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GOLDEN OLDIE

Republication of: Geometrical theorems on Einstein's cosmological equations (By E. Kasner)

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Abstract The seminal paper by Kasner (Am. J. Math. **43**, 217–221, 1921) is reprinted here, together with an editorial comment on its lasting scientific relevance, and a biography of the author.

Keywords Einstein's equations · Cosmological solutions · Kasner solutions · Golden Oldie

Editor's note

John Wainwright

The well-known Kasner solutions of the vacuum Einstein field equations are of importance in general relativity in that they serve to determine the behaviour of generic cosmological solutions of the Einstein field equations on approach to the singularity. Kasner's paper, written in 1921, gives the first derivation of these solutions, starting with the assumptions that the metric is diagonal and that it depends on only one coordinate (see Sect. 3 in the paper).

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In the paper the solution was written for a positive definite metric, but there is the implication that the signature can be Lorentzian, with the possibility that x^1 be a timelike or a spacelike coordinate.[1] The metric was given in the form

$$ds^2 = x_1^{2a_1} dx_1^2 + x_1^{2a_2} dx_2^2 + x_1^{2a_3} dx_3^2 + x_1^{2a_4} dx_4^2,$$

where

$$a_2 + a_3 + a_4 = 1 + a_1, \quad a_2^2 + a_3^2 + a_4^2 = (1 + a_1)^2.$$

Thirty years later Taub [22] gave the Kasner solutions, which form a 1-parameter family of solutions, in the following now-familiar form

$$ds^2 = -dt^2 + t^{2p_1} dx_1^2 + t^{2p_2} dx_2^2 + t^{2p_3} dx_3^2, \tag{1}$$

where the constants p_1, p_2 and p_3 satisfy

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1.$$

Taub, who was unaware of Kasner's paper, derived these solutions during a systematic investigation of spatially homogeneous vacuum solutions, i.e., solutions that admit a three-parameter group of isometries acting on spacelike hypersurfaces. He showed that (1) was the general solution in which the group is of Bianchi type I (see Sect. 6). In this context one thinks of the Kasner solutions as representing idealized universes that are expanding in a highly anisotropic manner and in which the matter has a negligible effect on the dynamics.

To the best of this writer's knowledge, Lifshitz and Khalatnikov [15] were the first to show that the Kasner solutions[2] have a much broader significance than might be expected, in that they approximate a general class of cosmological solutions near the initial singularity. These investigations culminated in the discovery [3] that the evolution of a sufficiently general cosmological solution towards the initial singularity has an oscillatory nature, with the solution passing through an infinite series of epochs in time, during each of which the metric is approximated by a Kasner metric, now called *Kasner epochs*. This behaviour has its simplest manifestation in the non-rotating spatially homogeneous universes of Bianchi types VIII and IX, and in fact the oscillatory behaviour in vacuum Bianchi IX universes was discovered independently by Misner [18], using a Hamiltonian formulation of the Einstein field equations. He coined the term "Mixmaster universe" to describe the mixing effect in these models. The geometrical reason for the oscillatory behaviour is that spatial curvature destabilizes the

[1] The Kasner metrics with spacelike coordinate, in which case the spacetime is static, have also been studied in the literature. We refer to Harvey [13], (see in particular his Eq. (34)), and to Stephani et al. [21] (see in particular, Sect. 13.3.2, p. 197, and Table 18.2, p. 285), for details and further references.

[2] These authors give credit to Taub [22] for the discovery of the solutions, but in a paper published two years later [16] they note that "This solution was apparently first obtained by Kasner (1921)" (see p. 195). It thus appears that Kasner's paper was "discovered" by relativists in the early 1960s.

Kasner solutions, causing a general solution that is close to one Kasner solution to be repelled and then attracted to a different Kasner solution. During the transition[3] between Kasner solutions the general solution is approximated by a Taub vacuum solution of Bianchi type II (first given by Taub [22] Eqs. (7.1)–(7.2)). There is a simple law relating successive Kasner epochs, called the *Kasner map* (see, e.g., [4,26, p. 236]). In the Hamiltonian formulation, the evolution of spatially homogeneous cosmologies is represented as the motion of a particle in a time-dependent potential, and the Kasner-to-Kasner transitions are described heuristically as "bounces off potential walls", with the Kasner map being encoded as a "bounce law" (for a simple description, see Uggla [23], pp. 222–226).

