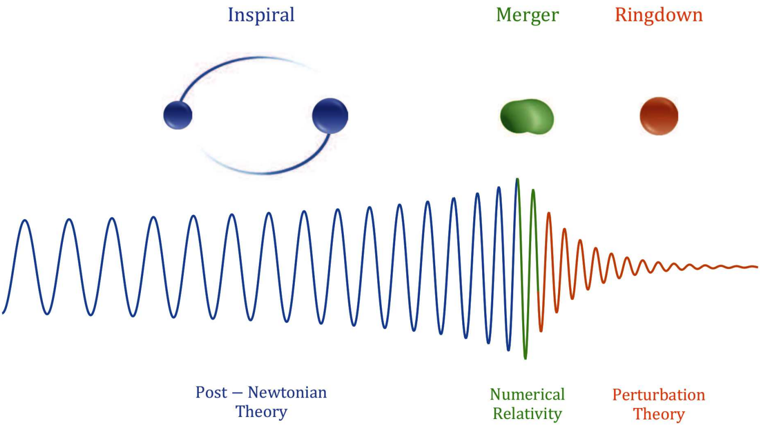
Binary system

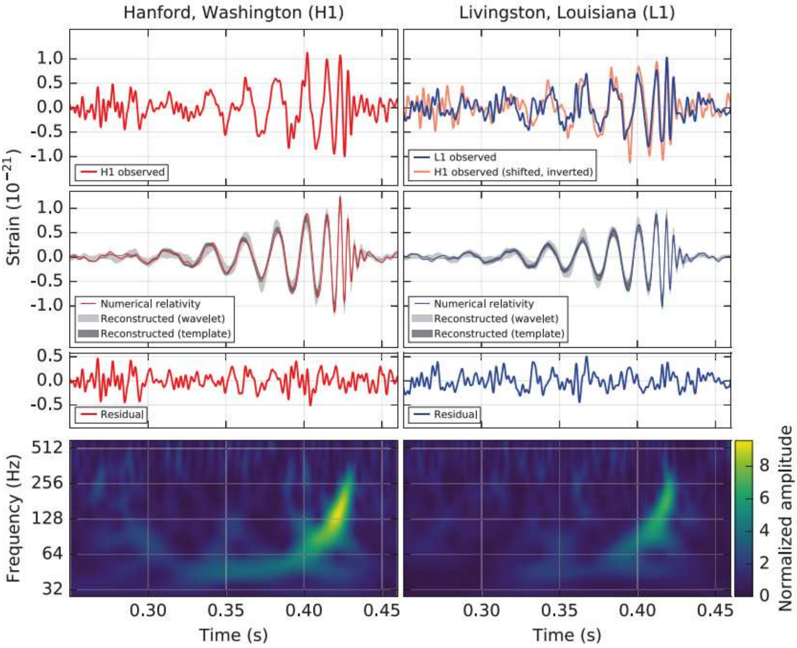
24.11.20, 1.12.20

General schedule

- ⋆ History
- ⋆ Introduction to general relativity
- ⋆ Detection principles
- ⋆ Detectors
- ⋆ Binary black-hole system
  - ⋆ First look at data and waveforms,
  - ⋆ Newtonian approximation to binary system,
  - ⋆ Material objects: neutron stars.
- ⋆ Bursts and continuous waves
- ⋆ Rates and populations, stochastic GW background, cosmology
- ⋆ Testing general relativity
- ⋆ Data analysis: waveforms and detection
- ⋆ Data analysis: parameter estimation

Last orbits of a binary system





Hanford, Washington (H1)
Livingston, Louisiana (L1)
Strain (10^{-21})
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
Residual
Frequency (Hz)
Time (s)
Normalized amplitude

GW150914: parameters

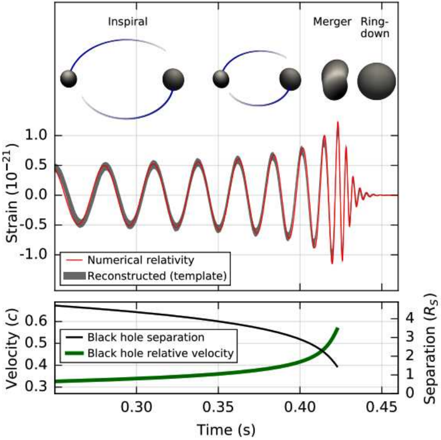
False alarm probability <1 in 5 million
False alarm rate <1 in 200000 years

- ⋆ $M_1 = 36^{+5}_{-4}\ \mathrm{M}_\odot$, $M_2 = 29^{+4}_{-4}\ \mathrm{M}_\odot$,
- ⋆ Final black hole parameters:
  - ⋆ mass $M = 62^{+4}_{-4}\ \mathrm{M}_\odot$,
  - ⋆ spin $a = 0.67^{+0.05}_{-0.07}$,
- ⋆ Distance: $410^{+160}_{-180}\ \mathrm{Mpc}$
  i.e. 1 billion 300 million light years,
  redshift $z = 0.09^{+0.03}_{-0.04}$ (assuming standard cosmology).

(uncertainties define 90% credible intervals)

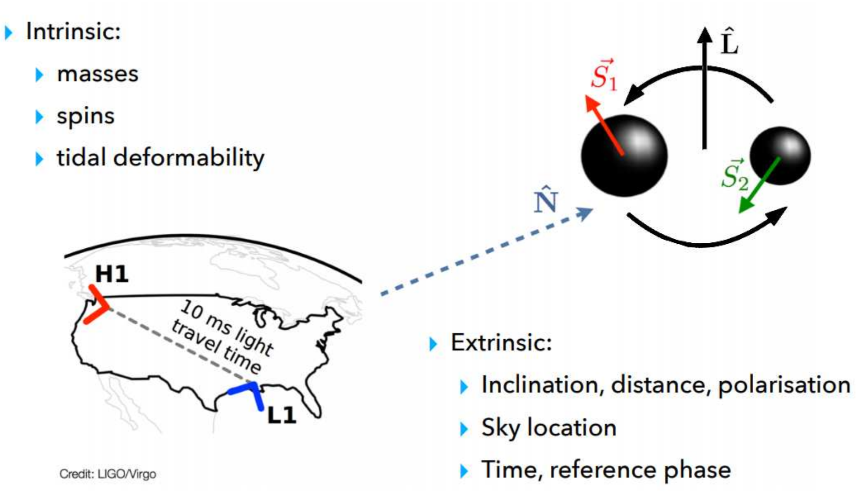
GW150914: parameters

- ⋆ Duration: 0.2 s,
- ⋆ Final orbital velocity: $> 0.5\ \mathrm{c}$,
- ⋆ Total energy emitted in waves: $E = mc^2 = 3^{+0.5}_{-0.5}\ \mathrm{M}_\odot \mathrm{c}^2$,
- ⋆ Peak "brightness": $3.6^{+0.5}_{-0.4} \times 10^{49}\ \mathrm{Joule/s}$ ($200^{+30}_{-30}\ \mathrm{M}_\odot \mathrm{c}^2/\mathrm{s}$),
  → much more than all the stars in the Universe radiate in EM!



