General relativity

3.11.20

General schedule

- History
- Introduction to general relativity
  - What gravity really is (according to Einstein),
  - Connection and curvature,
  - Einstein equations.
- Detection principles
- Detectors
- Binary black-hole system
- Bursts and continuous waves
- Rates and populations, stochastic GW background, cosmology
- Testing general relativity
- Data analysis: waveforms and detection
- Data analysis: parameter estimation

Why relativity? Maxwell and Newton incompatible

Maxwell's equations describe electromagnetism and optical phenomena within the theory of waves:

- ⋆ A special medium, *"luminiferous æther"*, needed to propagate the waves; Æther weakly interacts with matter, is carried along with astronomical objects,
- ⋆ Light propagates with a finite speed, but this speed is *not invariant* in all frames,
- ⋆ Especially, Maxwell's equations are *not invariant* under Galilean transformations between, say, inertial coordinate frames O and O':

$$x' = x - vt, \quad t' = t$$

- ⋆ To make electromagnetism compatible with classical Newton's mechanics, light has speed $c = 3 \times 10^8$ m/s only in frames where source is at rest.

Why relativity? Maxwell and Newton incompatible

Albert Einstein (1905): Maxwell's unification of electricity and magnetism is complete by showing that the two fields is really one.

Special relativity is based on two postulates:

- ⋆ the laws of physics are invariant (i.e., give the same results) in all inertial systems (non-accelerating frames of reference), $\rightarrow$ no experiment can measure absolute velocity,
- ⋆ the speed of light in vacuum is the same for all observers.

Lorentz transformation instead of Galilean:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$
$$x' = \gamma\,(x - vt)$$
$$\text{with } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

- ⋆ length contraction $\Delta l' = \Delta l/\gamma$,
- ⋆ time dilation $\Delta t' = \Delta t\gamma$,
- ⋆ "relativistic mass" $m\gamma$,
- ⋆ mass–energy equivalence $E = mc^2$,
- ⋆ universal speed limit,
- ⋆ relativity of simultaneity.

Gravity and acceleration

What is the difference between Newtonian and Einsteinian theory?

- ⋆ **Newton viewpoint**: mass tells gravity how to exert a force, force tells mass how to accelerate

$$F = -\frac{GM_g m_g}{r^2}, \quad F = m_i a$$

$$a = -\frac{GM_g}{r^2}\frac{m_g}{m_i}$$

- ⋆ is gravitational mass m_g equal to inertial mass m_i?

- ⋆ **Einstein viewpoint**: Mass (energy) tells spacetime how to curve, curved spacetime tells mass (energy) how to move (J. Wheeler) - geometry is related to mass distribution.

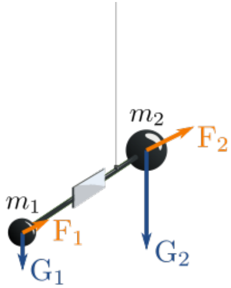
Equivalence principle

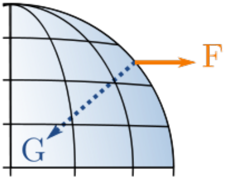
Weak equivalence principle: testing the equivalence of gravitational mass and inertial mass

Eötvös parameter η for two different test bodies A and B (aluminum and gold, for example):

$$\eta(A, B) = 2\frac{\left(\frac{m_g}{m_i}\right)_A - \left(\frac{m_g}{m_i}\right)_B}{\left(\frac{m_g}{m_i}\right)_A + \left(\frac{m_g}{m_i}\right)_B}$$

From the times of Galileo (no difference „by eye”) till present (Eöt-Wash group) $\eta < 10^{-13}$





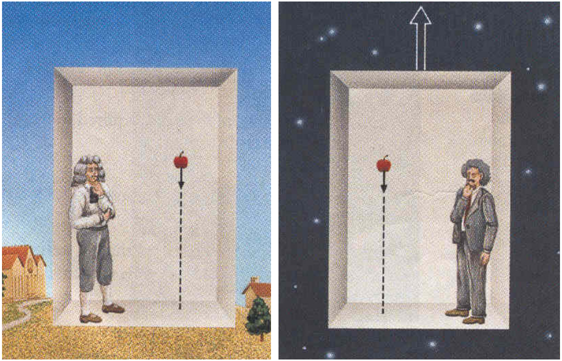
Equivalence principle

Strong equivalence principle:

- ⋆ The outcome of any local experiment (gravitational or not) in a free-falling laboratory is independent of the velocity of the laboratory and its location in spacetime,
- ⋆ the laws of gravitation are independent of velocity and location,
- ⋆ Locally, the effects of gravitation (motion in a curved space) are the same as that of an accelerated observer in flat space.

