Detection principles

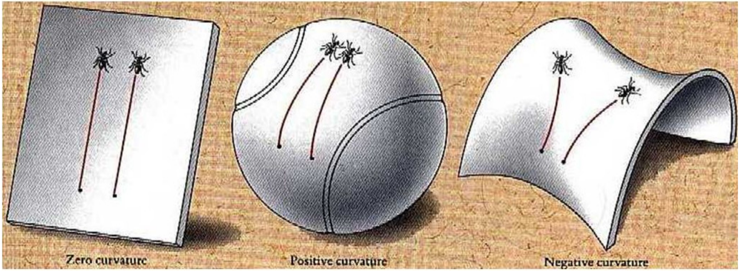
10.11.20

General schedule

- ⋆ History
- ⋆ Introduction to general relativity
- ⋆ Detection principles
  - ⋆ Deviation of geodesics,
  - ⋆ Wave equation from Einstein's equation,
  - ⋆ What is actually measured?
- ⋆ Detectors
- ⋆ Binary black-hole system
- ⋆ Bursts and continuous waves
- ⋆ Rates and populations, stochastic GW background, cosmology
- ⋆ Testing general relativity
- ⋆ Data analysis: waveforms and detection
- ⋆ Data analysis: parameter estimation

Geodesic deviation in curved spacetime

- ⋆ In general relativity, trajectories of freely-falling particles are geodesics (the equivalent of straight lines in curved spacetime)
  - → Newton's 1st law: Unless acted upon by a non-gravitational force, a test mass will follow a geodesic.
- ⋆ The curvature of spacetime is revealed by the behavior of neighbouring geodesics
- ⋆ Non-zero curvature ↔ acceleration of geodesic deviation ↔ non-uniform gravitational field



Geodesic deviation: Newtonian viewpoint

Two test masses falling towards the Earth from points P_1 and P_2, initially separated by ξ_0 (distance to the center r_0).

Finding similar triangles:

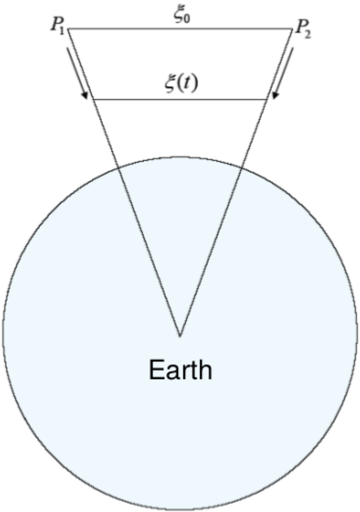
$$\frac{\xi_0}{r_0} = \frac{\xi(t)}{r(t)} = k,$$

so if

$$\ddot{r} = -\frac{GM}{r^2}, \text{ then } \ddot{\xi} = k\ddot{r} = -k\frac{GM}{r^2}.$$

or

$$\ddot{\xi} = -\frac{\xi}{r}\frac{GM}{r^2} = -\frac{GM\xi}{r^3}.$$



Geodesic deviation: Newtonian viewpoint

$$\frac{d^2\xi}{d(ct)^2} = -\frac{GM\xi}{c^2 r^3} = -\frac{\xi}{a(r)^2}.$$

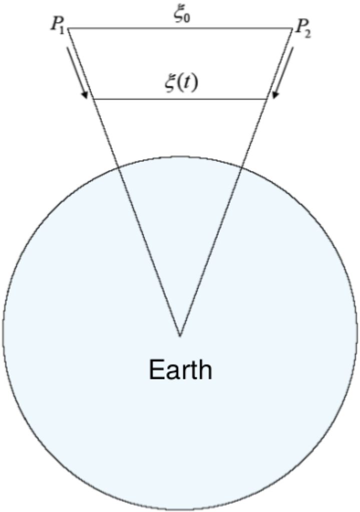
At the surface of the Earth ($r = R = 6370$ km):

$$\frac{1}{a(R)} = \frac{1}{\mathcal{R}^2} = \frac{GM}{c^2 R^3} = 2 \times 10^{-23}\ \mathrm{m}^{-2}.$$

$\mathcal{R}$ represents the radius of curvature of spacetime at the Earth's surface:

$$\mathcal{R} \sim 2 \times 10^{11}\ \mathrm{m} \gg R.$$

$\rightarrow$ spacetime near Earth is nearly flat.



