}_1 \cdot \overrightarrow{\nabla}\right) P_0 + \kappa P_0 \left(\overrightarrow{\nabla} \cdot \overrightarrow{\mathrm{v}}_1\right) = 0$$
- $i\omega P_1 + \mathrm{v}_{1,z} P_0' + \kappa P_0 \left(ik_x \mathrm{v}_{1,x} + ik_y \mathrm{v}_{1,y} + ik_z \mathrm{v}_{1,z} \right) = 0$
- substituting $\mathrm{v}_{1,x}$, $\mathrm{v}_{1,y}$ and the hydrostatic equilibrium $P_0' = -g\rho_0$:
  $$i\omega P_1 - g\rho_0 \mathrm{v}_{1,z} - \frac{ik_{xy}^2}{\omega} \frac{\kappa P_0}{\rho_0} P_1 + ik_z \kappa P_0 \mathrm{v}_{1,z} = 0$$
- substituting local speed of sound $\mathrm{v}_{s,0}^2(z) = \dfrac{\kappa P_0(z)}{\rho_0(z)}$:
  $$i\omega P_1 - g\rho_0 \mathrm{v}_{1,z} - \frac{ik_{xy}^2}{\omega} \mathrm{v}_{s,0}^2 P_1 + ik_z \mathrm{v}_{s,0}^2 \rho_0 \mathrm{v}_{1,z} = 0$$
- $\left(\omega^2 - k_{xy}^2 \mathrm{v}_{s,0}^2\right) P_1 = -\left(ig + k_z \mathrm{v}_{s,0}^2\right) \omega \rho_0 \mathrm{v}_{1,z}$ hence $P_1 = -\dfrac{ig + k_z \mathrm{v}_{s,0}^2}{\omega^2 - k_{xy}^2 \mathrm{v}_{s,0}^2} \omega \rho_0 \mathrm{v}_{1,z}$.

THREE APPROXIMATIONS

- $i\omega\rho_0 \mathrm{v}_{1,z} = -P_1' - g\rho_1$

  $$\rho_1 = \left(\frac{i}{\lambda_\rho} - k_z\right)\frac{\rho_0}{\omega}\mathrm{v}_{1,z} + \frac{k_{xy}^2}{\omega^2}P_1$$

  $$P_1 = -\frac{ig + k_z \mathrm{v}_{s,0}^2}{\omega^2 - k_{xy}^2 \mathrm{v}_{s,0}^2}\omega\rho_0 \mathrm{v}_{1,z}$$

  calculating P_1' is going to be unpleasant, and dispersion relation can be complex

- Three approximations:
  - short-wavelength: $k_{xy} \gg 1/|\lambda_\rho|$ and $|k_z| \lesssim k_{xy}$
  - incompressible (highly subsonic): $\mathrm{v}_{s,0} \gg \mathrm{v}_\rho$
  - isothermal: $\mathrm{v}_{s,0}' = 0$

SHORT-WAVELENGTH LIMIT

- $k_{xy} \gg 1/|\lambda_\rho|$ and $|k_z| \lesssim k_{xy}$
- $P_1 = -\omega\rho_0 \frac{ig + k_z \mathrm{v}_{s,0}^2}{\omega^2 - k_{xy}^2 \mathrm{v}_{s,0}^2} \mathrm{v}_{1,z} \simeq \frac{ig + k_z \mathrm{v}_{s,0}^2}{k_{xy}^2 \mathrm{v}_{s,0}^2} \omega\rho_0 \mathrm{v}_{1,z}$ (we need the smaller ig term for ρ_1)
- $P_1' \simeq ik_z P_1 \simeq i\omega \frac{k_z^2}{k_{xy}^2} \rho_0 \mathrm{v}_{1,z}$
- $\rho_1 = \left(\frac{i}{\lambda_\rho} - k_z\right) \frac{\rho_0}{\omega} \mathrm{v}_{1,z} + \frac{k_{xy}^2}{\omega^2} P_1 \simeq i \left(\frac{1}{\lambda_\rho} + \frac{g}{\mathrm{v}_{s,0}^2}\right) \frac{\rho_0}{\omega} \mathrm{v}_{1,z}$
- $i\omega\rho_0 \mathrm{v}_{1,z} = -P_1' - g\rho_1 \simeq -i\omega \frac{k_z^2}{k_{xy}^2} \rho_0 \mathrm{v}_{1,z} - ig \left(\frac{1}{\lambda_\rho} + \frac{g}{\mathrm{v}_{s,0}^2}\right) \frac{\rho_0}{\omega} \mathrm{v}_{1,z}$
- the dispersion relation: $\omega^2 \left(1 + \frac{k_z^2}{k_{xy}^2}\right) \simeq -\frac{g}{\lambda_\rho} - \frac{g^2}{\mathrm{v}_{s,0}^2}$
- Instability $\omega^2 < 0$ requires that $\mathrm{v}_{s,0}^2/(g\lambda_\rho) > -1$. Any $\lambda_\rho > 0$ (density increasing against $\vec{g}$) is unstable, but also $\lambda_\rho < 0$ can be unstable for $g(-\lambda_\rho) \equiv \mathrm{v}_\rho^2 > \mathrm{v}_{s,0}^2$ (supersonic free-fall).