Einstein: equivalence principle

Einstein (1907), "the happiest thought of his life":

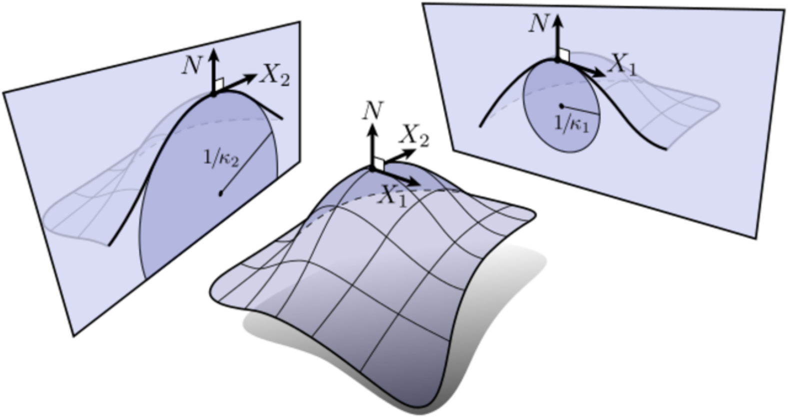


Gravitation is a form of acceleration; locally, the effects of gravitation (motion in a curved space) are the same as those of an accelerated observer (in flat space).

How it is to be free?

General relativity was not easy to acknowledge, because of various freedoms:

- ⋆ Choice of coordinate systems (rectilinear, curvilinear),
- ⋆ Choice of reference frames (inertial, non-inertial),
- ⋆ Fields (scalar, vector) change from point to point,
- ⋆ Curved spacetime itself changes from point to point.



http://brickisland.net/cs177

A curved 2D surface: at a given point, **principal curvatures** denoted κ_1 and κ_2, are the maximum and minimum values of the curvature $\rightarrow$ various notions describing the curvature: Gauss (intrinsic to the surface, $K = \kappa_1 \kappa_2$), extrinsic (requires an idea of embedding space exterior to the surface).

Laws in the inertial frame

2nd Newton's law ($F = ma$):

$$\underbrace{\ddot{x}^k = \frac{1}{m} F^k(\mathbf{x}, t)}_{\text{inertial frame}} + \underbrace{\text{Coriolis, centrifugal, } \ldots}_{\text{non-inertial frame add-ons}}$$

Let's assume:

- ⋆ the simplest case of an affine space (flat space of Euclides, Galileo and Newton),
- ⋆ a rectilinear (Cartesian) coordinate system in a 4-dimensional space, $(y^a) = (t, x, y, z)$.

In an inertial frame, trajectory of a body follows a straight line (1st Newton law):

$$\ddot{y}^a = 0 \quad \text{(free fall)}$$

How does it look like in a different (maybe curvilinear) coordinate system (x^k)?

- ⋆ $\ddot{x}$ is a second derivative with respect to an independent time variable, e.g. proper time τ,
- ⋆ Assuming Einstein's summation convention: $x_a x^a = x_0 x^0 + x_1 x^1 + x_2 x^2 + x_3 x^3$.