Geodesic deviation: Newtonian viewpoint

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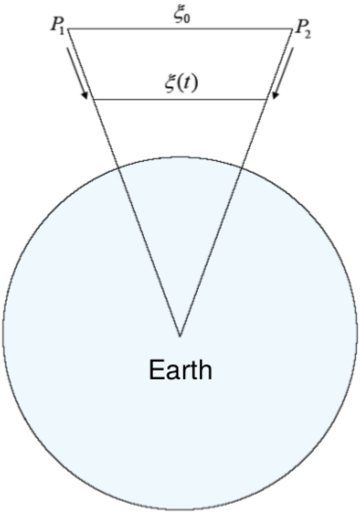
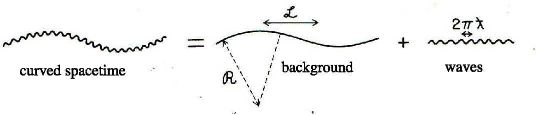
- ⋆ $\frac{GM\xi}{c^2 r^3}$ has a form of a tidal force,
- ⋆ Relation to the Riemann tensor: $\mathcal{R}^{-2} = |R_{\alpha\beta\gamma\delta}|$.



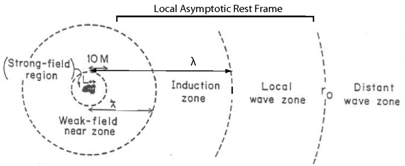
What is a wave?

General relativity is nonlinear & fully dynamical $\rightarrow$ not so clear a distinction between waves and the rest of the metric. Speaking about waves is ”safe” in certain limits:

- $\star$ linearized theory,
- $\star$ as small perturbations of a smooth background metric (gravitational lensing of waves, cosmological perturbations),



- $\star$ in weak field (far-zone, i.e., much more than one wavelength distant from the source):



Gravitational waves in linearized general relativity

Einstein's equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu},$$

where

- ⋆ $g_{\mu\nu}$ is the spacetime metric; $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$,
- ⋆ $R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}$ is the Ricci tensor,
- ⋆ $R_{\rho\mu\sigma\nu}$ is the Riemann tensor ($R_{\rho\mu\sigma\nu} \equiv 0$ means flat spacetime),
- ⋆ $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar,
- ⋆ $T_{\mu\nu}$ is the energy-momentum tensor (describing matter content).

Let's assume that gravitational waves ($h_{\mu\nu}$) are a small addition to otherwise stationary metric $\eta_{\mu\nu}$ (for example, Minkowski):

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \text{and} \quad |h_{\mu\nu}| \ll 1$$

Riemann tensor in linearized general relativity

$$R^{\nu}_{\mu\rho\sigma} = \partial_\rho \Gamma^{\nu}_{\mu\sigma} - \partial_\sigma \Gamma^{\nu}_{\mu\rho} + \Gamma^{\nu}_{\lambda\rho}\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\nu}_{\lambda\sigma}\Gamma^{\lambda}_{\mu\rho}, \quad \text{and} \quad R_{\nu\mu\rho\sigma} = g_{\nu\rho} R^{\rho}_{\mu\rho\sigma},$$

where the connection coefficients are expressed using the metric as *the Christoffel symbols*:

$$\Gamma^{\nu}_{\mu\rho} = \frac{1}{2} g^{\nu\lambda}(g_{\lambda\mu,\rho} + g_{\lambda\rho,\mu} - g_{\mu\rho,\lambda}),$$