INCOMPRESSIBLE (SUBSONIC) LIMIT

- strongly subsonic free-fall velocity $\mathrm{v}_{s,0} \gg \mathrm{v}_\rho$ leads to incompressibility:

$$\vec{\nabla} \cdot \vec{\mathrm{v}}_1 = -\frac{ik_{xy}^2}{\omega\rho_0}P_1 + ik_z \mathrm{v}_{1,z} = \left(\frac{k_{xy}^2 \mathrm{v}_\rho^2 - ik_z|\lambda_\rho|\omega^2}{k_{xy}^2 \mathrm{v}_{s,0}^2 - \omega^2}\right)\frac{\mathrm{v}_{1,z}}{|\lambda_\rho|} \to 0$$

- $P_1 = -\dfrac{ig + k_z \mathrm{v}_{s,0}^2}{\omega^2 - k_{xy}^2 \mathrm{v}_{s,0}^2}\omega\rho_0 \mathrm{v}_{1,z} \simeq \dfrac{k_z\omega}{k_{xy}^2}\rho_0 \mathrm{v}_{1,z}$

- $P_1' \simeq \dfrac{k_z\omega}{k_{xy}^2}(\rho_0 \mathrm{v}_{1,z})' = \dfrac{k_z\omega}{k_{xy}^2}\left(\dfrac{1}{\lambda_\rho} + ik_z\right)\rho_0 \mathrm{v}_{1,z}$

- $\rho_1 = \left(\dfrac{i}{\lambda_\rho} - k_z\right)\dfrac{\rho_0}{\omega}\mathrm{v}_{1,z} + \dfrac{k_{xy}^2}{\omega^2}P_1 \simeq \dfrac{i\rho_0}{\omega\lambda_\rho}\mathrm{v}_{1,z}$

- $i\omega\rho_0 \mathrm{v}_{1,z} = -P_1' - g\rho_1 \simeq -\dfrac{k_z\omega}{k_{xy}^2\lambda_\rho}\rho_0 \mathrm{v}_{1,z} - \dfrac{ik_z^2\omega}{k_{xy}^2}\rho_0 \mathrm{v}_{1,z} - \dfrac{ig}{\omega\lambda_\rho}\rho_0 \mathrm{v}_{1,z}$

- $\omega^2\left(1 - \dfrac{ik_z}{k_{xy}^2\lambda_\rho} + \dfrac{k_z^2}{k_{xy}^2}\right)\mathrm{v}_{1,z} \simeq -\dfrac{g}{\lambda_\rho}\mathrm{v}_{1,z}$

