Gen Relativ Gravit (2009) 41:421–432
DOI 10.1007/s10714-008-0721-6

GOLDEN OLDIE EDITORIAL

Editorial note to: J. N. Goldberg and R. K. Sachs, A theorem on Petrov types

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Published online: 19 December 2008

Keywords Goldberg–Sachs theorem · Vacuum metric · Petrov types · Golden Oldie

Part 1: Explanation of some details of derivation

By Andrzej Krasiński

Assumptions about properties of congruences of curves in a spacetime have powerful implications for the Weyl tensor. A well-known example is the conclusion that follows from the propagation equations of kinematical tensors in relativistic hydrodynamics [1]: If, in a perfect fluid spacetime, there exists a family of timelike curves with zero shear, rotation and acceleration, then the spacetime must be conformally flat. The theorem proven in the paper reprinted here is another example, where limitations imposed on the Weyl tensor by properties of congruences of null curves are discussed. Namely, a geodesic and shearfree null congruence exists in a vacuum spacetime if and only if its Weyl tensor is algebraically special. The congruence is then the degenerate principal null congruence of the Weyl tensor. (The theorem is further extended to include electromagnetic field with geodesic rays and a null field.)

The republication of the original paper can be found via doi:10.1007/s10714-008-0722-5.

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Although rather abstract and formal, this theorem played a powerful role in deriving the Kerr metric (see [2]), and was useful in deriving several other exact solutions of Einstein's equations [3].

The original publication is rather concise, so its calculations may be difficult to verify. Therefore, in this note we provide some details omitted by the authors. A still more detailed derivation, by the same method but in a different notation, is presented in Ref. [4].

A few rewriting errors of the original paper were corrected by the editor. All the corrections are marked by editorial footnotes, and some of them are explained in the note below. The authors' footnotes have asterisks as footnote marks, the editorial footnotes are numbered. Other features of the original article are faithfully reproduced in this reprint.

1. Section 1, Equation (1.8). The explanation provided in the paragraph around Eq. (1.8) may be too brief for first-time readers. Here is a more detailed derivation.

A definition of a projection on a subspace locally orthogonal to a null vector k has to take into account the fact that this hypersurface contains k. Therefore, given k^a, we first define a second null vector ℓ^a that is tangent to the same light cone and obeys:

$$\ell^a k_a = 1, \quad \ell^a \ell_a = 0. \tag{1}$$

Note that ℓ^a is not defined uniquely. If m^a is an arbitrary spacelike vector of unit length ($m^a m_a = 1$) orthogonal both to k^a and to ℓ^a, then $\ell'^a = \ell^a - \frac{1}{2} b^2 k^a + b m^a$ obeys (1) as well, where b is an arbitrary parameter.

Then we define the projection tensor on the (2-dimensional) surface that is orthogonal to both ℓ^a and k^a:

$$p_{ab} = g_{ab} - \ell_a k_b - k_a \ell_b \Longrightarrow p_{ab} k^b = p_{ab} \ell^b = 0. \tag{2}$$

This surface does not include ℓ^a or k^a—because a vector that is orthogonal to two linearly independent null vectors must be spacelike.

Now assume that k^a is a null geodesic vector *field*, and it is affinely parametrised so that $k^r k_{a;r} = 0$. We then define

$$A_{ab} \stackrel{\text{def}}{=} k_{r;s} p^r{}_a p^s{}_b. \tag{3}$$

Then the following holds:

$$k_{a;b} = A_{ab} + a_a k_b + k_a b_b, \tag{4}$$

where

$$\begin{aligned} a_a &\stackrel{\text{def}}{=} \ell^s k_{a;s} - \frac{1}{2} k_{r;s} \ell^r \ell^s k_a, \\ b_a &\stackrel{\text{def}}{=} \ell^s k_{s;a} - \frac{1}{2} k_{r;s} \ell^r \ell^s k_a. \end{aligned} \tag{5}$$

It follows that $a_a k^a = b_a k^a = 0$. Now we decompose A_{ab} into the trace θ (expansion), the trace-free symmetric part σ_{ab} (shear) and the antisymmetric part ω_{ab} (rotation):

$$A_{ab} = \omega_{ab} + \sigma_{ab} + p_{ab}\theta \tag{6}$$

(for some reason, tradition requires one to write the last term without the coefficient 1/2 that would be natural here). The geometric interpretation of rotation, expansion and shear is similar to that in hydrodynamics and agrees with the everyday meaning of these words; the respective changes apply to images of an object projected by the family of light rays on 2-surfaces orthogonal to the family. Like in hydrodynamics, rotation and shear do or do not vanish simultaneously with the scalars defined below. Also, the following equations are useful:

$$\begin{aligned} &p^a{}_r p^r{}_b = p^a{}_b, \quad g^{ab} p_{ab} = p^{ab} p_{ab} = 2, \\ &p^{ab}\sigma_{ab} = g^{ab}\sigma_{ab} = 0, \\ &k^b\omega_{ab} = k^b\sigma_{ab} = \ell^b\omega_{ab} = \ell^b\sigma_{ab} = 0. \end{aligned} \tag{7}$$