We are interested in expressions linear in $h_{\mu\nu}$ (h^2 too small to be important). Therefore:

$$\Gamma^{\nu}_{\mu\rho} = \frac{1}{2} \eta^{\nu\lambda}(h_{\lambda\mu,\rho} + h_{\lambda\rho,\mu} - h_{\mu\rho,\lambda})$$

and

$$R^{\nu}_{\mu\rho\sigma} = \partial_\rho \Gamma^{\nu}_{\mu\sigma} - \partial_\sigma \Gamma^{\nu}_{\mu\rho} + \mathcal{O}(h^2) \rightarrow$$

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_{\rho\nu} h_{\mu\sigma} + \partial_{\sigma\mu} h_{\nu\rho} - \partial_{\rho\mu} h_{\nu\sigma} - \partial_{\sigma\nu} h_{\mu\rho}\right).$$

Convenient choice of variables and gauge

To simplify, the *trace-reversed* tensor $\overline{h}^{\mu\nu}$ is introduced:

$$\overline{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h, \quad \text{where} \quad h = \eta_{\alpha\beta}\, h^{\alpha\beta} \quad \text{and} \quad \overline{h} = -h.$$

With this change & after some algebra, the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = \frac{8\pi\, G}{c^4}\, T_{\mu\nu}$$

are

$$\Box \overline{h}_{\nu\sigma} + \eta_{\nu\sigma}\, \partial^\rho\, \partial^\lambda \overline{h}_{\rho\lambda} - \partial^\rho\, \partial_\nu \overline{h}_{\rho\sigma} - \partial^\rho\, \partial_\sigma \overline{h}_{\nu\rho} + \mathcal{O}(h^2) = -\frac{16\pi G}{c^4}\, T_{\nu\sigma},$$

where $\Box = \eta_{\rho\sigma}\, \partial^\rho\, \partial^\sigma$ is the d'Alambert operator (wave operator). In Cartesian terms

$$\Box = \eta_{\rho\sigma}\, \partial^\rho\, \partial^\sigma = -\frac{1}{c^2}\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2.$$

Gauge freedom

Further simplification is the use of *gauge freedom*; by imposing the Lorenz (de Donder, harmonic) gauge condition,

$$\partial_\nu \bar{h}^{\mu\nu} = 0,$$

we get

$$\boxed{\Box \bar{h}_{\nu\sigma} = -\frac{16\pi G}{c^4} T_{\nu\sigma}}.$$

Gauge fixing

Chosing a gauge (gauge fixing) is a way to (sometimes partially) deal with redundant degrees of freedom in field variables. For example in EM, for a vector potential A^μ, the *Lorenz gauge* condition is

$$\partial_\mu A^\mu = 0.$$

Note, transformations of the type $A^\mu \to A^\mu + \partial^\mu f$ are still possible, with f a scalar function (harmonic, $\partial_\mu \partial^\mu f \equiv 0$).

Coordinate freedom

We are also allowed to make infinitesimal coordinate transformations,

$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x^{\beta})$, with ξ^{α} small in the sense that $\xi^{\alpha} \ll 1$, $|\partial_{\beta}\xi^{\alpha}| \ll 1$.

This imply

$$\frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta} + \partial_{\beta}\xi^{\alpha}, \quad \text{and} \quad \frac{\partial x^{\alpha}}{\partial x'^{\beta}} = \delta^{\alpha}_{\beta} - \partial_{\beta}\xi^{\alpha} + \mathcal{O}((\partial\xi)^2).$$

Recalling that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $g'_{\mu\nu}(x') = \dfrac{\partial x^{\alpha}}{\partial x'^{\mu}} \dfrac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}(x)$,

$$g'_{\alpha\beta} = \eta_{\alpha\beta} + \underbrace{h_{\alpha\beta} - \partial_{\alpha}\xi_{\beta} - \partial_{\beta}\xi_{\alpha}}_{h'_{\alpha\beta}} + \mathcal{O}(h\partial\xi, (\partial\xi)^2) \qquad (\xi_{\alpha} = \eta_{\alpha\beta}\xi^{\beta}).$$