It is initially surprising that the Kasner solutions, which are vacuum solutions, should determine the dynamics of cosmological models with matter, on approach to the singularity. This aspect of the behaviour hinges on a conjecture[4] by Belinskii, Khalatnikov and Lifshitz (BKL) to the effect that "matter does not matter" close to the cosmological initial singularity, i.e., matter is not dynamically significant in that epoch (see [16], p. 200 and [4], p. 532 and p. 538).

The Kasner solutions arise very naturally when one formulates the Einstein field equations for non-tilted spatially homogeneous cosmologies as a dynamical system, as was first done by Collins [9] for some special classes of models. The Kasner solutions are then represented as *equilibrium points* (i.e., fixed points) of the dynamical system, reflecting the fact that they are *self-similar* solutions, i.e., admit a homothetic vector field, as was first pointed out by Eardley [10]. In the dynamical systems formulation of Collins, as developed further by Wainwright and Hsu [25], the set of Kasner solutions is represented by a circle of equilibrium points, the *Kasner circle*. The Kasner equilibrium points are saddle points, and their instability manifests itself in the Taub orbits that join a given Kasner point to a different Kasner point. The oscillatory approach to the singularity is now described by the behaviour of orbits in the state space: during the asymptotic stage the orbit of a typical model of Bianchi type VIII or IX is approximated by an infinite sequence of Taub orbits successively joining Kasner points [17]. It was thus conjectured that the union of the Kasner circle and the set of all Taub orbits forms the past attractor in the state space of non-tilted spatially homogeneous cosmologies, called the *Mixmaster attractor* ([26], Sect. 6.4).

The research described above concerning the role of the Kasner solutions in the oscillatory approach to the singularity has been heuristic, relying on approximations of various sorts and on numerical simulations. In 2001, however, Ringström was able to use the dynamical systems formulation to prove that typical orbits in the state space for Bianchi IX models are past asymptotic to the Mixmaster attractor [20], which represents a major step in providing a rigorous foundation for the oscillatory approach to the singularity.

The discussion so far has concerned the occurrence of Mixmaster dynamics in spatially homogeneous solutions. The scope of the discussion, and hence of the significance of the Kasner solutions, is increased dramatically by a second conjecture of

[3] This transition is described in detail in [3], Eqs. (2.11)–(2.19).

[4] Exceptions to this conjecture are provided by a stiff fluid or a massless scalar field (see, e.g., [1]).

BKL to the effect that near a cosmological initial singularity, Einstein's field equations effectively reduce to ODE, i.e., the spatial derivatives are dominated by the time derivatives, and hence have a negligible effect on the dynamics (see [5], p. 656). One thus expects that Mixmaster dynamics will occur in sufficiently general spatially inhomogeneous solutions, but that the specific sequence of Kasner epochs will depend on which fundamental world line one follows into the past. The situation can be summarized in the following conjecture: *for almost all cosmological solutions of Einstein's field equations, a spacelike initial singularity is vacuum-dominated, local and oscillatory* [24]. Equivalently one can say that *local Mixmaster dynamics* occurs [7]. There is now a rich variety of numerical and heuristic work supporting the conjecture, in addition to the initial BKL work (see [5] for a review of this research), much of it based on the Hamiltonian approach. We refer in particular to Berger et al. [8] for analytical and numerical results supporting the conjecture in vacuum solutions with a T^2 isometry group, and to Berger [6] for further discussion and references (see Sect. 3). Numerical simulations supporting the conjecture in vacuum cosmological solutions with no symmetry have been reported by Garfinkle [11].

One of the difficulties in proving the BKL conjecture is in giving a precise formulation of their claims. In order to address this difficulty, Uggla et al. [24] reformulated Einstein's field equations by introducing scale-invariant variables that have the property that the variables are bounded asymptotically on approach to the singularity. This made it possible to formulate the BKL conjecture in terms of specific limits, which may represent a first step in providing a proof. The discussion of the validity of the BKL conjecture is, however, complicated by the fact that as the singularity is approached localized spatial structure may form along individual fundamental worldlines, indicating that spatial derivatives are not negligible along these worldlines, with the result that local Mixmaster dynamics does not occur. This behaviour has been studied analytically [19] and numerically [2,12] in vacuum Gowdy solutions. Whether or not this behaviour is restricted to a set of fundamental worldlines of measure zero is not known at present.[5]

In conclusion, it is clear that the Kasner solutions, despite their extreme simplicity, are playing a fundamental role in the ongoing efforts to understand the asymptotic dynamics of cosmological solutions of the Einstein field equations, on approach to the singularity.