The scalars of rotation, expansion and shear are then

$$\omega^2 \stackrel{\text{def}}{=} \frac{1}{2}\omega_{ab}\omega^{ab} = \frac{1}{2}k_{[a;b]}k^{a;b}, \tag{8}$$

$$\theta = \frac{1}{2}k^m{}_{;m}\,, \tag{9}$$

$$\sigma^2 \stackrel{\text{def}}{=} \frac{1}{2}\sigma_{ab}\sigma^{ab} = \frac{1}{2}k_{(a;b)}k^{a;b} - \theta^2. \tag{10}$$

As can be seen, these quantities depend only on k^a, not on the auxiliary field ℓ^a.

Now we set up a field of null vector bases over the spacetime that will include the k^a and ℓ^a. The other vectors in the basis, m^a and $\overline{m}^a$, will be complex conjugate to each other, orthogonal both to k^a and to ℓ^a, and will obey relations similar to (1):

$$\begin{aligned} g_{ab}m^a m^b = g_{ab}\overline{m}^a\overline{m}^b = 0, \quad g_{\alpha\beta}m^a\overline{m}^b = 1, \\ g_{ab}m^a k^b = g_{ab}m^a\ell^b = 0, \quad g_{ab}\overline{m}^a k^b = g_{ab}\overline{m}^a\ell^b = 0. \end{aligned} \tag{11}$$

(As explained in the paper, these basis vectors are not uniquely defined—see Eq. (1.9) of the paper. The freedom of transformations includes the non-uniqueness of ℓ^a mentioned after Eq. (1) above.) We label the basis vectors as follows (hats denote tetrad indices; in what follows the hats over tetrad indices will be omitted when there is no risk of confusion):

$$e_{\hat{3}}{}^a = k^a, \quad e_{\hat{4}}{}^a = \ell^a, \quad e_{\hat{1}}{}^a = m^a, \quad e_{\hat{2}}{}^a = \overline{m}^a. \tag{12}$$

The $e_\alpha{}^a$ coincide with the $k_\alpha{}^a$ of the paper, but we will keep the $e_\alpha{}^a$ notation to avoid confusion with the vector k^a introduced before. The matrix of scalar products among the basis vectors (the tetrad metric) is then given by (1.7) in the paper. Because of the form of this matrix, the upper and lower tetrad indices are related to each other as follows:

$$v_{\widehat{3}} = v^{\widehat{4}}, \quad v_{\widehat{4}} = v^{\widehat{3}}, \quad v_{\widehat{2}} = v^{\widehat{1}} = \overline{v_{\widehat{1}}}, \quad v_{\widehat{1}} = v^{\widehat{2}} = \overline{v_{\widehat{2}}}. \tag{13}$$

Some of the Ricci rotation coefficients[1] are related in simple ways to physical quantities, and this is one of the useful features of this tetrad. For the considerations in the paper, the following formulae are useful.

The tetrad components of the "acceleration" of k^a are

$$e_\alpha{}^a k_{a;r} k^r = \Gamma_{\widehat{3}\alpha\widehat{3}}. \tag{14}$$

Thus, if k^a is assumed geodesic and affinely parametrised, the first of (2.4) follows.

In order to calculate the expansion, we do the following operations

$$k^m{}_{;m} = g_{mn} k^{m;n} = \eta_{\alpha\beta} e^\alpha{}_m e^\beta{}_n k^{m;n},$$

where $\eta_{\alpha\beta}$ is the flat Lorentzian metric; and then we explicitly run through all the values of α and β. Most of the terms are zero or cancel out, and what remains is

$$\theta = \frac{1}{2} k^{r;s} \left(m_r \overline{m}_s + \overline{m}_r m_s\right) = \frac{1}{2} \left(\Gamma_{312} + \Gamma_{321}\right) \equiv \frac{1}{2} \left(\Gamma_{312} + \overline{\Gamma}_{312}\right). \tag{15}$$

The tetrad components of the shear tensor are found directly from (6), with use of (15) and of the orthogonality relations (11). Their only non-zero components are:

$$\begin{aligned} \sigma_{\widehat{1}\widehat{1}} &= k_{a;b} m^a m^b = \Gamma_{311} \equiv -\Gamma_{131}, \\ \sigma_{\widehat{2}\widehat{2}} &= k_{a;b} \overline{m}^a \overline{m}^b = \Gamma_{322} \equiv -\Gamma_{232}. \end{aligned} \tag{16}$$

The component $\sigma_{\widehat{1}\widehat{1}}$ defined above coincides with the complex scalar σ in (1.8) in the paper.