Because $|\partial_{\beta}\xi^{\alpha}| \ll 1$ the metric perturbation $h'_{\alpha\beta}$ is small, the approximation is still valid. Applied to metric perturbation $\bar{h}'_{\alpha\beta}$:

$$\bar{h}'_{\alpha\beta} = \bar{h}_{\alpha\beta} - \partial_{\alpha}\xi_{\beta} - \partial_{\beta}\xi_{\alpha} + \eta_{\alpha\beta}\partial_{\mu}\xi^{\mu}.$$

Dealing with deegrees of freedom

- $\star$ A symmetric 4×4 tensor $h_{\mu\nu}$ has 10 degrees of freedom,
- $\star$ 4 d.o.f. used by imposing the Lorenz gauge ($\partial_\nu \overline{h}^{\mu\nu} = 0$).

$$\text{In vacuum } T_{\mu\nu} \equiv 0, \text{ so } \quad \Box \overline{h}_{\nu\sigma} = 0.$$

$\rightarrow$ speed of the wave equals speed of light c. Remaining 6 d.o.f.: in Lorenz gauge one can always consider coordinate transformations

$$\overline{h'}_{\nu\sigma} = \overline{h}_{\nu\sigma} + \xi_{\mu\nu}, \quad \text{where} \quad \xi_{\mu\nu} = \eta_{\mu\nu}\,\partial_\rho \xi^\rho - \xi_{\mu,\nu} - \xi_{\nu,\mu} \;\rightarrow\; \Box \xi_{\mu\nu} = 0$$

$\Box \xi_{\mu\nu} = 0$ means fixing 4 of 6 remaining d.o.f.

Transverse-traceless gauge

A good choice is the *transverse-traceless gauge*, with ξ^t such that $\bar{h} = 0$, and ξ^i such that $\bar{h}^{it} = 0$, $\partial_t \bar{h}^{tt} = 0$:

$$\underbrace{\bar{h}^{tt} = 0, \quad \bar{h}^{it} = 0}_{\text{purely spatial}}, \quad \underbrace{\partial_i \bar{h}^{ij} = 0}_{\text{Lorenz gauge}}, \quad \underbrace{\bar{h}^i_i = 0}_{\text{trace = 0}}.$$

This is the definition of the *transverse-traceless* tensor $\bar{h}^{TT}_{ij}$ (and since it's traceless, $\bar{h}^{(TT)}_{\mu\nu} = h^{(TT)}_{\mu\nu}$).

TT is not a necessary, but a convenient choice: $\bar{h}^{TT}_{\mu\nu}$ contains only physical (non-gauge) information about the radiation.

Plane gravitational waves in the TT gauge

Let's consider a plane wave: $\bar{h}_{\mu\nu} = \Re\left(A_{\mu\nu}\exp\left(ik^\alpha x_\alpha\right)\right)$

- ⋆ Because it is a solution to Einstein (wave) equation: $k_\alpha k^\alpha = 0 \quad \rightarrow \quad \omega = k^t = \sqrt{k_x^2 + k_y^2 + k_z^2}$.
- ⋆ Choice of Lorenz gauge: $\partial_\alpha \bar{h}^{\mu\alpha} = A_{\mu\alpha}k^\alpha = 0$.

Using remaining freedom, applying the transverse-traceless (TT) gauge for a wave traveling in the z direction we get:

- ⋆ $k^t = k^z = \omega, \quad k^x = k^y = 0, \quad A_{\alpha z} = 0$,
- ⋆ $A^\mu_\mu = \eta^{\mu\nu}A_{\mu\nu} = 0, \quad A_{\alpha t} = 0$.

which means $\bar{h}^{(TT)}_{\mu\nu} = A^{(TT)}_{\mu\nu}\cos\left(\omega\left(t - z\right)\right)$, with

$$A^{(TT)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{(TT)}_{xx} & A^{(TT)}_{xy} & 0 \\ 0 & A^{(TT)}_{xy} & -A^{(TT)}_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(2 remaining d.o.f.)