(uncertainties define 90% credible intervals)

Binary system: 15+ parameters

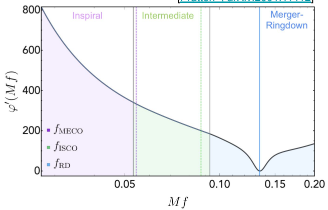


Frequency dependence in waveform

- Inspiral-Merger-Ringdown (IMR) waveform model written as frequency-dependent amplitude and phase

$$\tilde{h}(f) = \mathcal{A}(f)\, e^{i\varphi(f)}$$

[Pratten+, arXiv:2001.11412]



- Parameterize phase corrections in 3 distinct regions:
  - Inspiral

$$\varphi_{\rm Ins}(f) = \varphi_{\rm ref} + 2\pi f t_{\rm ref} + \varphi_{\rm Newt}(Mf)^{-5/3} + \varphi_{0.5{\rm PN}}(Mf)^{-4/3} + \varphi_{1{\rm PN}}(Mf)^{-1} + \varphi_{1.5{\rm PN}}(Mf)^{-2/3} + \cdots$$

Coefficients analytically known in GR

- *Phenom*enological Coefficients
  - Intermediate

$$\varphi_{\rm Int} = \frac{1}{\eta}\left(\beta_0 + \beta_1 f + \beta_2 \log(f) - \frac{\beta_3}{3} f^{-3}\right)$$

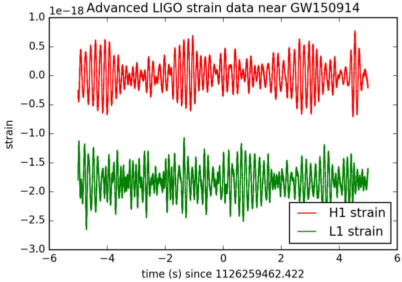
  - Merger-Ringdown

$$\varphi_{\rm MR} = \frac{1}{\eta}\left\{\alpha_0 + \alpha_1 f - \alpha_2 f^{-1} + \frac{4}{3}\alpha_3 f^{3/4} + \alpha_4 \tan^{-1}\left(\frac{f - \alpha_5 f_{\rm RD}}{f_{\rm damp}}\right)\right\}$$

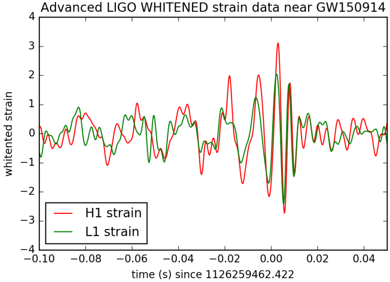
Caution: Coefficients calibrated against NR but are not expressed in parameters relevant to GR or modified theories of gravity...

LIGO–G2002002

How the data looks like



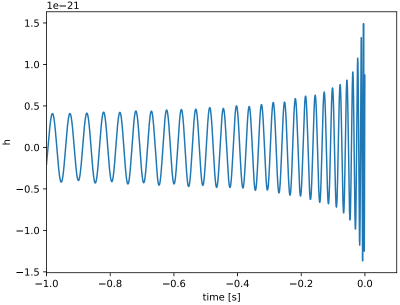
The data are dominated by the **low frequency noise** (L1 offset by -2×10^{-18} due to very low frequency oscillations).

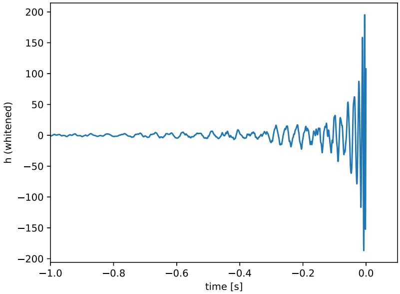


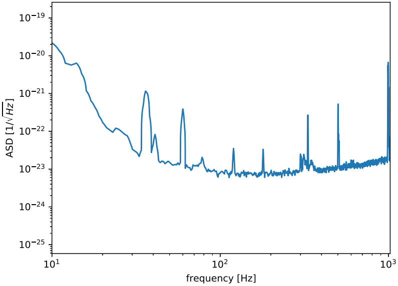
If one knows the signal is there:

- ⋆ filtering the frequencies outside the desired band with bandpass filter,
- ⋆ suppressing the instrumental lines,
- ⋆ whitening: dividing the data by the noise ASD in the Fourier domain to normalize the power for all frequencies for an easier comparison.

Raw vs whitened waveform

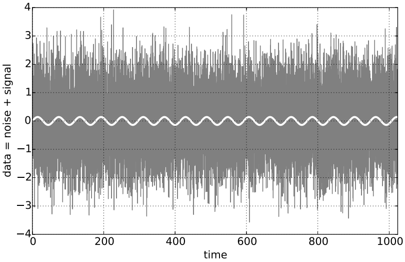


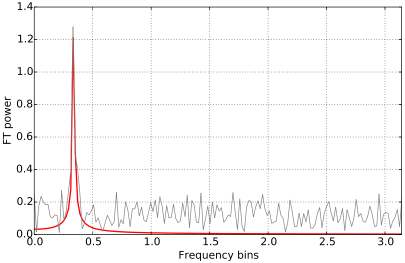




Whitening: $h_w(f) \rightarrow h(f)/ASD(f)$

Matched filtering: a monochromatic signal





In this case a Fourier transform is sufficient to detect the signal (simplest matched filter method):

$$F = \left| \int_0^{T_0} x(t) \exp(-i\omega t) dt \right|^2$$

Signal-to-noise $SNR = h_0 \sqrt{\frac{T_0}{S_0}}$

T_0 - time series duration, S_0 - spectral density of the data.

Matched filtering

Assuming a signal model h, looking for the "best match" correlation $C(t)$ in data stream x, for a given time offset t

$$C(t) = \int_{-\infty}^{\infty} \underbrace{x(t')}_{\text{Data}} \quad \times \quad \underbrace{h(t'-t)}_{\text{Template with time offset } t} \quad dt'$$

Rewrite correlation using Fourier transforms

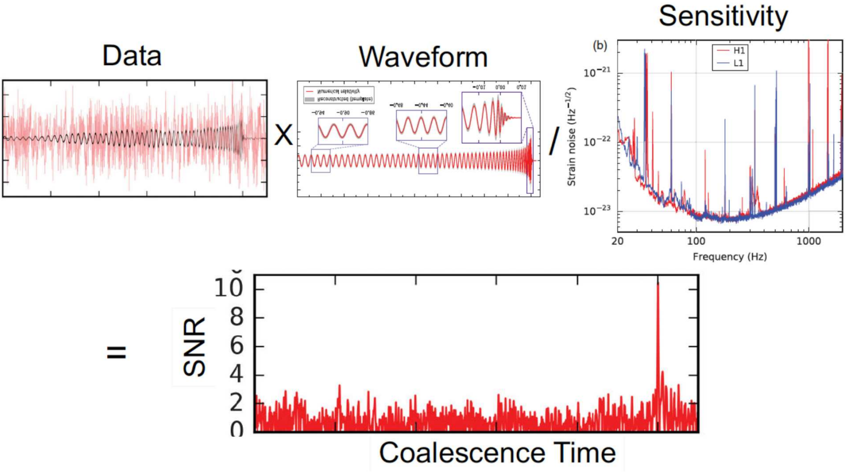
$$C(t) = 4\int_0^{\infty} \tilde{x}(f)\tilde{h}^*(f)e^{2\pi i f t}df$$

(an inverse FT of $\tilde{x}(f)\tilde{h}^*(f)$). In practice, optimal matched filtering with the frequency weighting

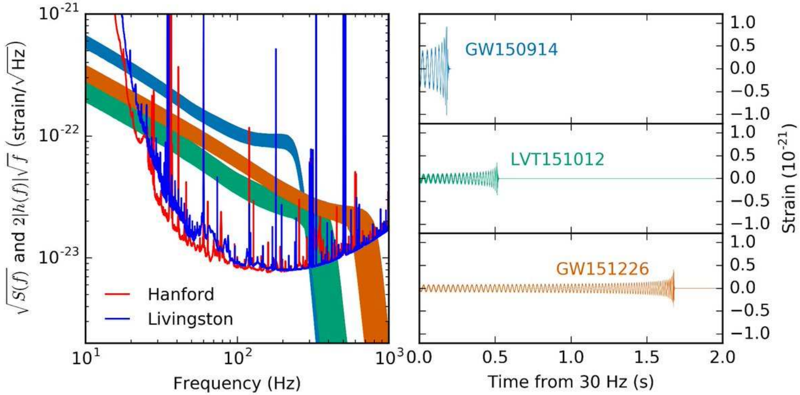
$$C(t) = (x|h) = 4\int_0^{\infty} \frac{\tilde{x}(f)\tilde{h}^*(f)}{S_n(f)}e^{2\pi i f t}df, \quad \text{with } S_n(f) \text{ (noise PSD).}$$

Matched filter SNR: $\rho = (x|h)/\sqrt{(h|h)}$, Optimal SNR: $\rho_{opt} = \sqrt{(h|h)}$

Matched filter in pictures



LIGO O1: 2 ("and a half") events



Optimal signal-to-noise ρ : $\rho^2 = \int_0^\infty \left(\frac{2|\tilde{h}(f)|\sqrt{f}}{\sqrt{S_n(f)}} \right)^2 d\ln(f)$

(GW150914: $\rho \simeq 24$, GW151226: $\rho \simeq 13$, LVT151012: $\rho \simeq 10$)

Initial estimates

For a spherical wave of amplitude $h(r)$,

- $\star$ flux of energy is $F(r) \propto h^2(r)$,
- $\star$ the luminosity $L(r) \propto 4\pi r^2 h^2(r)$.