Expressing (y^a) in (x^k):

$$\dot{y}^a = \frac{\partial y^a}{\partial x^k} \dot{x}^k, \quad \text{and} \quad \ddot{y}^a = \frac{\partial y^a}{\partial x^k} \ddot{x}^k + \frac{\partial^2 y^a}{\partial x^k \partial x^l} \dot{x}^k \dot{x}^l \equiv 0.$$

We want $\ddot{x}^m$ in (x^k) coordinates, so using the following relation between (y^a) and (x^k) systems,

$$\frac{\partial x^m}{\partial y^a} \frac{\partial y^a}{\partial x^k} = \frac{\partial x^m}{\partial x^k} = \delta_k^m,$$

We get

$$\ddot{x}^m + \frac{\partial x^m}{\partial y^a} \frac{\partial^2 y^a}{\partial x^k \partial x^l} \dot{x}^k \dot{x}^l = 0.$$

Connection coefficients

Defining

$$\Gamma^m_{kl} := \frac{\partial x^m}{\partial y^a}\frac{\partial^2 y^a}{\partial x^k \partial x^l}, \qquad (\Gamma^m_{kl} \equiv \Gamma^m_{lk}, \text{ due to symmetry of derivatives})$$

we get the equation of motion (equation of geodesics):

$$\ddot{x}^m + \Gamma^m_{kl}\dot{x}^k\dot{x}^l = 0.$$

- ⋆ In general case, coefficients Γ^m_{kl} measure a departure of the (x^k) frame from linearity ("inertiality"),
- ⋆ Non-linear addition $\Gamma^m_{kl}\dot{x}^k\dot{x}^l$ contains all the *apparent* forces (Coriolis, etc.), related to non-inertial nature of the coordinate system (frame),
- ⋆ in 3+1 spacetime, Γ^m_{lk} has $4 \times 10 = 40$ independent components.

Gravitation: Newton vs Einstein

Newton:

- ⋆ Space is euclidean, time is absolute, there is no relation between them
- ⋆ Gravitation is a force acting between masses
- ⋆ Laws of motion expressed in *the* rectilinear inertial frame

Einstein:

- ⋆ Space and time are related
- ⋆ 4-dimensional space-time is curved by masses, and gravitation is an effect of this curvature
- ⋆ Spacetime is *curved*, so rectilinar coordinate systems are not even possible

Gravitation = a field of local inertial frames

In general relativity, there is no *global* inertial frame, but in every point in spacetime there is a *local* inertial frame.

An inertial frame is an *equivalence class* of inertial coordinate systems: coordinate systems (x^k) and (y^a) belong to the same equivalence class

$$(x^k) \sim_{\mathbf{x}} (y^a)$$

iff, in the neighborhood of point $\mathbf{x} \in M$,

$$\frac{\partial^2 y^a}{\partial x^k \partial x^l}(\mathbf{x}) = 0.$$

Gravity as apparent force

Rewriting the geodesic equation in a form of Newton's equation of motion,

$$\ddot{x}^m = -\Gamma^m_{kl}\dot{x}^k\dot{x}^l,$$

with the right side describing gravitational forces in (x^k) coordinate frame (depends on this choice).

⋆ Locally, gravitational forces can be eliminated ($\Gamma^m_{kl} = 0$) by choosing an inertial frame,

→ this is the core idea behind the "free-falling lift" *Gedankenexperiment* (nowadays, the space station at the orbit), and gravity as an *apparent* force,

⋆ Gravitation in a curved space can be eliminated *locally* (from point to point), but not *globally*: it is present in the *curvature* of spacetime, i.e., in the global structure of free-falling trajectories.

Is there a way to distinguish real acceleration from apparent one, caused by a choice of coordinates and frames?

How to quantify curvature?

Detecting the true (coordinate independent) departure from flatness:

- ⋆ If spacetime is flat in the neighborhood of $\mathbf{x} \in M$, then we could chose a coordinate system in which $\Gamma^m_{kl} = 0$ in that neighborhood.
- ⋆ Is it easy, hard or even possible to select such a coordinate system? So far we know how to chose the inertial frame in which $\Gamma^m_{kl}(\mathbf{x}) = 0$, i.e., only at $\mathbf{x}$...
- ⋆ Something less ambitious: is it possible to zero the derivatives of Γ^m_{kl},

$$\frac{\partial \Gamma^m_{kl}}{\partial x^n} = \partial_n \Gamma^m_{kl} := \Gamma^m_{kln} \text{ at } \mathbf{x}?$$