If a direction of propagation is n^i, then $n^i\, h_{ij}^{\mathrm{TT}} = 0$ in the TT gauge (the gravitational wave is described by 2×2 matrix in the plane orthogonal to the direction of propagation **n**). For propagation along the z-axis

$$h_{\mu\nu}^{(TT)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx}^{(TT)} & A_{xy}^{(TT)} & 0 \\ 0 & A_{xy}^{(TT)} & -A_{xx}^{(TT)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos(\omega(t - z)),$$

where $A_{xx}^{(TT)}$ and $A_{xy}^{(TT)}$ are two independent polarization states, usually called A_+ and $A_\times$; sometimes called the *helicity states* - change under rotation of ϕ around **n** as

$$h \to e^{i\mathbf{S}\cdot\mathbf{n}\phi} A \quad \text{where} \quad \mathbf{S} = \text{particle spin}$$

$$h_\times \pm i\, h_+ \to e^{\mp 2i\,\phi}\,(A_\times \pm i\, A_+).$$

Gravitational waves in the TT gauge

For a free test particle initially at rest, in the coordinate system corresponding to the TT gauge, it stays at rest: coordinates do not change, particles remain attached to initial positions.

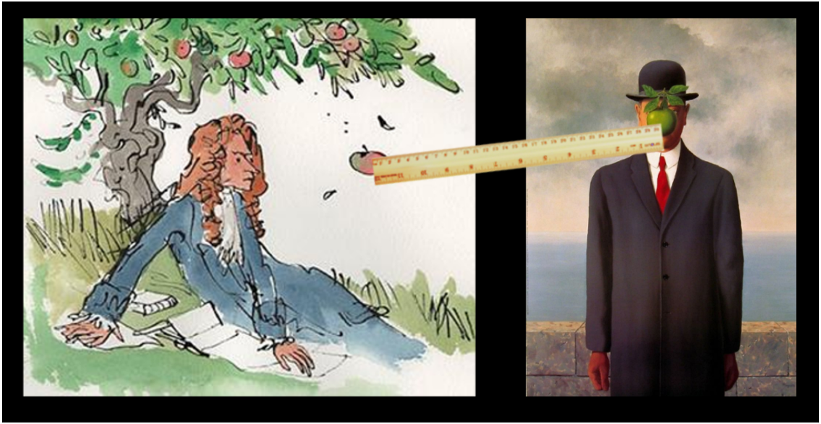
TT gauge represents a coordinate system comoving with freely-falling particles.

How to know that something is changing?

What about the **proper (spacetime) distance** between neighbouring particles?

Detection principle: *proper* distance measurement

"How to measure distance if the ruler also changes length?"



(Quentin Blake "Izaak Newton")

(Rene Magritte "The Son of Man")

Proper distance between test particles

Two test particles, initially at rest, one at $x = 0$ and the other at $x = \epsilon$, in spacetime described by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

The proper distance between them is

$$\Delta s = \int |g_{\mu\nu} dx^\mu dx^\nu|^{1/2} = \int_0^\epsilon |g_{xx}|^{1/2} \approx \epsilon \sqrt{g_{xx}(x = 0)}.$$

If $g_{xx}\,(x = 0) = \eta_{xx} + h_{xx}^{(TT)}\,(x = 0)$,
then

$$\Delta s \approx \epsilon \left(1 + \frac{1}{2} h_{xx}^{(TT)}(x = 0)\right),$$

which, in general, is varying in time (as h_{xx}^{TT} is).

Geodesic deviation in general relativity

The *geodesic deviation* equation is

$$\frac{D^2\xi^\alpha}{D\tau^2} = R^\alpha_{\beta\gamma\delta}u^\beta u^\gamma \xi^\delta,$$

with $D/D\tau$ a covariant derivative, and $u^\mu = \partial\xi^\mu/\partial\tau$, a 4-velocity along the geodesic.
An analogue of an evolution of distance ξ between two distant test particles due to the tidal force.