INCOMPRESSIBLE (SUBSONIC) LIMIT

- $\omega^2 \left(1 - \frac{ik_z}{k_{xy}^2 \lambda_\rho} + \frac{k_z^2}{k_{xy}^2} \right) \mathrm{v}_{1,z} \simeq -\frac{g}{\lambda_\rho} \mathrm{v}_{1,z}$
- $\omega^2 \left(\mathrm{v}_{1,z} - \frac{\mathrm{v}'_{1,z}}{k_{xy}^2 \lambda_\rho} - \frac{\mathrm{v}''_{1,z}}{k_{xy}^2} \right) \simeq -\frac{g}{\lambda_\rho} \mathrm{v}_{1,z}$
- to obtain a real dispersion relation, we substitute $\mathrm{v}_{1,z} = f_0(z)\xi_1$ with $\xi_1 \propto \exp(ik_z z)$ (a redefined k_z). Note that $\mathrm{v}'_{1,z} = (f'_0 + ik_z f_0)\xi_1$ and $\mathrm{v}''_{1,z} = (f''_0 + 2ik_z f'_0 - k_z^2 f_0)\xi_1$.
- $\omega^2 \left(f_0 - \frac{f'_0 + ik_z f_0}{k_{xy}^2 \lambda_\rho} - \frac{f''_0 + 2ik_z f'_0 - k_z^2 f_0}{k_{xy}^2} \right) \xi_1 \simeq -\frac{g}{\lambda_\rho} f_0 \xi_1$

  the imaginary terms cancel out for $f'_0/f_0 = -1/(2\lambda_\rho)$, which means that $f_0 \propto 1/\sqrt{\rho_0}$ and $f''_0/f_0 = 1/(4\lambda_\rho^2)$.
- $\omega^2 \left(1 + \frac{1}{4k_{xy}^2 \lambda_\rho^2} + \frac{k_z^2}{k_{xy}^2} \right) \simeq -\frac{g}{\lambda_\rho}$
- Instability $\omega^2 < 0$ requires simply that $\lambda_\rho > 0$ (density increasing against $\vec{g}$).

ISOTHERMAL LIMIT

- $v'_{s,0} = 0$. However, since $(v_{s,0}^2)' = -\kappa g - \dfrac{v_{s,0}^2}{\lambda_\rho}$, this implies a particular density height $\lambda_\rho = -\dfrac{v_{s,0}^2}{\kappa g} < 0$ and free-fall velocity $v_\rho^2 = \dfrac{v_{s,0}^2}{\kappa}$ (compressible).

- The calculations are rather lengthy, involving the $v_{1,z} = f_0(z)\xi_1$ substitution, the resulting dispersion relation is:

$$(-\omega^2)\left[1 + \frac{1}{k_{xy}^2 v_{s,0}^2 - \omega^2}\left(k_z^2 v_{s,0}^2 + \frac{\kappa^2 g^2}{4 v_{s,0}^2}\right)\right] + (\kappa - 1)\frac{k_{xy}^2 g^2}{k_{xy}^2 v_{s,0}^2 - \omega^2} = 0$$

- For $\omega^2 < 0$ and $\kappa > 1$ all terms are positive, so there is no unstable solution, consistent with $\lambda_\rho < 0$.

SUMMARY

- Short-wavelength limit ($k_{xy} \gg 1/|\lambda_\rho|$ and $|k_z| \lesssim k_{xy}$):

$$\omega^2 \left(1 + \frac{k_z^2}{k_{xy}^2} \right) \simeq -\frac{g}{\lambda_\rho} - \frac{g^2}{\mathrm{v}_{\mathrm{s},0}^2}$$

- Subsonic / incompressible limit ($\mathrm{v}_{\mathrm{s},0} \gg \mathrm{v}_\rho$):

$$\omega^2 \left(1 + \frac{1}{4k_{xy}^2\lambda_\rho^2} + \frac{k_z^2}{k_{xy}^2} \right) \simeq -\frac{g}{\lambda_\rho}$$

- Isothermal limit ($\mathrm{v}'_{\mathrm{s},0} = 0$, hence $\lambda_\rho = -\mathrm{v}_{\mathrm{s},0}^2/\kappa g$):

$$(-\omega^2)\left[1 + \frac{1}{k_{xy}^2\mathrm{v}_{\mathrm{s},0}^2 - \omega^2} \left(k_z^2 \mathrm{v}_{\mathrm{s},0}^2 + \frac{\kappa^2 g^2}{4\mathrm{v}_{\mathrm{s},0}^2} \right) \right] + (\kappa - 1)\frac{k_{xy}^2 g^2}{k_{xy}^2\mathrm{v}_{\mathrm{s},0}^2 - \omega^2} = 0$$