- ⋆ If for (x^k), $\Gamma^m_{kl}(\mathbf{x}) = 0$, but $\Gamma^m_{kln}(\mathbf{x}) \neq 0$, is there (y^a), for which both $\tilde{\Gamma}^a_{bc}(\mathbf{x}) = 0$ and $\tilde{\Gamma}^a_{bcd}(\mathbf{x}) = 0$?
- ⋆ We have selected (x^k) and (y^a) to be both inertial, because $\Gamma^m_{kl}(\mathbf{x}) = \tilde{\Gamma}^a_{bc}(\mathbf{x}) = 0$, so they belong to the same equivalence class, which means

$$\frac{\partial^2 x^m}{\partial y^b \partial y^c} = \frac{\partial^2 y^a}{\partial x^k \partial x^l} = 0.$$

How to quantify curvature?

Transformation law between connections is (given here without derivation):

$$\tilde{\Gamma}^a_{bc} = \frac{\partial y^a}{\partial x^m} \frac{\partial x^k}{\partial y^b} \frac{\partial x^l}{\partial y^c} \Gamma^m_{kl} + \frac{\partial y^a}{\partial x^m} \frac{\partial^2 x^m}{\partial y^b \partial y^c},$$

(btw. it is obvious that connections are not tensors from the existence of the second, non-tensor term).

The derivative $\partial_d \tilde{\Gamma}^a_{bc} := \tilde{\Gamma}^a_{bcd}$ is

$$\tilde{\Gamma}^a_{bcd} = \frac{\partial y^a}{\partial x^m} \left(\frac{\partial x^k}{\partial y^b} \frac{\partial x^l}{\partial y^c} \frac{\partial x^n}{\partial y^d} \Gamma^m_{kln} + \frac{\partial^3 x^m}{\partial y^b \partial y^c \partial y^d} \right).$$

Can we chose the (y^a) coordinates such that $\tilde{\Gamma}^a_{bcd} = 0$?

Since the third derivative is symmetric, we can only remove the symmetric part of Γ^m_{kln}.

How to quantify curvature?

Completely symmetric part of Γ^m_{kln} is

$$\Gamma^m_{(kln)} := \frac{1}{3!}\left(\Gamma^m_{kln} + \Gamma^m_{nkl} + \Gamma^m_{lnk} + \Gamma^m_{nlk} + \Gamma^m_{lkn} + \Gamma^m_{knl}\right) = \frac{1}{3}\left(\Gamma^m_{kln} + \Gamma^m_{nkl} + \Gamma^m_{lnk}\right).$$

Also

$$\frac{\partial x^k}{\partial y^{(b}} \frac{\partial x^l}{\partial y^c} \frac{\partial x^n}{\partial y^{d)}} \Gamma^m_{kln} = \frac{\partial x^k}{\partial y^b} \frac{\partial x^l}{\partial y^c} \frac{\partial x^n}{\partial y^d} \Gamma^m_{(kln)}.$$

Therefore

$$\tilde{\Gamma}^a_{(bcd)} = \frac{\partial y^a}{\partial x^m}\left(\frac{\partial x^k}{\partial y^b} \frac{\partial x^l}{\partial y^c} \frac{\partial x^n}{\partial y^d} \Gamma^m_{(kln)} + \frac{\partial^3 x^m}{\partial y^b \partial y^c \partial y^d}\right),$$

so

$$\tilde{\Gamma}^a_{bcd} - \tilde{\Gamma}^a_{(bcd)} = \frac{\partial y^a}{\partial x^m} \frac{\partial x^k}{\partial y^b} \frac{\partial x^l}{\partial y^c} \frac{\partial x^n}{\partial y^d}\left(\Gamma^m_{kln} - \Gamma^m_{(kln)}\right).$$

If there is a non-symmetric part of Γ^m_{kln}, then it cannot be removed by a choice of coordinates. Note however, that the above looks like a transformation law for a tensor!

Curvature tensor

In an arbitrary inertial frame, the *curvature tensor* is

$$K^m_{kln} := \Gamma^m_{kln} - \Gamma^m_{(kln)} = \partial_n \Gamma^m_{kl} - \partial_{(n}\Gamma^m_{kl)}.$$

If it is non-zero in some inertial frame, then it cannot be zeroed in another $\rightarrow$ the spacetime is not flat!