A simplified case:

- ⋆ two test particles, both initially at rest ($u^\alpha = (1, 0, 0, 0)$),
- ⋆ one located at $x = 0$ and the other at $x = \epsilon$ (distance between particles $\xi^\alpha = (0, \epsilon, 0, 0)$),
- ⋆ Weak-field limit (proper time $\tau \approx$ coordinate time t).

Geodesic deviation in general relativity (weak field)

The general equation

$$\frac{D^2\xi^\alpha}{D\tau^2} = \frac{\partial^2\xi^\alpha}{\partial t^2} = R^\alpha_{\beta\gamma\delta} u^\beta u^\gamma \xi^\delta$$

is, due to our simplifications,

$$\frac{\partial^2\xi^\alpha}{\partial t^2} = \epsilon R^\alpha_{ttx} = -\epsilon R^\alpha_{txt},$$

with two interesting directions ($\alpha = x$ or $\alpha = y$):

$$R^x_{txt} = \eta^{xx} R_{xtxt} = -\frac{1}{2} h^{(TT)}_{xx,tt},$$
$$R^y_{txt} = \eta^{yy} R_{ytxt} = -\frac{1}{2} h^{(TT)}_{xy,tt},$$

that is,

$$\frac{\partial^2\xi^x}{\partial t^2} = \frac{1}{2}\epsilon\frac{\partial^2 h^{(TT)}_{xx}}{\partial t^2}, \quad \frac{\partial^2\xi^y}{\partial t^2} = \frac{1}{2}\epsilon\frac{\partial^2 h^{(TT)}_{xy}}{\partial t^2}.$$

Geodesic deviation in general relativity (weak field)

A more general case:
with $x = \epsilon \cos\theta,\ y = \epsilon \sin\theta,\ z = 0,$

$$
\begin{aligned}
\frac{\partial^2 \xi^x}{\partial t^2} &= \frac{1}{2}\epsilon \cos\theta \frac{\partial^2 h_{xx}^{(TT)}}{\partial t^2} + \frac{1}{2}\epsilon \sin\theta \frac{\partial^2 h_{xy}^{(TT)}}{\partial t^2}, \\
\frac{\partial^2 \xi^y}{\partial t^2} &= \frac{1}{2}\epsilon \cos\theta \frac{\partial^2 h_{xy}^{(TT)}}{\partial t^2} - \frac{1}{2}\epsilon \sin\theta \frac{\partial^2 h_{xx}^{(TT)}}{\partial t^2}.
\end{aligned}
$$

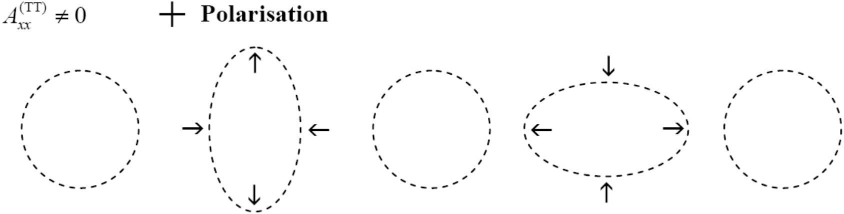
with solutions, for the plane wave in the z direction,

$$
\begin{aligned}
\xi^x &= \epsilon \cos\theta + \frac{1}{2}\epsilon A_{xx}^{(TT)} \cos\theta \cos\omega t + \frac{1}{2}\epsilon A_{xy}^{(TT)} \sin\theta \cos\omega t, \\
\xi^y &= \epsilon \sin\theta + \frac{1}{2}\epsilon A_{xy}^{(TT)} \cos\theta \cos\omega t - \frac{1}{2}\epsilon A_{xx}^{(TT)} \sin\theta \cos\omega t.
\end{aligned}
$$