Edward Kasner: a brief biography

Andrzej Krasiński (compiled from Refs. [27,28] below)

Edward Kasner was born in New York City on 2 April 1878. Kasner once said that the first time he heard the word "mathematician" was at the age of 8, when his teacher at Public School No. 2 told him he would become a great one. At the age of 13, he entered the College of the City of New York to pursue a five-year course of study that combined the then standard high school and undergraduate curricula. He completed

[5] I thank Woei Chet Lim for helpful comments on this matter.

this program on schedule to earn his B.S. in 1896. He followed this with graduate work at Columbia University, where he studied principally under the Göttingen-trained American mathematician Frank Nelson Cole. Kasner took his Columbia A.M. and Ph.D. in mathematics in 1897 and 1899, respectively. Columbia University's records suggest that his was only the second or third doctorate in mathematics awarded there. In 1899 he proceeded to Göttingen for additional postgraduate training and he spent the 1899–1900 academic year mostly in the lecture rooms of Felix Klein and David Hilbert.

Kasner returned to New York to take up his new post as tutor in mathematics at Barnard College in 1900. He remained in that position until 1905, receiving then a promotion to the rank of instructor and to that of adjunct professor one year later. His undergraduate teaching aimed not so much to produce future mathematicians as to instill an appreciation of the subject in his students. To this end, he instituted the "mathematics bulletin board" on which he regularly posted students' work of special interest or merit. It became a recognized, though unofficial, honour for students to find their work so displayed. Kasner also participated in the monthly meetings of the American Mathematical Society (AMS), held in New York City, frequently presenting talks on his latest research work in the field of differential geometry. In 1904 he was asked to give a keynote address on geometry at the International Congress of Arts and Sciences associated with the St. Louis World's Fair. He chose as his topic "Present Problems of Geometry", and he shared the platform with Henri Poincaré. In 1906 he was elected to the vice presidency of the American Association for the Advancement of Science (AAAS), to the chair of the AAAS's Section on Mathematics and Astronomy and to the vice presidency of the AMS in 1908. The AMS selected him to deliver the prestigious Colloquium Lectures of the AMS in the summer of 1909. He spoke on his then recent work in the trajectories of dynamics, and his lectures came out in book form in 1913 as "Differential Geometric Aspects of Dynamics" (published again in 1934).

Kasner shifted his position permanently from Barnard to Columbia in 1910, when the Columbia trustees offered him a full professorship. He remained on the Columbia faculty for the rest of his career during which he pursued four areas of research: differential geometry applied to dynamics (until 1920), the geometrical aspects of Einsteinian relativity (1920–1927), the analysis of the polygenic functions of a complex variable (1927–1940), and the geometrical analysis of horn angles (1940–1955). In 1917 he was elected to the National Academy of Sciences.

In addition to his research work at Columbia, Kasner also devoted himself to the training of graduate students, regularly running his "Seminar in Differential Geometry". As Jesse Douglas, one of his pupils, has described it, "Kasner's mode of presentation was ideal for those seeking fruitful research problems. The theme of his advanced teaching, in any mathematical situation under discussion, was this: 'Have we a problem here? Is there something still incomplete, something of significance or interest left to be investigated? Is there a property stated which is not characteristic?—then find a characteristic property' ". Kasner thus challenged his auditors to do mathematics actively by critically questioning and examining—not merely accepting—the mathematics received.

