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Editor's Note: On Some Static Solutions of Einstein's Gravitational Field Equations in a Spherically Symmetric Case. On a New Cosmological Solution of Einstein's Field Equations of Gravitation.

by Hidekazu Nariai

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The solution presented in Nariai's papers is an illustration to a certain text-book problem that has been persistently gotten wrong in most relativity textbooks, even the recent ones (the *Exact solutions* ..., Ref. 1, p. 155–158, being a notable exception). From the assumption of spherical symmetry, the following metric form follows:

$$ds^2 = \alpha(t,r)dt^2 + 2\beta(t,r)dtdr + \gamma(t,r)dr^2 + \delta(t,r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \quad (1)$$

where α, β, γ and δ are arbitrary functions, ϑ and φ are coordinates on a sphere, r is a parameter labelling the spheres and t is a time coordinate. The functions α, β, γ and δ are determined up to the transformations:

$$t = f(t', r'), \qquad r = g(t', r'), \quad (2)$$

where f and g are arbitrary functions subject only to $\partial(f, g)/\partial(t', r') \neq 0$. Then, the argument goes, we can use the transformations (2) to simplify two of the functions α, β, γ and δ as we wish, so we choose f and g so that $\beta = 0$ and $\delta = -r^2$ after the transformation. This is where the error is. Such a transformation exists only if the gradient of δ is a spacelike vector, which is a coordinate-independent property. The cases that are then left out of sight are:

(i) The gradient of δ being a timelike vector. Then coordinates can be chosen so that $\delta = -t^2$. This case contains, among other things, the Kantowski–Sachs [2] class of metrics (and its inhomogeneous generalizations in the $\beta' = 0$ subfamily of the Szekeres [3]–Szafron [4] metrics, see Ref. 5). Also, this case contains the Schwarzschild metric extended into the black hole region $r < 2m$. Strangely, this extension *is* discussed in most textbooks, but its inconsistency with $\delta = -r^2$ remains unnoticed.

(ii) The gradient of δ being a nonzero null vector.[1] The choice $\delta = -r^2$ is then possible, but it automatically implies $\alpha = 0$ in the new coordinates, and so one cannot achieve $\beta = 0$ in addition. In vacuum with zero cosmological constant, no solution of Einstein's equations exists in this case, but this editor is not aware of any further study of this class of metrics.

(iii) The gradient of δ being zero, i.e. δ being a constant. This is a coordinate-independent property in the class (1)–(2), and so no condition can be imposed on δ by coordinate transformations in this case. Again, no vacuum solution with $\Lambda = 0$ exists, and the vacuum solution with $\Lambda \neq 0$ was found by Nariai in the papers reprinted here (see also Ref. 6 and Ref. 1, p. 155–158).

In Nariai's first paper, the solution is hidden among other results. The main purpose of the paper was to obtain a collection of static spherically symmetric solutions of Einstein's equations, and the solution in question came up as just one element of the collection. The key observation is made in the phrase containing eq. (3), and the solution itself is given by eq. (35). The second paper is all devoted to investigating its properties.

Later research has brought more information on the solution (see Ref. 6). Nariai found it having assumed staticity from the beginning. However, this assumption is not necessary. It was shown in Ref. 6 that the collection of all spherically symmetric vacuum solutions of Einstein's equations with the cosmological constant consists of two metrics; one of them is the Kottler solution (i.e. the Schwarzschild solution generalized for Λ), and the other is the following:

$$\begin{aligned} ds^2 = {} & \{a(t)\cos[\ln(r/l)] + b(t)\sin[\ln(r/l)]\}^2 dt^2 \\ & - (l/r)^2 dr^2 - l^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \end{aligned} \tag{3}$$

[1] Here we refer to the case when the function δ has a nonzero null gradient in an open four-dimensional region. This should not be mixed up with the situation in the Schwarzschild solution, where the said gradient is null on a 3-dimensional hypersurface, the event horizon.

where $\Lambda = l^{-2}$. The time-dependence of (3) is spurious; a and b can be made constant by a coordinate transformation (see Ref. 6), and the resulting metric is equivalent to eq. (1) in the second paper.