The + polarisation

$A_{xx}^{(TT)} \neq 0$, $A_{xy}^{(TT)} = 0$

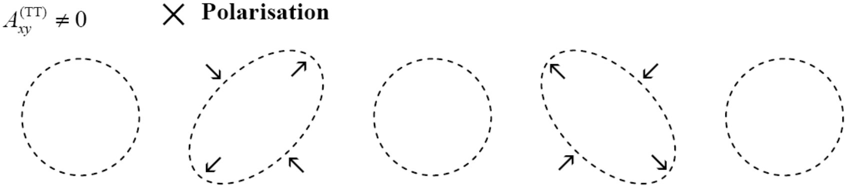
$$\begin{aligned} \xi^x &= \epsilon \cos\theta \left(1 + \frac{1}{2} A_{xx}^{(TT)} \cos\omega t\right), \\ \xi^y &= \epsilon \sin\theta \left(1 - \frac{1}{2} A_{xx}^{(TT)} \cos\omega t\right). \end{aligned}$$



The $\times$ polarisation

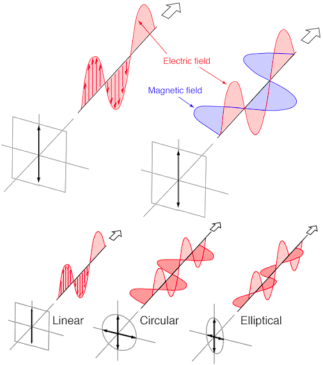
$A_{xy}^{(TT)} \neq 0$, $A_{xx}^{(TT)} = 0$

$$\begin{aligned} \xi^x &= \epsilon \cos\theta + \frac{1}{2}\epsilon \sin\theta A_{xy}^{(TT)} \cos\omega t, \\ \xi^y &= \epsilon \sin\theta - \frac{1}{2}\epsilon \cos\theta A_{xy}^{(TT)} \cos\omega t. \end{aligned}$$

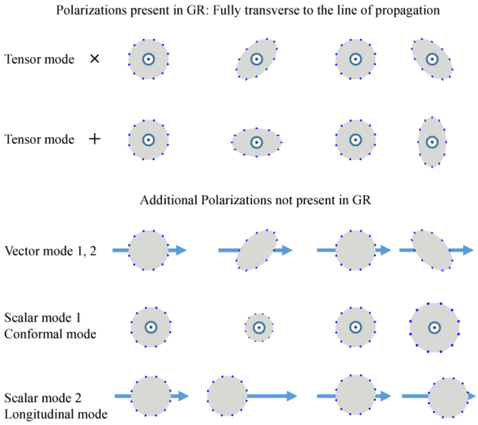


Polarizations: EM vs GW

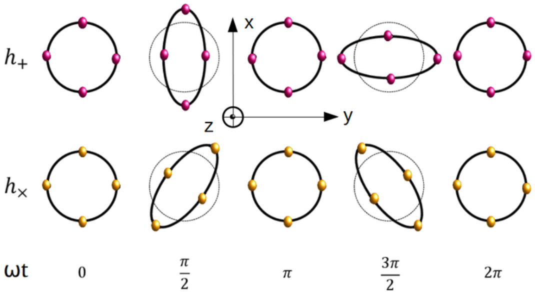
Electromagnetic waves:



Gravitational waves:



Polarizations: EM vs GW



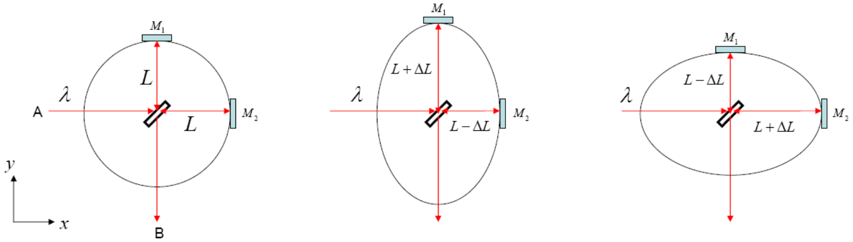
⋆ $+$ and $\times$ patterns are orthogonal polarization states (by analogy with EM waves, where the two linear polarizations added with phase difference $\pm\pi/2$ to obtain circularly polarized waves),

⋆ GW is invariant under rotations of π about direction of propagation (EM waves are invariant under rotations of 2π),

→ in analogy to quantum mechanics: helicity $2\pi/s$, where s is the particle spin ($s = 1$ for a photon, $s = 2$ for hypothetical graviton).