For the tetrad components of the rotation tensor we find

$$\omega_{12} = -\omega_{21} = \frac{1}{2} \left(\Gamma_{312} - \Gamma_{321}\right) = \overline{\omega}_{21} = -\overline{\omega}_{12}, \tag{17}$$

all other components being zero. Thus, the only non-vanishing component of the rotation tensor is pure imaginary, and, moreover, as follows from (8) and (11), it is connected to the scalar of rotation ω by ${\omega_{12}}^2 = -\omega^2$; thus $\omega_{12} = \mathrm{i}\omega$. As one can see then, the expansion and the rotation are, respectively, the real and the imaginary part of the same complex quantity

$$z \stackrel{\text{def}}{=} \theta + \mathrm{i}\omega = \Gamma_{312} \equiv -\Gamma_{132}. \tag{18}$$

[1] Their definition in the paper is not a universally accepted one, some authors define $\Gamma_{\alpha\beta\gamma}$ with the opposite sign.

Projecting (6) on all the tetrad vectors and using all of the above, Eq. (1.8) in the paper follows.

2. Section 2, Equations (2.5)–(2.6). The paragraph containing Eqs. (2.5) and (2.6) in the paper does not give a readable hint on how to derive these equations. Here is the derivation.

Take the Ricci identity for the field k^a:

$$k_{a;bc} - k_{a;cb} = R_{rabc} k^r. \tag{19}$$

In the further calculations, projections of this equation on different combinations of the tetrad vectors will be considered. The following equations will be useful ($k^a{}_{;b}\, k^b = 0$ being assumed):

$$g^{ab} k_{a;bc} = 2\theta_{,c}\,, \tag{20}$$

$$k^c g^{ab} k_{a;cb} = -k^{c;b} k_{b;c} = -2\left(\sigma^2 - \omega^2 + \theta^2\right), \tag{21}$$

$$k^c m^a \overline{m}^b k_{a;bc} = k^c \left(m^a \overline{m}^b k_{a;b}\right)_{;c} - k^c m^a{}_{;c}\, \overline{m}^b k_{a;b} - k^c m^a \overline{m}^b{}_{;c}\, k_{a;b}, \tag{22}$$

$$k^c m^a \overline{m}^b k_{a;cb} = \overline{m}^b \left(k^c m^a k_{a;c}\right)_{;b} - \overline{m}^b k^c{}_{;b}\, m^a k_{a;c}\,, \tag{23}$$

$$m^a{}_{;c}\, k_{a;b} = g^{as} m_{s;c} m_{a;b} = \eta^{\alpha\beta} e_\alpha{}^a e_\beta{}^s m_{s;c} m_{a;b}. \tag{24}$$

Equations (20) and (21) follow from (9) and (4)–(6). In simplifying the right-hand sides of (22) and (23) the definition of $\Gamma_{\alpha\beta\gamma}$ will be used, and in simplifying the right-hand side of (24) the terms corresponding to different values of α and β will be written out explicitly.

Contract (19) with $k^c g^{ab}$; the result is the equation of evolution of θ, analogous to the Raychaudhuri equation:

$$k^c \theta_{,c} + \sigma^2 - \omega^2 + \theta^2 = \frac{1}{2} R_{rc} k^r k^c. \tag{25}$$

Now contract (19) with $k^c \left(m^a \overline{m}^b - m^b \overline{m}^a\right)$ and use the fact that $R_{r[ab]s}\, k^r k^s \equiv 0$. The result is

$$k^s \Gamma_{3[12],s} + \frac{1}{2}\left[(\Gamma_{312})^2 - (\Gamma_{321})^2\right] = 0. \tag{26}$$

In consequence of (17), Eq. (26) can be written as

$$\mathrm{i} k^s \omega_{,s} + 2\mathrm{i}\theta\omega = 0. \tag{27}$$

Equations (25) and (27) can be written as one complex equation, Eq. (2.5) in the paper (actually, in the original text, the last term $+\sigma^2$ was missing, it was restored here).