The manifold of the Nariai solution is a Cartesian product of two surfaces of the same positive constant curvature, one with signature (+ −), the other with signature (+ +) (a sphere) (see Ref. 6). The solution can be reparametrized so that the limit $\Lambda \to 0$ (i.e. $l \to \infty$) becomes meaningful; the following coordinate transformation makes it possible:

$$r = le^{r'/l}, \qquad \vartheta = \pi/2 + \vartheta'/l, \qquad \varphi = \varphi'/l. \tag{4}$$

Then, dropping primes, the result is

$$ds^2 = [a(t)\cos(r/l) + b(t)\sin(r/l)]^2 dt^2 - dr^2 - d\vartheta^2 - \cos^2(\vartheta/l)d\varphi^2, \tag{5}$$

and the limit $l \to \infty$ is the Minkowski metric.

A certain misunderstanding concerning the so-called Bertotti [7]–Robinson [8] solution can be explained by this opportunity. The two solutions are not the same, contrary to common wisdom. The Bertotti solution is a generalization of the Nariai solution for electromagnetic field in the source. It has the same geometric structure as the Nariai solution, but the two curvatures are different, the first one is not necessarily positive, and their difference is proportional to the electromagnetic field. The Robinson solution [8] is the limit $\Lambda = 0$ of the Bertotti solution (it is conformally flat), and does not contain the Nariai solution as a subcase; the two curvatures in it are of the same absolute value, but of opposite signs. Another related problem is the so-called Birkhoff theorem (see Ref. 9, p. 167, for a discussion), which is not as strong as textbooks like to imply. The following two formulations of it are met in the literature: Every spherically symmetric solution of the Einstein equations in vacuum is:

(i) Static.

(ii) Equivalent to the Schwarzschild solution under a coordinate transformation.

The first formulation is false (the Schwarzschild solution taken inside the black hole region is the counterexample; unless the value of Λ is negative with a sufficiently large absolute value so that the horizon disappears). The second formulation is correct if $\Lambda = 0$, but false if $\Lambda \neq 0$; in the second case the Nariai solution is the counterexample.

Bonnor in Ref. 9, p. 167, proposed bypassing the problem by adding "Every *physically significant*..."; the physical significance requirement was meant to exclude the Nariai solution, very nearly rediscovered in Ref. 9.

However, it is fair to say that the *physical* meaning of the Nariai solution is still unknown; the original paper and Ref. 6 only investigated its geometry.

— *Andrzej Krasiński, Associate Editor*

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Brief biography

Hidekazu Nariai was born in Taisha-machi, Hikawa-gun, Shimane prefecture (west Japan) on February 2, 1924. He graduated from the Matsue high school in Shimane prefecture and the Tohoku university in Miyagi prefecture (east Japan). He obtained his PhD degree at this university and stayed there as an assistant. In 1953 he moved to Takehara city in Hiroshima prefecture (west Japan) as a research associate in the Research Institute for Theoretical Physics, Hiroshima university and continued his academic career there as a lecturer, an associate professor, professor (1972–1986), and emeritus professor (1987–1990). He died and was buried in Takehara city on December 5, 1990. In June 1990 his Institute (RITP) and the Research Institute for Fundamental Physics, Kyoto university (RIFP) were united into a single Institute —- the Yukawa Institute for Theoretical Physics, Kyoto university (YITP).

His many and valuable works are concerned with (1) exact solutions including the Nariai solutions, (2) cosmological turbulence theory, (3) cosmological perturbation theory including Nariai and Ueno's theory of the cosmological Newtonian approximation, (4) quantum field theory in the expanding universe including Nariai and Kimura's theory, (5) junction conditions and general-relativistic dynamics in collapsing stars, and (6) renormalized gravitation theory with higher-order Lagrangians as an extension of the Einstein theory. At present the Nariai solutions seem to be most famous among his many works.

— *K. Tomita*

A more extended biography of H. Nariai can be found in Ref. 1.

— *Ed.*

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