Conservation of energy (flux through surface at r):

$$\Longrightarrow h(r) \propto 1/r.$$

Radiating modes: quadrupole and higher

For a mass distribution $\rho(r)$, conserved moments:

- $\star$ monopole $\int \rho(r) d^3r$ - total mass-energy (energy conservation),
- $\star$ dipole $\int \rho(r) r d^3r$ - center of mass-energy (momentum conservation).

Quadrupolar nature of GWs

In electromagnetism, radiation due time changing electric dipole moment $\mathbf{d} = e\mathbf{x}$:

$$\text{Luminosity} \propto \ddot{\mathbf{d}}$$

Gravitational-wave emission in the dipole mode would mean the changing in time mass dipole moment:

$$\mathbf{d} = \sum_i m_i \mathbf{x}_i \quad \rightarrow \quad \dot{\mathbf{d}} = \underbrace{\sum_i m_i \dot{\mathbf{x}}_i}_{\textit{Momentum}}$$

Conservation of momentum means no mass dipole GW radiation. Likewise, for the current dipole moment

$$\mathcal{D} = \underbrace{\sum_i m_i \mathbf{x}_i \times \dot{\mathbf{x}}_i}_{\textit{Angular momentum}}$$

the conservation of angular momentum means no current dipole GW radiation.

Estimate of wave amplitude

The wave equation for GWs,

$$\Box \overline{h}^{\alpha\beta} = \frac{16\pi G}{c^4} T^{\alpha\beta}$$

is an analogue to the Maxwell equation (Gauss law) in the Lorenz gauge, $(1/c^2)\partial_t \phi + \nabla \cdot \mathbf{A} = 0$:

$$\nabla \cdot \mathbf{E} = \Box \phi = 4\pi\rho, \quad \text{where} \quad \mathbf{E} = -\nabla\phi + \partial_t \mathbf{A}.$$

By analogy between solutions

$$\phi(t, \mathbf{r}) = \int \frac{\rho(t - R/c, \mathbf{x})}{R} dV, \quad \overline{h}^{\alpha\beta} = \frac{4G}{c^4} \int \frac{T^{\alpha\beta}(t - R/c, \mathbf{x})}{R} dV.$$

with $R = |\mathbf{r} - \mathbf{x}|$. Far from the compact source ($r \gg \mathbf{x}$), the solution is described by the *far-field solution*:

$$\overline{h}^{\alpha\beta}(t, \mathbf{r}) = \frac{4G}{c^4 r} \int T^{\alpha\beta}(t - R/c, \mathbf{x}) dV.$$

Estimate of wave amplitude

Using the energy-momentum conservation:

$$T^{\alpha\beta}_{,\beta} = 0 \quad \rightarrow \quad \bar{h}^{ij} \approx -\frac{2G}{c^4 r}\frac{d^2 I^{ij}}{dt^2},$$

where I^{ij} is the moment of inertia tensor (related to quadrupole moment tensor),

$$I_{ij} = \int \left(r^2\delta_{ij} - x_i x_j\right)\rho(r)dV, \quad Q_{ij} = \int \left(3x_i x_j - r^2\delta_{ij}\right)\rho(r)dV.$$

For two masses m separated by a on circular orbit in the $x - y$ plane with angular frequency ω around their center of mass,

$$I^{xx} \propto \int \rho x^2 dV = 2m\left(\frac{a}{2}\cos(\omega t)\right)^2 = \frac{1}{4}ma^2\left(1 + \cos(2\omega t)\right),$$

which leads to

$$\bar{h}^{xx} = \frac{2Gma^2\omega^2}{c^4 r}\cos(2\omega t).$$

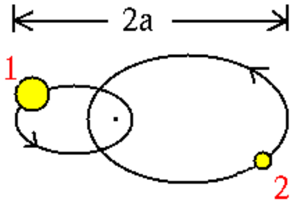
- ⋆ GW quadrupole radiation is at twice the orbital frequency (in the first approximation),
- ⋆ Amplitude $\propto Gma^2\omega^2/(c^4 r)$.

Gravitational waves: some estimates

GWs correspond to accelerated movement of masses.

Consider a binary system of m_1 and m_2, semiaxis a with

- ⋆ total mass $M = m_1 + m_2$,
- ⋆ reduced mass $\mu = m_1 m_2 / M$,
- ⋆ mass quadrupole moment $Q \propto Ma^2$,
- ⋆ Kepler's third law $GM = a^3\omega^2$.



$$h(r) \propto \frac{1}{r}\frac{\partial^2(Ma^2)}{\partial t^2} = \boxed{\frac{G^2}{c^4}\frac{1}{r}\frac{M\mu}{a} = \frac{G^{5/3}}{c^4}\frac{1}{r}M^{2/3}\mu\,\omega^{2/3}.}$$

Gravitational waves: quadrupole approximation

The quadrupole approximation (slowly-moving sources, Einstein 1918), wave amplitude is

$$h^{\mu\nu} = \frac{2}{r}\frac{G}{c^4}\ddot{Q}^{\mu\nu}, \quad \text{or, in terms of kinetic energy,} \quad h \sim \frac{E_{kin.}^{nsph.}}{r}.$$

Resulting GW luminosity is

$$L_{GW} \equiv \frac{dE_{GW}}{dt} \approx \frac{1}{5}\frac{G}{c^5}\langle \dddot{Q}^{\mu\nu}\dddot{Q}_{\mu\nu}\rangle$$

$$\propto \frac{G}{c^5}Q^2\omega^6 \propto \frac{G^4}{c^5}\left(\frac{M}{a}\right)^5 \propto \frac{c^5}{G}\left(\frac{R_s}{a}\right)^2\left(\frac{v}{c}\right)^6.$$

($R_s = 2GM/c^2$, $c^5/G \simeq 3.6 \times 10^{52}$ Joule/s)

Binary system: evolution of the orbit

Waves are emitted at the expense of the orbital energy:

$$E_{orb} = -\frac{Gm_1 m_2}{2a}, \qquad \frac{dE_{orb}}{dt} \equiv \frac{Gm_1 m_2}{2a^2}\dot{a} = -\frac{dE_{GW}}{dt}.$$

Evolution of the semi-major axis:

$$\frac{da}{dt} = -\frac{dE_{GW}}{dt}\frac{2a^2}{G\underbrace{m_1 m_2}_{\mu M}} \rightarrow \frac{da}{dt} = -\frac{64}{5}\frac{G^3}{c^5}\frac{\mu M^4}{a^3}.$$

The system will coalesce after a time τ,

$$\tau = \frac{5}{256}\frac{c^5}{G^3}\frac{a_0^4}{\mu M^4},$$

where a_0 is the initial separation.