Equation (2.6) in the paper follows on contracting (19) with $k^c m^a m^b$ and using (22)–(24). The complex conjugate of (2.6) also holds, and it follows from (19) contracted with $k^c \overline{m}^a \overline{m}^b$.

3. Section 2, Equation (2.10). The derivation of Eq. (2.10) involves a trick. From writing out $R_{31} = 0$ we have $R_{31} = R_{4313} + R_{1312} = 0$—and then we apply the first of (2.1) to calculate R_{1343} instead of R_{4313}. (The equality $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$ is not an identity when the rotation coefficients are used as basic objects; it has to be imposed as an extra condition.) This trick enables us to efficiently use the assumptions $\Gamma_{314} = 0$ and $\Gamma_{3\alpha 3} = 0$.

4. Section 2, after (2.10). The deductions following (2.5')–(2.10) in the paper are described somewhat too briefly, and are thus difficult to follow. Here is a more detailed presentation. We use the tetrad in which $\Gamma_{314} = 0$ and assume that the field $k = e_{\widehat{3}}$ is geodesic, affinely parametrised and shearfree ($\Gamma_{3\alpha 3} = 0 = \Gamma_{311} = \Gamma_{322}$). Also, $\Gamma_{411} = 0$ by (2.9).

Applying the integrability condition (2.12) to the set (2.5')—(2.10) one gets

$$k^s\left(\Gamma_{413} - \Gamma_{341}\right)_{,s} = -2z\Gamma_{413} + \left(\Gamma_{123} - \overline{z}\right)\left(\Gamma_{413} - \Gamma_{341}\right). \tag{28}$$

Then, from the equation $R_{31} = R^3{}_{313} + R^2{}_{312} = 0$ (in tetrad indices), using the first of (2.1), we get

$$-k^s\Gamma_{341,s} = \Gamma_{341}\left(z + \overline{z} - \Gamma_{123}\right) - \overline{z}\Gamma_{413} - m^s z_{,s}\,. \tag{29}$$

We use (29) to eliminate $k^s\Gamma_{341,s}$ from (28), and then use (2.10) to eliminate $m^s z_{,s}$ from the result. This results in Eq. (2.13) (corrected here; in the original paper the last term before 0 was absent).

Next, writing out $R_{11} = R^3{}_{113} + R^4{}_{114} = 2R_{4113} = 0$ with use of the first of (2.1), we obtain (2.14). The integrability condition of (2.13)–(2.14) leads to

$$\begin{aligned}\Gamma_{413}\left(-k^s\Gamma_{121,s} + m^s\Gamma_{123,s}\right) = \Gamma_{413}\left[\Gamma_{341}\Gamma_{123} - 9\Gamma_{413}z\right.\\ \left. + \overline{z}\left(\Gamma_{121} + \Gamma_{413}\right) + \Gamma_{123}\left(-\Gamma_{121} + \Gamma_{413}\right)\right]\end{aligned} \tag{30}$$

(in obtaining the above, one must use (2.5'), (2.10), (2.13) and (2.14) to eliminate the derivatives of z and of Γ_{413}). Writing out $R_{13} = R_{3134} + R_{2131} = 0$ with use of (2.1), we get

$$-k^s\Gamma_{121,s} + m^s\Gamma_{123,s} = \Gamma_{413}\left(z + \overline{z} + \Gamma_{123}\right) + \Gamma_{341}\Gamma_{123} - \Gamma_{121}\left(\Gamma_{123} - \overline{z}\right). \tag{31}$$

Substituting (31) in (30), we obtain $z{\Gamma_{413}}^2 = 0$, as stated in the paper. (Actually, the original paper said $-10z^2\Gamma_{413} = 0$, but the exponent was misplaced.)

5. Section 2, Equation (2.15). The details of the proof in the paragraph containing (2.15) are as follows. In the tetrad used in the paper, the conditions for the Weyl tensor to be algebraically special, with k_3 being the double Debever vector, are (in tetrad indices):

$$C_{133\alpha} = C_{233\alpha} = 0. \tag{32}$$

Using this, the tetrad components of $C^{\rho}{}_{\alpha\rho\beta} = 0$ imply

$$\begin{aligned}
-C_{3434} &= C_{3142} + C_{3241} = -C_{1212} = 0,\\
-C_{3441} &= C_{4112}, \qquad C_{3442} = C_{4212},\\
C_{3141} &= C_{3112} = C_{3242} = C_{3212} = C_{4142} = 0.
\end{aligned} \tag{33}$$