PHYSICAL PRINCIPLE

- Consider the short-wavelength limit $k_x \gg 1/|\lambda_\rho|$ with $k_y = k_z = 0$.
- Recall the basic equations:

$$i\omega\frac{\rho_1}{\rho_0} = -ik_x \mathrm{v}_{1,x} - \frac{\mathrm{v}_{1,z}}{\lambda_\rho}$$

$$i\omega\frac{P_1}{\rho_0} = -\mathrm{v}_{\mathrm{s},0}^2\, ik_x \mathrm{v}_{1,x} + g\mathrm{v}_{1,z}$$

$$i\omega \mathrm{v}_{1,x} = -ik_x\frac{P_1}{\rho_0}$$

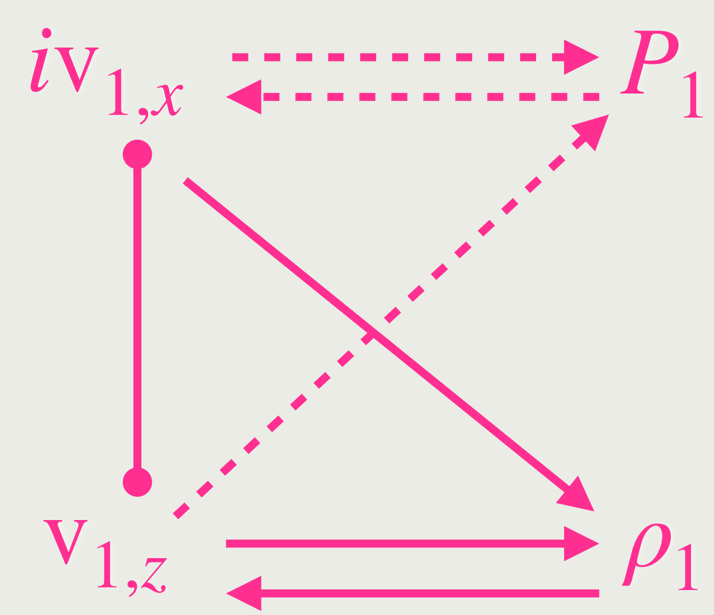
$$i\omega \mathrm{v}_{1,z} = -\frac{P_1'}{\rho_0} - g\frac{\rho_1}{\rho_0}$$

- Note that $\frac{P_1}{\rho_1} \sim \frac{\omega^2}{k_x^2} \sim \frac{1}{k_x^2\tau_\rho^2} \ll \mathrm{v}_\rho^2$, hence we can neglect the pressure gradient term in the $\mathrm{v}_{1,z}$ equation. Also note that $\left|k_x \mathrm{v}_{\mathrm{s},0}^2 \mathrm{v}_{1,x}\right| = \left|\frac{k_x^2}{\omega}\mathrm{v}_{\mathrm{s},0}^2\frac{P_1}{\rho_0}\right| \gg \left|\omega\frac{P_1}{\rho_0}\right|$.

$$i\omega\frac{\rho_1}{\rho_0} = -ik_x \mathrm{v}_{1,x} - \frac{\mathrm{v}_{1,z}}{\lambda_\rho}$$

$$\mathrm{v}_{\mathrm{s},0}^2\, ik_x \mathrm{v}_{1,x} \simeq g\mathrm{v}_{1,z}$$

$$i\omega \mathrm{v}_{1,z} \simeq -g\frac{\rho_1}{\rho_0}$$



PHYSICAL PRINCIPLE

- Incompressible loop:
  $v_{1,z} \rightarrow \rho_1 \rightarrow v_{1,z}$
  for $\lambda_\rho > 0$, $v_{1,z} > 0$ brings lower ρ_0 from below, triggering $\rho_1 < 0$. Perturbed gravitational force points upwards, increasing $v_{1,z}$.

- Compressible loop:
  $v_{1,z}(\rightarrow P_1) \rightarrow iv_{1,x} \rightarrow \rho_1 \rightarrow v_{1,z}$
  $v_{1,z} > 0$ triggers $P_1 > 0$, which triggers a horizontal sound wave with $iv_{1,x} > 0$, which triggers $\rho_1 < 0$, etc.

$$i\omega \frac{\rho_1}{\rho_0} = - ik_x v_{1,x} - \frac{v_{1,z}}{\lambda_\rho}$$

$$v_{s,0}^2 \, ik_x v_{1,x} \simeq g v_{1,z}$$

$$i\omega v_{1,z} \simeq - g \frac{\rho_1}{\rho_0}$$