Binary system: chirp mass

Waves are emitted at the expense of the orbital energy:

$$E_{orb} = -\frac{Gm_1 m_2}{2a}, \qquad \frac{dE_{orb}}{dt} \equiv \frac{Gm_1 m_2}{2a^2}\dot{a} = -\frac{dE_{GW}}{dt}.$$

Resulting evolution of the orbital frequency ω:

$$\dot{\omega}^3 = \left(\frac{96}{5}\right)^3 \frac{\omega^{11}}{c^{15}} G^5 \mu^3 M^2 = \left(\frac{96}{5}\right)^3 \frac{\omega^{11}}{c^{15}} G^5 \mathcal{M}^5,$$

where $\mathcal{M} = \left(\mu^3 M^2\right)^{1/5} = (m_1 m_2)^{3/5}/(m_1 + m_2)^{1/5}$ is the chirp mass.
GWs frequency from a binary system is primarily twice the orbital frequency ($2\pi f_{GW} = 2\omega$). $\mathcal{M}$ is a directly measured quantity:

$$\mathcal{M} = \frac{c^3}{G}\left(\frac{5}{96}\pi^{-8/3} f_{GW}^{-11/3} \dot{f}_{GW}\right)^{3/5}.$$

Binary system: energy emitted in GWs

End of the chirp f^c_{GW} is related to critical distance between masses a_{fin}:

$$a_{fin} = R_{s1} + R_{s2} = \frac{2G}{c^2}\left(m_1 + m_2\right).$$

It can be used to estimate the total mass M:

$$M = m_1 + m_2 \approx \frac{c^3}{2\sqrt{2}G\pi}\frac{1}{f^c_{GW}}.$$

Energy emitted during the life of the binary system (rest-mass energy + orbital energy):

$$E = E_{rm} + E_{orb} = \left(m_1 + m_2\right)c^2 - \frac{Gm_1 m_2}{2a}.$$

(for $m_1 = m_2$, $a_{fin} = 2R_s = 4Gm_1/c^2$, $\Delta E \approx 6\%$).

Final black hole spin

Orbital angular momentum J (major semi-axis $a = a_1 + a_2$):

$$J = m_1 a_1^2 \omega + m_2 a_2^2 \omega = m_1 m_2 \sqrt{\frac{Ga}{m_1 + m_2}}.$$

Dimensionless spin magnitude χ of an object with J and M:

$$\chi = \frac{cJ}{GM^2}.$$

For $m_1 = m_2 = m$,

- ⋆ $a = a_{fin}$, $\chi \approx 0.35$,
- ⋆ $a = 2r_{isco}$ (innermost stable circular orbit around BH of mass m), $\chi \sim 0.7$.

Circularization of orbit by GW radiation

Peters (1964, p. 103):

The equation of the relative orbit of the motion is

$$r = \frac{a\,(1 - e^2)}{1 + e\cos(\Psi - \Psi_o)} \quad . \qquad (5.36)$$

Equation 5.43 can be integrated to get a(e) during the collapse of a system. The integration is tedious but straight forward. a(e) is then found to be

$$a = \frac{c_o\, e^{12/19}}{(1 - e^2)} \left[1 + \frac{121}{304} e^2\right]^{\frac{870}{2299}} , \qquad (5.48)$$

where c_o is determined by the initial conditions $a = a_o$ when $e = e_o$. a is plotted against e in Figure 5.

For small e, this reduces to

$$a = c_o\, e^{12/19} \quad , \quad e^2 \ll 1 ,$$

For a - semi-major axis, e - eccentricity,

$$\frac{e}{e_0} \approx \left(\frac{a}{a_0}\right)^{19/12}$$

Reduction of a by ≈ 2 means reduction in eccentricity by ≈ 3
$\rightarrow$ radiation reaction quickly circularizes the orbit.

Parameter estimation basics (GW510914)

GW amplitude dependence for a binary system

$$h \propto \mathcal{M}^{5/3} \times f_{GW}^{2/3} \times r^{-1}$$

where $\mathcal{M}$ is the chirp mass, $\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$, known from the observations:

$$\mathcal{M} = \frac{c^3}{G}\left[\frac{5}{96}\pi^{-8/3} f_{GW}^{-11/3} \dot{f}_{GW}\right]^{3/5}$$

From higher-order post-Newtonian corrections: $q = m_2/m_1$, spin components parallel to the orbital angular momentum...

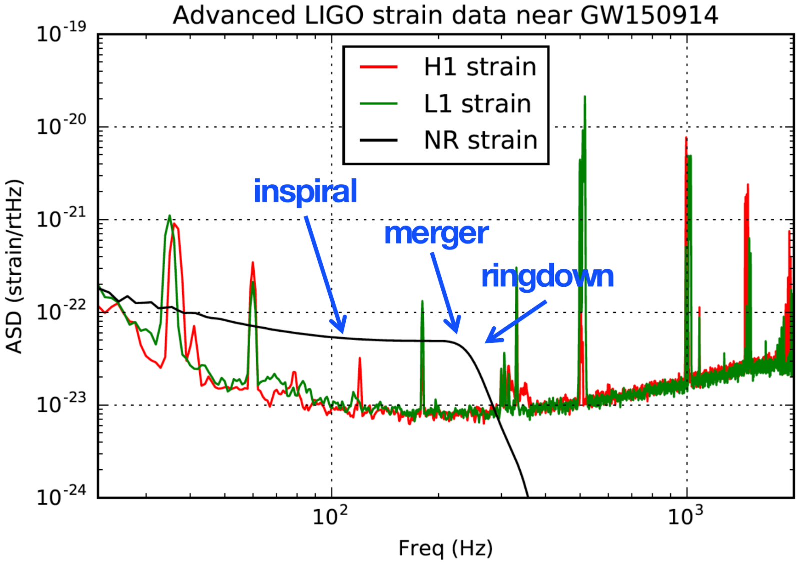
$$\mathcal{M} \simeq 30M_\odot \implies M = m_1 + m_2 \simeq 70M_\odot \quad (\text{if } m_1 = m_2,\ M = 2^{6/5}\mathcal{M})$$

8 orbits observed until 150 Hz (orbital frequency 75 Hz):

- ⋆ Binary neutron star system is compact enough, but too light,
- ⋆ Neutron star-black hole system for a given total mass - black hole too big, would merge at lower frequency.

→ Black hole binary.

Binary inspiral vs the sensitivity curve



Binary inspiral vs the sensitivity curve

The so-called *Newtonian* signal at instantaneous frequency f_{GW} is

$$h = Q(\text{angles}) \times \mathcal{M}^{5/3} \times f_{GW}^{2/3} \times r^{-1} \times e^{-i\Phi}.$$

where the signal's phase is

$$\Phi(t) = \int 2\pi f_{GW}(t')dt'.$$

The relation between f_{GW} and t

$$\pi\mathcal{M} f_{GW}(t) = \left(\frac{5\mathcal{M}}{256(t_c - t)}\right)^{3/8}$$

The orbital velocity is $v \propto (\pi M f_{GW})^{1/3}$ because from Kepler's 3rd law ($\omega^2 a^3 = GM$), one gets $\omega = 2\pi f = \pi f_{GW}$, $v = \omega a$ $\rightarrow v^3 = \pi G M f_{GW}$.

Binary inspiral vs the sensitivity curve

Matched filtering means that the signal is integrated with a proper phase as it sweeps through the range of frequencies.

Sensitivity curves most often show the effective (match-filtered) h_{eff}, and not the instantaneous h.

Dimensonal estimation of the frequency slope:

$$N_{cycles} \approx f_{GW}^2 \times \left(\frac{df_{GW}}{dt} \right)^{-1}$$

$$h_{eff} \propto \sqrt{N_{cycles}}\, h \propto \sqrt{f_{GW} t}\, h \propto \sqrt{f_{GW} \times f_{GW}^{-8/3}} \times f_{GW}^{2/3} \propto \boxed{f_{GW}^{-1/6}}.$$

Binary inspiral vs the sensitivity curve

Actually used in estimating the SNR is the frequency-domain match-filtering signal model $\tilde{h}(f) \simeq h_{eff}$ (Fourier transform of $h(t)$),

$$\tilde{h}(f) = Q(angles)\sqrt{\frac{5}{24}}\pi^{-2/3}\frac{\mathcal{M}^{5/6}}{r}f_{GW}^{-7/6}e^{-i\Psi(f)},$$

where the frequency domain phase Ψ is (in point-particle approximation):

$$\Psi(f) \equiv \Psi_{PP}(f) = 2\pi f t_c - \phi_c - \frac{\pi}{4} + \frac{3M}{128\mu v^{5/2}}\sum_{k=0}^{N}\alpha_k v^{k/2}.$$