Useful properties:

- $\star$ Symmetric in first two 'downstairs' indicies: $K^m_{kln} = K^m_{lkn}$,
- $\star$ Completely symmetric part vanishes (first Bianchi identity): $K^m_{(kln)} = \frac{1}{3}\left(K^m_{kln} + K^m_{nkl} + K^m_{lnk}\right) = 0.$

In an arbitrary non-inertial frame, K^m_{kln} equals

$$K^m_{kln} = \Gamma^m_{kln} - \Gamma^m_{(kln)} + \underbrace{\Gamma^j_{kl}\Gamma^m_{nj} - \Gamma^j_{(kl}\Gamma^m_{n)j}}_{\text{non-inertial part}}.$$

Riemann tensor

An equivalent measure for the "departure from flatness" is the Riemann tensor, which exploits the anti-symmetric properties of K^m_{kln}:

$$R^m_{kln} := -2K^m_{k[ln]} = -K^m_{kln} + K^m_{knl},$$

Useful properties:

- ⋆ Anti-symmetric in last two 'downstairs' indicies: $R^m_{kln} = -R^m_{knl}$,
- ⋆ Completely anti-symmetric part vanishes (first Bianchi identity): $R^m_{[kln]} = \frac{1}{3}\left(R^m_{kln} + R^m_{nkl} + R^m_{lnk}\right) = 0$.

In an arbitrary non-inertial frame, R^m_{kln} equals

$$R^m_{kln} = \Gamma^m_{knl} - \Gamma^m_{kln} + \underbrace{\Gamma^j_{kn}\Gamma^m_{lj} - \Gamma^j_{kl}\Gamma^m_{nj}}_{\text{non-inertial part}}.$$

Riemann = curvature

The two notions are equivalent:

$$R^m_{kln} = -2K^m_{k[ln]}, \quad K^m_{kln} = -\frac{2}{3}R^m_{(kl)n},$$

although each one is suited for specific purposes:

Curvature tensor:

A measure of ”obstruction” against ”flattening” the coordinates in the neighborhood of a point.

Riemann tensor:

A measure of how a vector changes in a process of parallel transport along a closed curve.

Of course, general relativity is much more than connection & curvature, but these two concepts are sufficient to describe motion of test masses in curved spacetime (→ detect the curvature and its changes).

Other mathematical tools:

- ⋆ Vectors and forms, co- and contravariant objects,
- ⋆ Manifolds, fibres, bundles,
- ⋆ Metric tensor,
- ⋆ Parallel transport,
- ⋆ Covariant derivative,
- ⋆ Lie derivative,
- ⋆ Symmetries and Killing fields,
- ⋆ Principle of least action.

Special relativity in Minkowski spacetime

How we evaluate the distance in space in the usual 3D geometry?
Let's consider spherical coordinates,

$$
\begin{aligned}
x^1 &= r \sin\theta \cos\phi \\
x^2 &= r \sin\theta \sin\phi \\
x^3 &= r \cos\theta
\end{aligned}
$$

and call such an object, $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$

the *metric tensor*. An infinitesimal distance between (r, θ, ϕ) and $(r + dr, \theta + d\theta, \phi + d\phi)$ is then,

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

Special relativity in Minkowski spacetime

Let's consider now a 4D space, with a following coordinate system:

$$
\begin{aligned}
x^0 &= ct \text{ (= t for c=1)} \\
x^1 &= x \\
x^2 &= y \\
x^3 &= z
\end{aligned}
$$

and introduce the following *metric tensor*

$$
\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

that can be used to calculate the distances in an usual way

$$
ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = -dt^2 + dx^2 + dy^2 + dz^2.
$$

A manifold with the signature (-+++) - set of points in a topological space - is called a *pseudo-Riemannian* manifold: the metric tensor is not positive-definite.