For pure $+$ mode, fractional change in proper distance is

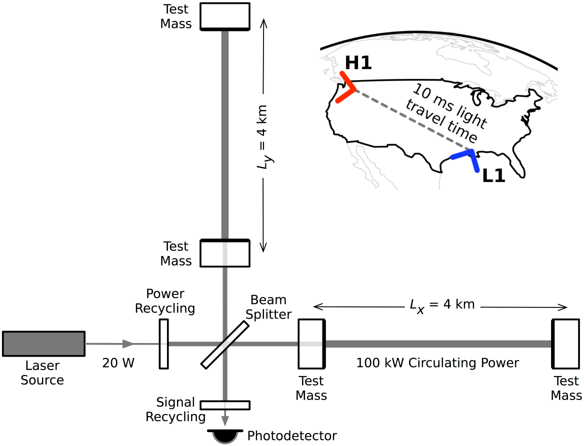
$$\frac{\Delta L}{L} = \frac{h}{2}$$



Gertsenshtein & Pustovit (1962) were first to suggest an interferometer to detect GWs. In the 70s Rainer Weiss had the same idea $\rightarrow$ LIGO

Detection principle: laser interferometry

"How to measure distance when the ruler also changes length?"



Changes in arms length are **very** small: $\delta L_x - \delta L_y = \Delta L < 10^{-18}$ m (smaller than the size of the proton). Wave amplitude $h = \Delta L / L \leq 10^{-21}$.

Change of arms' length $\leftrightarrow$ variation in light travel time

Change of the x-arm: $ds^2 = -c^2 dt^2 + (1 + h_{xx})\, dx^2 = 0.$

Assume $h(t)$ is constant during light's travel through interferometer, replace $\sqrt{1 + h_{xx}}$ with $1 + h_{xx}/2$, integrate from $x = 0$ to $x = L$:

$$\int dt = \frac{1}{c} \int \left(1 + \frac{1}{2} h_{xx}\right) dx \quad \rightarrow \quad t_x = h_{xx} L/2c.$$

Round-trip time in the x-arm: $t_x = h_{xx} L/c.$

Round-trip time in the y-arm: $t_y = -hL/c \qquad (h_{yy} = -h_{xx} = -h)$

Round-trip times difference: $\boxed{\Delta\tau = 2hL/c}$

Phase difference (dividing $\Delta\tau$ by the radian period of light $2\pi/\lambda$):

$$\boxed{\Delta\phi = \frac{4\pi}{\lambda} hL}.$$

- Do test masses move in response to a gravitational wave?
  - No, in the TT gauge (free-falling masses define the coordinates),
  - Yes, in the laboratory coordinates (masses move affected by tidal forces).
- Do light wavelength change in response to a gravitational wave?
  - No (see above),
  - Yes, stretch by h as the masses move (as in the cosmological redshift).
- If light waves are stretched by gravitational waves, how can light be used as a ruler?
  - Indeed, the instantaneous response of an interferometer to a gravitational wave is *null*.
  - But the light travels through the arms for some finite time allowing for the phase shift to build up.

(P. R. Saulson, 1997, *Am. J. Phys*. 65, 501)

Literature

- ⋆ Lecture notes of Sean Carroll (http://preposterousuniverse.com/grnotes)
- ⋆ ”General Relativity and Gravitational Waves”, Thomas Moore, http://pages.pomona.edu/~tmoore/LesHouches, lecture 5
- ⋆ ”The basics of gravitational wave theory”, É. Flanagan, S. Hughes, arXiv:gr-qc/0501041