Binary system: source distance estimate

- ⋆ At cosmological distances, the observed frequency f_{GW} is redshifted by $(1+z)$:

$$f \to f/(1+z)$$

- ⋆ There is no mass scale in vacuum GR, so redshifting of f_{GW} cannot be distinguished from rescaling the masses

  because of the expansion in powers of $v \propto (\pi M f_{GW})^{1/3}$

  $\Longrightarrow$ inferred masses are $m = (1+z)m^{source}$

→ Direct, independent **luminosity distance** measurement (but not z) from GW with f_{GW} and the strain h:

$$r = \frac{5}{96\pi^2}\frac{c}{h}\frac{\dot{f}_{GW}}{f_{GW}^3}.$$

Post-Newtonian expansion

The post-Newtonian approximation consists of a weak-field, slow-motion approximation to the true GR problem in which relevant quantities are expanded around the zeroth-newtonian order ones as a sum in powers of $(v/c)^{2n}$ or $(M/c^2 d)^n$, where n determines the n-PN order. For example, the well known Kepler law $(\omega d)^2 = GM/d$ is valid at zeroth order. GR corrections add higher powers of the PN-expansion so that in a general form we have

$$(\omega_{orb}^2 d) = \frac{GM}{d} + \sum_n a_n(\theta, \varphi) \left(\frac{GM}{dc^2}\right)^n .$$

In order to compute the orbital frequency ω_{orb} one needs expressions for the energy and flux of the system. Once these are obtained, and assuming that the system evolves adiabatically one can obtain $t(v)$ by applying the chain rule

$$\frac{dE}{dt} = \frac{dE}{dv}\frac{dv}{dt} = -\mathcal{L}(v) \Rightarrow \frac{dv}{dt} = -\frac{\mathcal{L}(v)}{dE/dv},$$

for afterwards obtaining an expression for the phase of the binary applying

$$\phi_{orb} = \int \omega_{orb}(t)dt, \qquad M\omega_{orb} = v^3.$$

The PN approximation provides an expansion of these and related quantities in powers of the expansion parameter v/c.

(J. Calderón Bustillo PhD thesis, Sect. 2)

Post-Newtonian expansion

The orbital phase $\phi(t)$ evolution is computed from the balance equation

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{L},}$$

which has the following PN expansion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_{\mathrm{N}} + \frac{1}{c^2}E_{\mathrm{1PN}} + \frac{1}{c^4}E_{\mathrm{2PN}} + \frac{1}{c^6}E_{\mathrm{3PN}} + \frac{1}{c^8}E_{\mathrm{4PN}} + \mathcal{O}\big((v/c)^9\big)\right)$$
$$= -\left(\mathcal{L}_{\mathrm{N}} + \frac{1}{c^2}\mathcal{L}_{\mathrm{1PN}} + \frac{1}{c^3}\mathcal{L}_{\mathrm{1.5PN}} + \frac{1}{c^4}\mathcal{L}_{\mathrm{2PN}} + \frac{1}{c^5}\mathcal{L}_{\mathrm{2.5PN}} + \frac{1}{c^6}\mathcal{L}_{\mathrm{3PN}} + \frac{1}{c^7}\mathcal{L}_{\mathrm{3.5PN}} + \frac{1}{c^8}\mathcal{L}_{\mathrm{4PN}} + \mathcal{O}\big((v/c)^9\big)\right).$$

P. Jaranowski (`bcc.impan.pl/13Gravitational/index.php/programme.html`)

Post-Newtonian expansion

Up to 3.5PN order, the energy and flux (non-spinning part) is:

$$
\begin{aligned}
E(x) = &-\frac{\mu c^2 x}{2}\Big\{1 + x\Big(-\frac{3}{4}-\frac{1}{12}\nu\Big) + x^2\Big(-\frac{27}{8}+\frac{19}{8}\nu-\frac{1}{24}\nu^2\Big) \\
&+ x^3\Big(-\frac{675}{64}+\Big[\frac{34445}{576}-\frac{205}{96}\pi^2\Big]\nu-\frac{155}{96}\nu^2-\frac{35}{5184}\nu^3\Big) \\
&+ x^{3/2}\Big[\frac{14}{3}S_\ell + 2\frac{\delta M}{M}\Sigma_\ell\Big] + x^{5/2}\Big[\Big(11-\frac{61}{9}\nu\Big)S_\ell + \frac{\delta M}{M}\Big(b-\frac{10}{3}\nu\Big)\Sigma_\ell\Big] \\
&+ x^{7/2}\Big[\Big(\frac{135}{4}-\frac{367}{4}\nu+\frac{29}{12}\nu^2\Big)S_\ell + \frac{\delta M}{M}\Big(\frac{27}{4}-39\nu+\frac{5}{4}\nu^2\Big)\Sigma_\ell\Big]\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{NS}(x) = &\frac{32c^5}{5G}\nu^2 x^5\Big\{1 + \Big(-\frac{1247}{336}-\frac{35}{12}\nu\Big)x + 4\pi x^{3/2} \\
&+ x^2\Big(-\frac{44711}{9072}+\frac{9271}{504}\nu+\frac{65}{18}\nu^2\Big) + \pi x^{5/2}\Big(-\frac{8191}{672}-\frac{583}{24}\nu\Big) \\
&+ x^3\Big[\frac{6643739519}{69854400}+\frac{16}{3}\pi^2-\frac{1712}{105}\gamma_E-\frac{856}{105}ln(16x) \\
&+ \Big(-\frac{134543}{7776}+\frac{41}{48}\pi^2\Big)\nu-\frac{94403}{3024}\nu^2-\frac{775}{324}\nu^3\Big]\Big\}.
\end{aligned}
$$

with

$$
\begin{aligned}
x &= \left(\frac{GM\omega_{orb}}{c^3}\right)^{2/3} & M &= m_1 + m_2 \\
\delta M &= m_1 - m_2 & \nu &= \mu/M = \frac{m_1 m_2}{M^2} \\
\vec{S} &\equiv \vec{S}_1 + \vec{S}_2 & \vec{\Sigma} &\equiv \left(\frac{\vec{S}_2}{m_2} - \frac{\vec{S}_1}{m_1}\right) \\
S_\ell &= \vec{S}_{\hat{\ell}} & \Sigma_\ell &= \Sigma_{\hat{\ell}}.
\end{aligned}
$$

(J. Calderón Bustillo PhD thesis, Sect. 2)

PN expansion: higher-oder modes

Once $\phi_{orb}(t)$ and $v(t)$ have been computed with a chosen PN approximant, it is convenient to decompose the GW strain $h = h_+ - ih_\times$ onto a -2 spherical harmonics basis

$$h(\theta,\varphi;t) = h_+ - ih_\times = \sum_{\ell\geq 2}\sum_{m=-\ell}^{m=\ell} Y^{-2}_{\ell,m}(\theta,\varphi)h_{\ell,m}(t).$$

The harmonics $Y^{-2}_{\ell,m}(\theta,\varphi)$ form a basis for functions defined on a sphere and due to their transformation properties under rotations are particularly suitable for the description of spin-2 fields as the gravitational field.

$$Y^{-2}_{\ell,m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}d_{\ell,m}(\theta)e^{im\phi},$$

with

$$d_{\ell,m} = \sum_{j=\max(0,m-2)}^{min(\ell+m,\ell-2)} \frac{(-1)^j}{j!}\frac{\sqrt{(\ell+m)!(\ell-m)!(\ell+2)!(\ell-2!)}}{(j-m+2)!(\ell+m-j)!(\ell-j-2)!}$$
$$\times\,(\cos\frac{\theta}{2})^{2\ell+m-2j-2}\times(\sin\frac{\theta}{2})^{2j-m+2}.$$

This allows to express the harmonics as $Y^{-2}_{\ell,m}(\theta,\varphi) = \mathcal{A}_{\ell,m}(\theta)e^{im\varphi}$, $\mathcal{A}_{\ell,m}(\theta)$ being real.

$$h^{\ell,m} = \int [h_+ - ih_\times]Y^{-2^*}_{\ell,m}(\theta,\phi)\sin\theta d\theta d\phi.$$

The resulting $h_{\ell,m}$ modes are usually expressed in two different ways. The one we will mainly use in this thesis separates them into a real amplitude $A_{\ell,m}$ and a phase $\phi_{\ell,m}$ as

$$h_{\ell,m} = A_{\ell,m}e^{i\phi_{\ell,m}}.$$

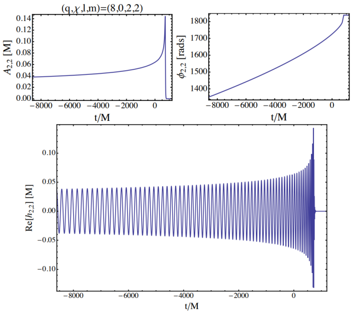


FIGURE 2.2: Amplitude, phase and real part of the (2, 2) mode of a $(q,\chi) = (8,0)$ system during the last orbits of the coalescence.