Comparing vectors in curved spaces

Consider an infinitesimal change of a vector **v** along a line parametrized by λ in a space with a coordinate basis **e**:

$$\frac{d\mathbf{v}}{d\lambda} = \frac{d(v^\alpha \mathbf{e}_\alpha)}{d\lambda} = \frac{dv^\alpha}{d\lambda}\mathbf{e}_\alpha + v^\alpha \frac{d\mathbf{e}_\alpha}{d\lambda}.$$

How the vectors from the coordinate basis change with λ?

$$\frac{d\mathbf{e}_\alpha}{d\lambda} = \frac{d\mathbf{e}_\alpha}{dx^\beta}\frac{dx^\beta}{d\lambda} \quad \text{with} \quad \frac{d\mathbf{e}_\alpha}{dx^\beta} = \underbrace{\Gamma^\gamma_{\alpha\beta}}_{\textit{Connection}} \mathbf{e}_\gamma$$

so we can write a *total* derivative

$$\frac{d\mathbf{v}}{d\lambda} = \left(\frac{dv^\alpha}{d\lambda} + \Gamma^\alpha_{\gamma\beta} v^\gamma \frac{dx^\beta}{d\lambda}\right)\mathbf{e}_\alpha \quad \text{or} \quad \frac{Dv^\alpha}{d\lambda} = \frac{dv^\alpha}{d\lambda} + \Gamma^\alpha_{\gamma\beta} v^\gamma \frac{dx^\beta}{d\lambda}.$$

In a curved space, the changes are because of

- ⋆ physical changes of a vector field between points,
- ⋆ curvilinear coordinates.

Comparing vectors in curved spaces

$\Gamma^{\alpha}_{\gamma\beta}$ (affine connection, Christoffel, Levi-Civita symbols) describe the effects of parallel transport in curved spaces; they are functions of the metric $g_{\alpha\beta}$:

$$\Gamma^{\alpha}{}_{\gamma\delta} = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\beta\gamma}}{\partial x^{\delta}} + \frac{\partial g_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial g_{\gamma\delta}}{\partial x^{\beta}} \right)$$

$$= \frac{1}{2} g^{\alpha\beta} (g_{\beta\gamma,\delta} + g_{\beta\delta,\gamma} - g_{\gamma\delta,\beta})$$

(symmetric in lower indices, $\Gamma^{\alpha}{}_{\gamma\delta} = \Gamma^{\alpha}{}_{\delta\gamma}$).

Comparing vectors in curved spaces

The total derivative, similar like in hydrodynamics, is

$$\frac{Dv^\alpha}{D\lambda} = \frac{dv^\alpha}{d\lambda} + \Gamma^\alpha_{\gamma\beta} v^\gamma \frac{dx^\beta}{d\lambda} \quad \text{or in vector notation} \quad \frac{D\mathbf{v}}{D\lambda} = \nabla_{\mathbf{u}} \mathbf{v}$$

with $u^\alpha = dx^\alpha / d\lambda$, the 4-velocity/tangent vector to the curve.

Often called the *covariant* derivative:

$$v^\alpha_{;\beta} = v^\alpha_{,\beta} + \Gamma^\alpha_{\gamma\beta} v^\gamma \quad \text{or} \quad \frac{Dv^\alpha}{D\lambda} = v^\alpha_{;\beta} u^\beta$$

Covariant derivative acting on the metric return 0 (metric compatibility):

$$g_{\alpha\beta;\gamma} = 0, \quad g^{\alpha\beta}_{;\gamma} = 0.$$

Riemann curvature tensor

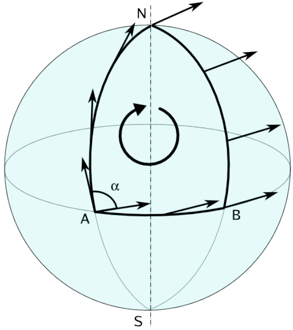
In the language of covariant derivatives along vector directions, $R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$ measures a failure of derivatives to commute.

- ⋆ Constructed from $g_{\mu\nu}$ and its first and **second** derivatives,
- ⋆ Imagine transporting a vector **V** around a closed loop by dx^σ, dx^μ and then dx^ν; the vector will change its components w.r.t. the original ones by ΔV^i.

Transport of a vector *parallel* to the connection.