Kasner often was asked to give mathematical presentations in kindergartens and private schools, building his lectures on such occasions around common and easily understood questions like "How many pennies can you place on the floor touching one penny?" With apparent ease, he would proceed to give the children in his audience an intuitive appreciation of deep problems in mathematics, such as the three-dimensional analog of this question, namely, sphere-packing. Kasner also offered, through New York's New School of Social Research, a popular evening course, "Fundamental Concepts of Mathematics", that aimed to introduce adult students to some of the key concepts of basic mathematics. He coauthored, with his student James Newman, the widely read book "Mathematics and the Imagination" (1940). There he popularized two terms he had challenged his young nephews to invent in the course of his ongoing mathematical discussions with them, namely, googol for 10^{100} and googolplex or 10^{googol}.

One of Kasner's great interests was painting and sketching in watercolor. During his visits to Europe he often attended a school of art in Paris. He was a life member of the Metropolitan Museum of Art in New York, where he was a regular visitor.

Kasner assumed the Robert Adrain Professorship in mathematics at Columbia in 1937. From this point on, he conducted much of his research in collaboration with his student John De Cicco. Kasner retired as Adrain Professor Emeritus in 1949. In 1954, on the occasion of the bicentennial convocation, the Columbia University awarded him the honorary doctorate in science (another recipient of the same honour, on the same day, was the Queen Mother Elizabeth of England). Never married, he died in New York City on 7 January 1955.

Among his most important works not mentioned above are "The Present Problems of Geometry", *Bulletin of the American Mathematical Society* **11** (1905): 283–314; and "The Theorem of Thomson and Tait and Natural Families of Trajectories", *Transactions of the American Mathematical Society* **11** (1910): 121–140. On Kasner's life, consult Ref 28 below, which includes a photograph, a technical discussion of Kasner's mathematical achievements, and a complete bibliography of his works. For a sense of Kasner's place in the American mathematical community, consult Della Dumbaugh Fenster and Karen Hunger Parshall, "A Profile of the American Mathematical Research Community: 1891–1906", in *The History of Modern Mathematics*, vol. **3** (1994), pp. 179–227. An obituary is in the New York Times 8 Jan. 1955. Ref. [29] below is an essay on Kasner, written while he was still professionally active, on the occasion of publication of "Mathematics and the Imagination".

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GEOMETRICAL THEOREMS ON EINSTEIN'S COSMOLOGICAL EQUATIONS.

By Edward Kasner.

I wish to generalize here some of my results published in the January and April numbers of the American Journal of Mathematics (Vol. 48, 1921, pp. 20, 126), relating to Einstein's original equations of gravitation (in space free from matter),

$$G_{\mu\nu} = 0. \tag{1}$$

Later Einstein introduced a so-called cosmological term involving a constant λ, the equations being then

$$G_{\mu\nu} - \lambda g_{\mu\nu} = 0. \tag{2}$$

More recently* he has employed the form

$$G_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} G = 0, \tag{3}$$

where G is the scalar curvature $g^{\alpha\beta} G_{\alpha\beta}$. I shall refer to (3) simply as the *cosmological equations*. Every solution of the former equations (1) is, of course, a solution of the latter (3), but not vice-versa. The ten equations (3), as Einstein shows, involve one extra dependence as compared with the ten equations (1).

§ 1. Five Dimensions.

I shall first take up the question of dimensionality (that is, class of the quadratic form). In the April paper† it was shown that no solution of (1) can represent a 4-spread imbedded in a 5-flat (except in the trivial case where the 4-spread is euclidean, that is, has zero Riemann curvature). We now inquire what solutions of (3) can be imbedded in a 5-flat, and shall find that there are actually two distinct possibilities.

Using the notation of the April paper, we write our spread in the form

$$w = f(x_1, x_2, x_3, x_4).$$

Referring to the formulas on p. 128, we have, in the standard coördinates there employed, involving the four principal curvatures k_i,

$$G_{12} = 0, \text{ etc.}; \qquad G_{11} = -k_1(k_2 + k_3 + k_4), \text{ etc.}$$

* *Berichte Berlin Akad. d. Wiss.* (1919). The complete equations when matter is present of course involve the energy tensor $T_{\mu\nu}$. See also Kopff, *Grundzüge* (1921), pp. 163–165, where the author refers to the field equations of the first, second, third kinds.

† "The Impossibility of Einstein Fields Immersed in Flat Space of Five Dimensions," Vol. 48, pp. 126–129.