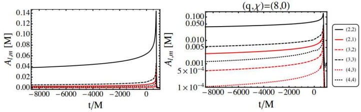
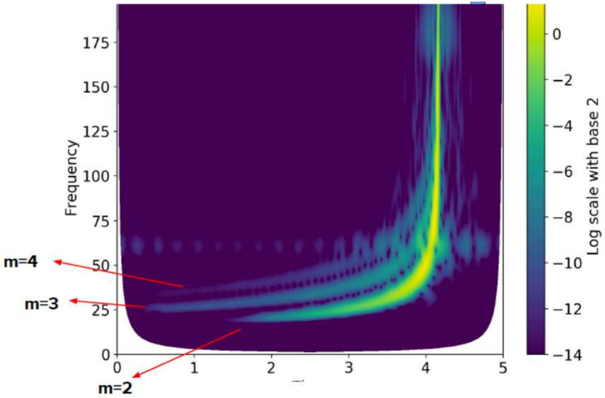


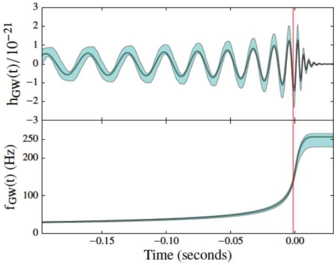
FIGURE 2.4: Amplitude of the (ℓ,m) modes of a $(q,\chi) = (8,0)$ system during the last orbits of the coalescence in lineal (left) and logarithmic (right) scale.

(J. Calderón Bustillo PhD thesis, Sect. 2)

PN expansion: higher-oder modes



Ringdown modes: BH spectroscopy



The final BH is highly perturbed (non-stationary, "has hair"):

- ⋆ BH radiates its excess energy in a GW "ringdown",
- ⋆ perturbation theory applicable to describe the "quasi-normal modes" (QNMs) → mode frequencies f_{lmn} and damping times τ_{lmn},
- ⋆ measuring two modes: calculation of M_f and χ_f, more than two modes: tests of GR.

$$h_+ = \frac{A(f_{lmn}, Q_{lmn})}{r}\left(1+\cos^2\iota\right)\exp\left(\frac{-\pi f_{lmn} t}{Q_{lmn}}\right)\cos\left(2\pi f_{lmn} t + \phi_{lmn}\right),$$

$$h_\times = \frac{A(f_{lmn}, Q_{lmn})}{r} 2\cos\iota\exp\left(\frac{-\pi f_{lmn} t}{Q_{lmn}}\right)\sin\left(2\pi f_{lmn} t + \phi_{lmn}\right),$$

$$\text{with } Q_{lmn} = \omega_{lmn}\tau_{lmn}/2.$$

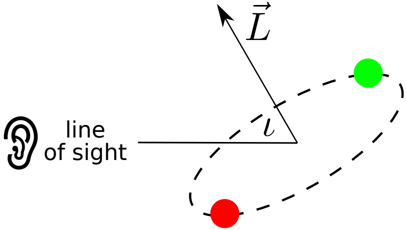
e.g. $f_{200} = 1.2 \times 10^3 \, \frac{10 M_\odot}{M}$ Hz, $\tau_{200} = 5.5 \times 10^{-4} \, \frac{M}{10 M_\odot}$ s.

Binary system: distance-inclination degeneracy

Luminosity distance $\sim 1/h$, and

$$h = h_+ F_+ + h_\times F_\times$$

depends on the inclination of the binary with repect to the "line of sight".



Two independent polarizations h_+ and $h_\times$:

$$\begin{aligned} h_+ &= \frac{2\mu}{r} \left(\pi M f_{GW}\right)^{2/3} \left(1 + \cos^2 \iota\right) \cos\left(2\phi(t)\right), \\ h_\times &= \frac{4\mu}{r} \left(\pi M f_{GW}\right)^{2/3} \cos\iota \sin\left(2\phi(t)\right). \end{aligned}$$

Efects of various parameters on inspiral waveform

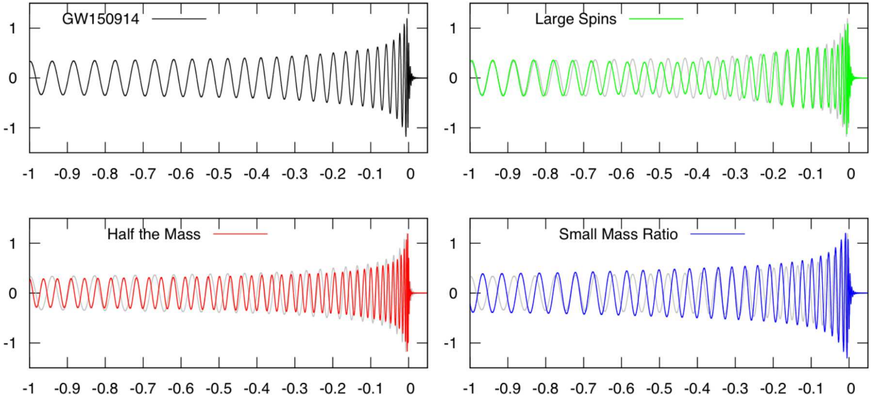


Illustration by N. Cornish and T. Littenberg

Binaries are standard sirens

Binaries are *clean* systems: we have accurate models even in full general relativity.

Loss of energy to GWs causes orbit to decay, orbital frequency to go up. So the GWs will chirp up in frequency. Chirp time $t_{\text{chirp}} \sim f / [\mathrm{d}f / \mathrm{d}t]$.

Signal contains both apparent brightness (from h and f) and intrinsic luminosity (from t_{chirp}), from which we can compute the distance to the source:

$$\text{Distance} \propto c \frac{}{\text{frequency}^2 \times t_{\text{chirp}}}$$

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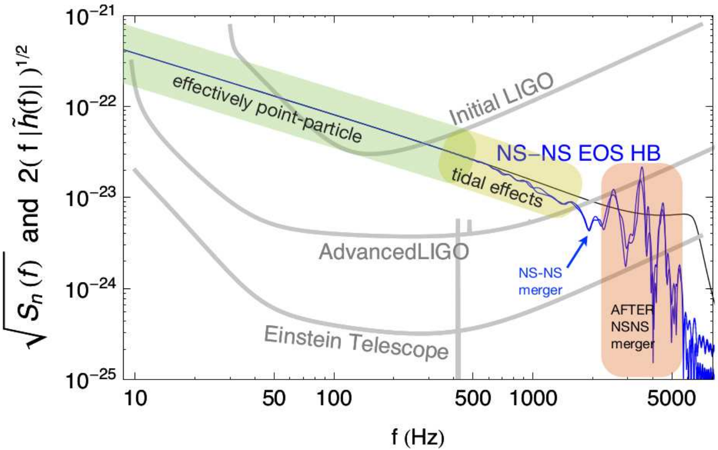
GW ASTRONOMY AND COSMOLOGY

27

LSC

Binary inspiral vs the sensitivity curve

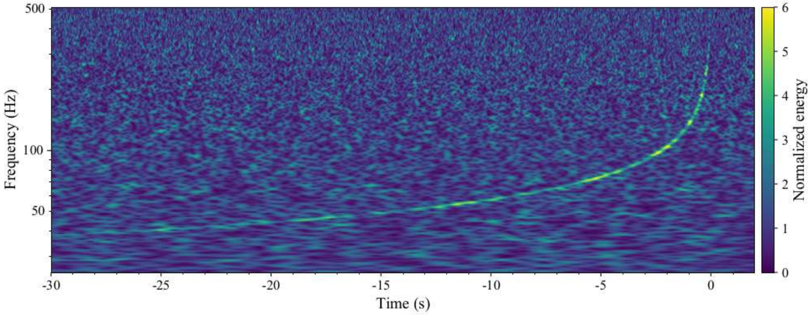


For extended-body interactions, phase evolution differs from point-particle description,

$$\Psi(f) = \Psi_{PP}(f) + \Psi_{tidal}(f)$$

Ψ_{tidal} breaks the v expansion degeneracy.