The Riemann tensor is roughly

$$R^\rho_{\sigma\mu\nu} = \Delta V^i / (dx^\sigma dx^\mu dx^\nu)$$



Riemann curvature tensor

In the language of covariant derivatives along vector directions, $R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$ measures a failure of derivatives to commute.

- ⋆ Constructed from $g_{\mu\nu}$ and its first and **second** derivatives,
- ⋆ Measures the *intrinsic* curvature $\rightarrow$ Gauss curvature, ”rotation” of parallel-transported vectors ($R \equiv 0 \iff$ space is flat),
- ⋆ Measures the tidal forces acting on a body moving on the geodesic $\rightarrow$ relative acceleration between nearby bodies (geodesic deviation),
- ⋆ in 3+1 spacetime, $R^{\rho}_{\sigma\mu\nu}$ has 256 components, only 20 independent (because of the following symmetries):

$$\begin{aligned} R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} &= -R_{\sigma\rho\mu\nu}, \\ R_{\rho\sigma\mu\nu} &= R_{\mu\nu\rho\sigma}, \\ R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\mu} + R_{\rho\nu\sigma\mu} &= 0. \end{aligned}$$

Useful second **Bianchi identity**: $\nabla_\gamma R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\gamma} + \nabla_\nu R_{\rho\sigma\gamma\mu} = 0.$

Ricci tensor and Ricci scalar

- ⋆ Ricci tensor is a contraction of the Riemann tensor:

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$$

$R_{\mu\nu}$ is kind of *average curvature*. It quantifies the amount by which a test volume differs from one in flat space,

In the vicinity of a given point, $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(x^2)$.

The difference in volume element: $dV = \left(1 - \frac{1}{6} R_{\mu\nu} x^{\mu} x^{\nu} + \mathcal{O}(x^3)\right) dV_{flat}$

- ⋆ Ricci scalar (scalar curvature) is contracted Ricci tensor:

$$R = R^{\mu}_{\mu}$$

used e.g., to compare areas of circles with those from flat space in n dimensions:

$$\frac{dS}{dS_{flat}} = 1 - \frac{R}{6n} r^2 + \mathcal{O}(r^4)$$

in 2D, $R = 2K$ (twice the Gauss curvature).

Useful second **Bianchi identity**: $\nabla^{\mu} R_{\alpha\mu} = \frac{1}{2} \nabla_{\alpha} R$

Energy-momentum tensor

The **energy-momentum** tensor (sometimes called the stress-energy tensor) contains mass-energy information. Most often used is the *perfect fluid* version,

$$T_{\mu\nu} = (\rho + p)u^\mu u^\nu + pg_{\mu\nu},$$

$\rightarrow$ neglect viscosity and elastic effects. Fluid which is isotropic in its rest frame ($g_{\mu\nu}u^\mu u^\nu = -1$)

$$T^\nu_\mu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

The conservation laws of $T_{\mu\nu}$ are analogs of conservation laws for energy and momenta from hydrodynamics, using the covariant derivative:

$$\nabla^\mu T_{\mu\nu} = 0.$$

Einstein equations

Using the just defined tensors, we arrive at

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

(10 equations in 3+1 dimensions).

Why like that? The equations should conserve energy & momentum. We would like to have

$$\nabla^\mu T_{\mu\nu} = 0. \quad \text{It implies} \quad \nabla^\mu G_{\mu\nu} = 0.$$

From the contracted Bianchi identity,

$$\nabla^\mu R_{\alpha\mu} = \frac{1}{2} \nabla_\alpha R \quad \rightarrow \quad \nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = 0.$$

Literature

- ⋆ Lecture notes of Sean Carroll (http://preposterousuniverse.com/grnotes)
- ⋆ Textbooks: Misner-Thorne-Wheeler, Wald,
- ⋆ Jerzy Kijowski, "Geometria różniczkowa jako narzędzie nauk przyrodniczych", Monografie CSZ, 2015 (in Polish),
- ⋆ SageManifolds examples: http://sagemanifolds.obspm.fr/examples.html

- ⋆ My old introduction to general relativity lecture slides: users.camk.edu.pl/bejger/raarcm/intro-gr.pdf