Also by the footnote on the same page, we have

$$G = -2(k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4).$$

Substituting in the cosmological equations and simplifying, we have the following set of equations for the determination of the four principal curvatures,

$$k_1(k_2 + k_3 + k_4) = k_2(k_3 + k_4 + k_1) = k_3(k_4 + k_1 + k_2) = k_4(k_1 + k_2 + k_3). \tag{4}$$

Subtracting say the second member from the first, we have

$$(k_1 - k_2)(k_3 + k_4) = 0, \tag{4'}$$

with five similar equations obtained by permuting the subscripts. Hence either

$$k_1 = k_2 \text{ or } k_3 = -k_4, \text{ etc.} \tag{4''}$$

If the common value of the four expressions in (4) is zero, then we have exactly the system (12) of the April paper, giving merely the trivial case where 3 or 4 of the k's vanish (which means that the manifold is euclidean). Otherwise we find from (4″) two possible types of solution

$$(a) \qquad k_1 = k_2 = k_3 = k_4 \neq 0,$$
$$(b) \qquad k_1 = k_2 = -k_3 = -k_4 \neq 0.$$

In the first case (a) the four principal curvatures are equal at every point, that is, every point is umbilical. It follows then from known theorems that the 4-spread must be a hypersphere. This, of course, checks up since it is known that a 4-dimensional hypersphere is actually imbedded in a 5-flat and is actually a solution of the cosmological equations. (It is sometimes referred to as DeSitter's "Spherical World.")

In the second possibility (b), we have, at every point, the four principal curvatures numerically equal, but two of them are positive and two are negative. We may say then that every point is *semi-umbilical.* It is immediately seen that the Riemann curvature of such a spread is not constant; for we find, in our special coördinates, that the conditions for constant Riemann curvature are

$$k_1k_2 = k_1k_3 = k_1k_4 = k_2k_3 = k_2k_4 = k_3k_4. \tag{5}$$

For the spherical solutions (a), these products are all equal; but in the new case (b), two of the products are positive and four are negative. We may term a spread of this new type, a *hyperminimal spread,* since we may think of it as a generalization of ordinary minimal surfaces, which have the property that the two principal curvatures are numerically equal, but opposite in sign. The actual existence of such hyperminimal spreads depends on the consistency of a certain set of three partial differential equations, of the second order, which can readily be written down. It may be that no solution exists, or that all the solutions are imaginary, but the possibility still remains open.

THEOREM I. *If a four-spread imbedded in a five-flat is to obey Einstein's cosmological equations (3), then either every point is umbilical (giving a hyper-sphere), or else every point is semi-umbilical (giving a possible type of hyper-minimal spread).*

§ 2. CONFORMAL REPRESENTATION.

I next take up the generalization of a theorem given in the January paper.* It was there shown that the only solutions of Einstein's original equations (1), which have the same light equation as the euclidean or Minkowski spread, are themselves euclidean; that is, if the spread is to be conformally representable on a 4-flat it must have zero Riemann curvature. We shall now prove:

THEOREM II. *The only spreads which obey the cosmological equations (3) and which are conformally representable on a 4-flat are those that have constant Riemann curvature (that is, the spread must be of spherical or pseudo-spherical character).*

For this purpose we use the notation of the earlier paper, in particular the formulas on pp. 22–23. Using the value of the tensor $G_{\mu\nu}$ there calculated, we find that the scalar curvature is

$$G = \frac{6}{\lambda}\{\Sigma N_{ii} + \Sigma N_i^2\}, \qquad \text{where} \qquad \lambda = e^{2N}.$$

Substituting then in the equations (3) we find the following system of equations for the determination of the unknown function N:

$$N_{12} - N_1 N_2 = 0, \qquad \text{etc.};$$

$$3N_{11} - N_{22} - N_{33} - N_{44} - 3N_1^2 + N_2^2 + N_3^2 + N_4^2 = 0, \qquad \text{etc.}$$

Employing the same transformation $N = -\log M$ as in the earlier paper, we obtain this simple system:†

$$M_{12} = 0, \qquad M_{13} = 0, \qquad \text{etc.};$$
$$M_{11} = M_{22} = M_{33} = M_{44}.$$

The general solution is obviously

$$M = a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5. \tag{6}$$

* "Einstein's Theory of Gravitation: Determination of the Field by Light Signals," Vol. 48, pp. 20–28. Apparently without knowledge of this paper, Ogura has recently given this special theorem in *Comptes Rendus*, Nov. 7, 1921.