GW170817: 17 August 2017, 14:41:04 CEST



- ⋆ Combined LIGO-Virgo signal-to-noise ratio: SNR=32.4 (strongest signal so far!),
- ⋆ False alarm rate: less than one in 80000 years,
- ⋆ Chirp mass $\mathcal{M} = 1.188^{+0.004}_{-0.002}\ M_\odot \to$ a very light system!
- ⋆ Distance $d = 40^{+8}_{-14}$ Mpc (90% credible intervals)

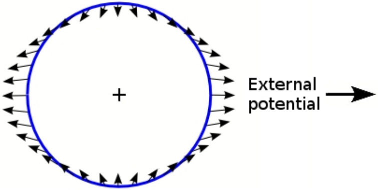
Signature of matter in binary NS waveforms

Tidal tensor $\mathcal{E}_{ij}$ of one of the components induces quadrupole moment Q_{ij} in the other:

$$Q_{ij} = -\lambda \mathcal{E}_{ij} \quad \rightarrow \quad \lambda = \frac{\text{size of quadrupole deformation}}{\text{strength of external tidal field}}$$

In lowest-order approximation: $\boxed{\lambda = \frac{2}{3} k_2 R^5}$

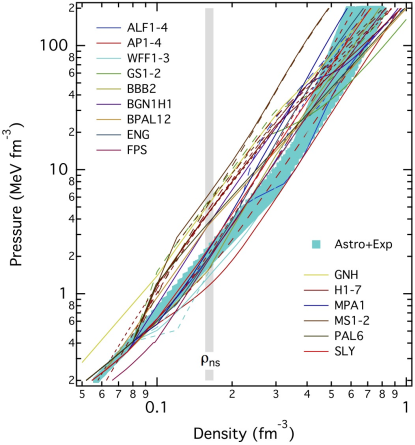
λ - tidal deformability,
$k_2 \in (0.05, 0.15)$ - Love number
(dependent on M and EOS).

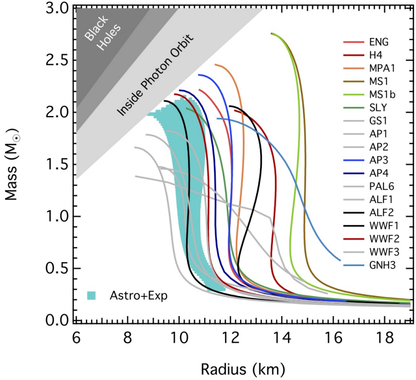


⋆ From the scaling this is a 5PN effect $(v/c)^{10}$

⋆ Convenient redefinition: $\boxed{\Lambda = G\lambda \left(\frac{GM}{c^2}\right)^{-5} \in (500, 3000)}$

Neutron stars: dense matter, $M(R)$





http://xtreme.as.arizona.edu/~fozel

Tolman-Oppenheimer-Volkoff equations (1939)

In spherically-symmetric spacetime,

$$ds^2 = e^{\nu(r)} c^2 dt^2 - \frac{dr^2}{1 - 2GM(r)/rc^2} - r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

equations of hydrostatic equilibrium are

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}\left(1 + \frac{P(r)}{\rho(r)c^2}\right)\left(1 + \frac{4\pi r^3 P(r)}{M(r)c^2}\right)\left(1 - \frac{2GM(r)}{c^2 r}\right)^{-1}$$

$$\frac{dM(r)}{dr} = 4\pi\rho(r)r^2,$$

and

$$\frac{d\nu(r)}{dr} = -\left(\frac{2}{P(r) + \rho(r)c^2}\right)\frac{dP(r)}{dr}.$$

$\star$ Every $P(\rho)$ relation (Equation Of State, EOS) results in the mass limit M_{max}.

Tidal deformability Λ

$$\lambda_{\rm td} = \frac{2}{3} R^5 k_2. \quad (7)$$

It represents the reaction of the star on the external tidal field (such as that in a tight binary system; e.g. Abbott et al. 2017). Influence of the tidal field is obtained in the lowest order approximation, by calculating the second (quadrupole) tidal Love number k_2 (Love 1911)

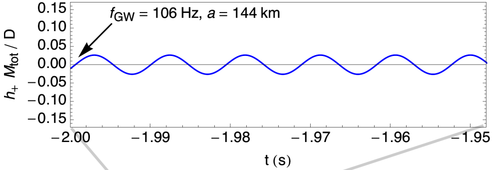
$$k_2 = \frac{8}{5} x^5 (1-2x)^2 (2 - y + 2x(y-1)) \Big(2x(6 - 3y + 3x(5y-8)) + 4x^3 \left(13 - 11y + x(3y-2) + 2x^2(1+y)\right) + 3(1-2x)^2(2 - y + 2x(y-1)) \ln(1-2x) \Big)^{-1}, \quad (8)$$

with the star's compactness $x = GM/Rc^2$, and y the solution of

$$\frac{{\rm d}y}{{\rm d}r} = -\frac{y^2}{r} - \frac{1 + 4\pi G r^2/c^2 (P/c^2 - \rho)}{(r - 2GM(r)/c^2)} y + \left(\frac{2G/c^2 (M(r) + 4\pi r^3 P/c^2)}{\sqrt{r}(r - 2GM(r)/c^2)} \right)^2 + \frac{6}{r - 2GM(r)/c^2} - \frac{4\pi G r^2/c^2}{r - 2GM(r)/c^2} \left(5\rho + 9P/c^2 + \frac{\left(\rho + P/c^2\right)^2 c^2}{\rho {\rm d}P/{\rm d}\rho} \right), \quad (9)$$

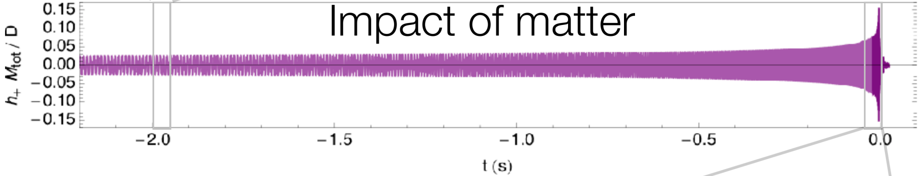
evaluated at the stellar surface (Flanagan & Hinderer 2008; Van Oeveren & Friedman 2017). In the following we use the normalised value of the $\lambda_{\rm td}$ parameter,

$$\Lambda = \lambda_{\rm td} \left(GM/c^2 \right)^{-5}. \quad (10)$$

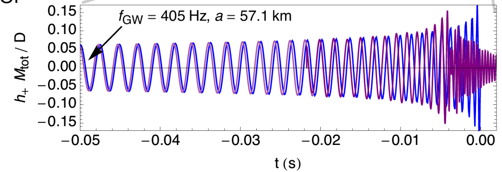


Hard to modify inspiral: transfer of ~10^{46} erg at ~100 Hz modifies phase by 10^{-3} radians (Crust shattering, Tsang et al 1110.0467)

Impact of matter



Tidal interactions lead to accumulated phase shift at higher frequencies.

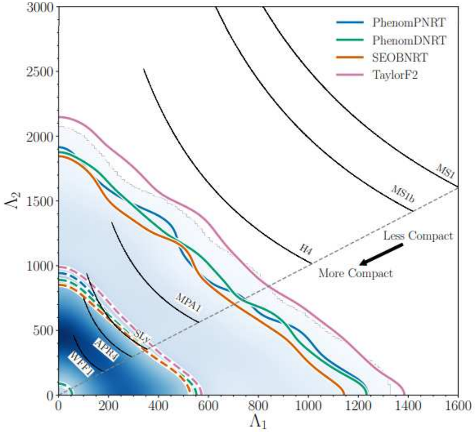


GW170817: constraints on dense matter

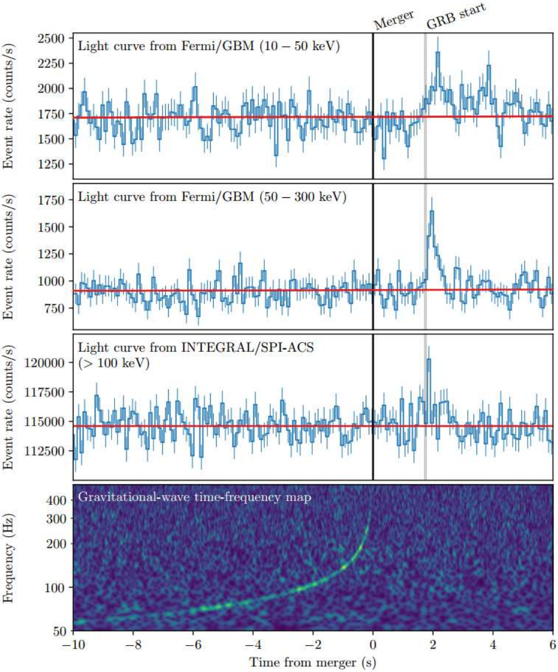
With minimal assumptions on the nature of compact objects:

- ⋆ Low-spin "realistic" priors (90% highest posterior density interval):

$$\tilde{\Lambda} = 300^{+420}_{-230}$$

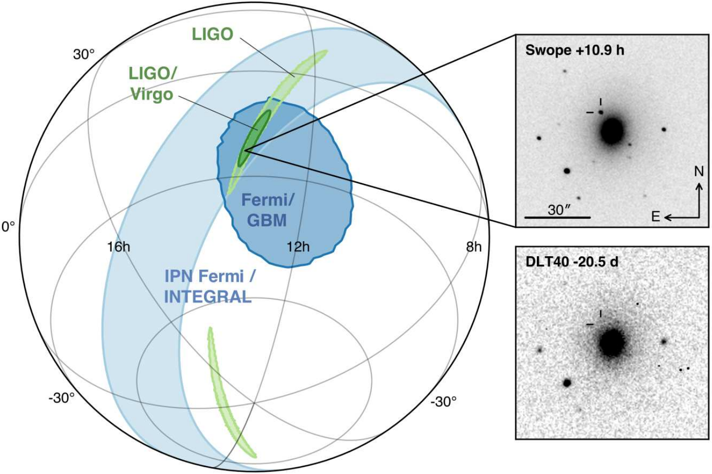


LVC: Phys. Rev. X 9, 011001 (2019)



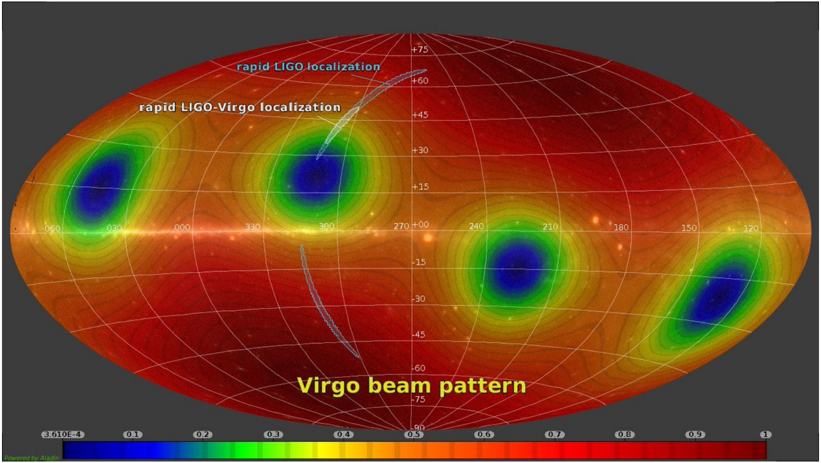
- ⋆ GW170817: **very long inspiral "chirp" (>100 s!)** firmly detected by the LIGO-Virgo network,
- ⋆ GRB 170817A: 1.74 ± 0.05 s later, weak **short gamma-ray burst** observed by Fermi (also detected by INTEGRAL).

GW170817: LIGO-Virgo triangulation



- ⋆ Fortunate orientation of Virgo w.r.t. signal → small sky patch: $28^{\circ 2}$,
- ⋆ New EM source in NGC 4993, consistent with GW distance 40^{+8}_{-14} Mpc,
- ⋆ Chance of temporal-spatial coincidence $< 5 \times 10^{-8}$.

GW170817: Virgo beam patterns



GW170817: speed of gravitation

Relative speed difference between GWs and photons:

$$\frac{v_{GW} - c}{c} = \frac{\Delta v}{c} \approx \frac{c\Delta t}{d}.$$

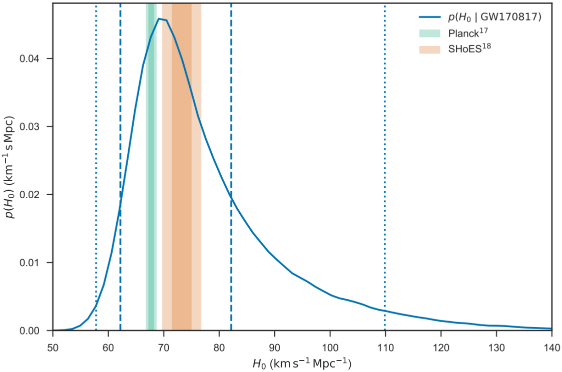
Assuming very conservative values:

- ⋆ Distance $d = 26$ Mpc (lower bound from 90% credible interval on luminosity distance derived from the GW signal),
- ⋆ Time delay $\Delta t = 10$ s

$$-3 \times 10^{-15} \leq \frac{\Delta v}{c} \leq 7 \times 10^{-16}$$

$$\boxed{v_{GW} = 299792458^{+0.000001}_{-0.000006} \text{ m/s} = c^{+0.000001}_{-0.000006} \text{ m/s}}$$

GW170817: Hubble parameter with a ”standard siren”



- ⋆ Hubble parameter defined as $v_H = H_0 d$,
- ⋆ $H_{0,GW} = 70.0^{+12.0}_{-8.0}$ km s^{-1} Mpc^{-1} (maximum a posteriori and 68% credible interval) = $\sim$14% at 1σ:
  - ⋆ $\sim$11% because of GW luminosity distance,
  - ⋆ The rest from the peculiar velocity of the galaxy.
- ⋆ Planck: 67.74 ± 0.46, SHoES: 73.24 ± 1.74 km s^{-1} Mpc^{-1}

Astrophysically-interesting parameters

- ⋆ Chirp mass $\mathcal{M} = \left(\mu^3 M^2\right)^{1/5} = (m_1 m_2)^{3/5}/(m_1 + m_2)^{1/5}$,

- ⋆ Mass ratio $q = m_2/m_1$ (at 1PN), alternatively $\nu = m_1 m_2/(m_1 + m_2)^2$,

- ⋆ Spin-orbit and spin-spin coupling (at 2PN and 3PN, resp.) $\rightarrow$

$$\chi_{eff} = (m_1\chi_{1z} + m_2\chi_{2z})/(m_1 + m_2)$$

  where χ_{iz} are spin components along system's total angular momentum,

- ⋆ Tidal deformability Λ (at 5PN) $\rightarrow$

$$\tilde{\Lambda} = \frac{16}{13}\frac{(m_1 + 12m_2)m_1^4\Lambda_1}{(m_1 + m_2)^5} + (1 \leftrightarrow 2)$$

- ⋆ Direct ”luminosity” (”loudness”) distance: binary systems are ”standard sirens”.

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- ★ "Ph 236: General Relativity", C. Hirata, http://www.tapir.caltech.edu/~chirata/ph236/2011-12 (energy of GWs, lect. 12-14)