† In the January paper, by a typographical error, an extra member $\frac{1}{2}M^{-1}\Sigma M_i^2$ was omitted in the second line of the corresponding set (10), p. 23, but the final result there given is correct.

Our differential form is therefore, since $\lambda = M^{-2}$,

$$ds^2 = M^{-2}(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2). \tag{7}$$

This is recognized as a manifold of constant Riemann curvature. (The curvature becomes zero only when the relation $a_1^2+a_2^2+a_3^2+a_4^2-4aa_5=0$ holds, verifying the special theorem of the earlier paper.)

Since a hypersphere can be mapped conformally on a 4-flat, it follows directly from Theorem II that the only cosmological solutions which can be represented conformally on a hypersphere are those of constant Riemann curvature.

We may also generalize the discussion of approximately-euclidean manifolds, given on pp. 27, 28 of the January paper, to approximately-spherical manifolds. The final result is

THEOREM III. *If two approximately-spherical spreads, both obeying the cosmological equations (3), admit conformal representation upon each other (thus having the same light equation), then they are necessarily isometric, except for a homothetic transformation.*

§ 3. SOLUTIONS DEPENDING ON ONE VARIABLE.

A simple example of a solution of (1) where the potentials involve only one of the variables is

$$x_4^{-2}dx_4^2 - x_1^4(dx_1^2 + dx_2^2 + dx_3^2). \tag{8}$$

All orthogonal solutions of this type (see *Bull. Amer. Math. Soc.*, vol. 27 (1920), p. 62) are easily shown to be reducible to the form

$$\begin{gathered} ds^2 = x_1^{2a_1}dx_1^2 + x_1^{2a_2}dx_2^2 + x_1^{2a_3}dx_3^2 + x_1^{2a_4}dx_4^2, \\ a_2 + a_3 + a_4 = a_1 + 1, \qquad a_2^2 + a_3^2 + a_4^2 = (a_1 + 1)^2. \end{gathered} \tag{9}$$

This can be put in the static form, and is completely determined by its light rays.

Analogous solutions of (3) are stated in an abstract printed in *Science* referred to below. We shall state the general result as

THEOREM IV. *All cosmological solutions for which the four potentials in the orthogonal form are functions of one of the four coördinates can be found explicitly by elementary algebraic and transcendental functions. The corresponding spreads can be imbedded in a 7-flat.*

§ 4. AN ALGEBRAIC SOLUTION.

We also state, omitting the easy proof,

THEOREM V. *If the quaternary form $ds^2 = g_{\mu\nu}dx_\mu dx_\nu$ is to be expressible as the sum of two binary forms, one involving say x_1, x_2, the other involving* *say* x_3, x_4, *and if the cosmological equations (3) are to be obeyed, then the only solution (except for a constant factor) is*

$$ds^2 = x_1^{-2}(dx_1^2 + dx_2^2) + x_3^{-2}(dx_3^2 + dx_4^2).$$

This can be imbedded in a 6-flat with cartesian coördinates $(X_1X_2X_3X_4X_5X_6)$, *the finite representation being*

$$X_1^2 + X_2^2 + X_3^2 = 1, \qquad X_4^2 + X_5^2 + X_6^2 = 1.$$

Excluding the obvious flat and spherical solutions, this is apparently the simplest solution of Einstein's equations which has thus far been obtained, and is the first case where the finite solution is an algebraic spread. In the example (8) given in § 3, the potentials $g_{\mu\nu}$ are algebraic, but not necessarily the corresponding finite spread.

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* The theorems of the present paper were first published in *Science*, Vol. 54 (Sept. 30, 1921), pp. 304–305. A typographical error on page 305 should be corrected as in formula (8) above. See also a paper appearing in the *Mathematischen Annalen*, entitled "The Solar Gravitational Field Completely Determined by its Light Rays." An independent proof of Theorem II, together with very elegant proofs of two of my previous results, applying to forms in n variables, is given in the paper by Prof. J. A. Schouten and Dr. D. J. Struick page 213 of this volume of this JOURNAL. The authors were kind enough to send me a copy of their manuscript.