Take the Bianchi identities $R_{ab[cd;e]} = 0$, contract them with g^{bc} and use the assumption $R_{ab} = 0$. In vacuum, the result is equivalent to $C^{r}{}_{abc;r} = 0$. By the equation preceding Eq. (2.1) in the paper, this is equivalent to:

$$e_a{}^s C^a{}_{bcd,s} - \Gamma^s{}_{rs} C^r{}_{bcd} + \Gamma^s{}_{br} C^r{}_{scd} + \Gamma^s{}_{cr} C^r{}_{bsd} + \Gamma^s{}_{dr} C^r{}_{bcs} = 0. \tag{34}$$

Take the components $(b, c, d) = (3, 3, 1)$ and $(3, 3, 2)$ of this equation and use (32)–(33). Then, join to this set the first of (33) and the identity $C_{3[\beta\gamma\delta]} = 0$. When $\Gamma_{133} \neq 0 \neq \Gamma_{233}$, a linear homogeneous set of algebraic equations results, whose solution is

$$C_{3412} = C_{3142} = C_{3241} = C_{3434} = C_{1212} = 0. \tag{35}$$

Now take the components (3, 3, 4) and (3, 1, 2) of (34), with (35) assumed. Taking into account the middle part of (33), with $\Gamma_{133} \neq 0 \neq \Gamma_{233}$, the solution of that set is

$$C_{3441} = C_{3442} = C_{4112} = C_{4212} = 0. \tag{36}$$

Take the components (3, 4, 1) and (3, 4, 2) of (34), with (35) and (36) assumed. They say that $\Gamma_{233} C_{4141} = 0 = \Gamma_{133} C_{4242}$, i.e. $C_{4141} = C_{4242} = 0$. Together with (33), (35) and (36) this means $C_{abcd} = 0$. With $R_{ab} = 0$, this is the Minkowski spacetime, in which a congruence of shearfree null geodesics does exist, so the theorem is trivially true.

Thus only $\Gamma_{133} = \Gamma_{233} = 0$ needs to be considered.

Now take (34) with the sets of indices (1, 3, 1) and (2, 3, 2), and use $\Gamma_{133} = \Gamma_{233} = 0$. Then join the first of (33) and again $C_{3[\beta\gamma\delta]} = 0$ to this set. If $\Gamma_{311} \neq 0 \neq \Gamma_{322}$ then (35) results once more.

The components (4, 3, 1) and (4, 3, 2) of (34) now say that $\Gamma_{311} C_{3442} = \Gamma_{322} C_{3441} = 0$. With the assumed $\Gamma_{311} \neq 0 \neq \Gamma_{322}$, and with (33), this implies (36) again. Then, the components (4, 3, 4) and (4, 1, 2) of (34) give a set of 2 equations, whose solution, with $\Gamma_{311} \neq 0 \neq \Gamma_{322}$ is $C_{4141} = C_{4242} = 0$, i.e. again $C_{abcd} = 0$. Thus the non-trivial solution of the set considered here is $\Gamma_{311} = \Gamma_{322} = 0$, which, as seen from (16), means that the shear of the k^a congruence is zero.

Part 2: Further developments

By Maciej Przanowski

The classical Goldberg–Sachs theorem found its important generalizations to the case of non-vacuum "physical" (i.e. Lorentzian) spacetimes as well as its counterparts in the cases of 4-dimensional Riemannian manifolds of non-Lorentzian signature or 4-dimensional complex analytic Riemannian manifolds called the *complex spacetimes*.

(a) *"Physical" (Lorentzian) spacetime* The most famous, perhaps, is here the Kundt–Thompson theorem [5,6]

THEOREM (Kundt–Thompson) *For a spacetime V_4 (i.e. a 4-dimensional differential manifold endowed with the metric of signature $+2$) two of the following statements yield the third*

(A) *The Weyl tensor is algebraically special in the sense of the Petrov classification*
(B) *There exists a shear-free null geodesic congruence*
(C)

$$\begin{aligned} V^{ea}V^{bc}C^{d}{}_{abc;d} &= 0 \quad \textit{for the types II or D,} \\ V^{bc}C^{d}{}_{abc;d} &= 0 \quad \textit{for the type III,} \\ V^{ea}C^{d}{}_{abc;d} &= 0 \quad \textit{for the type N} \end{aligned}$$

where C_{abcd} is the Weyl tensor, V_{ab} is a complex null bivector such that $V_{ab}k^b = 0$ with k^a being the multiple principal null vector in case (A) *and the vector tangent to the shear-free null geodesic congruence in case* (B)*, and ";" stands for the covariant derivative.* □

It is evident that the "generalization" is expressed by point (C); in this point some conditions on the covariant derivative of the Weyl tensor are imposed, but it is not assumed that the respective spacetime is vacuum, as it was in the classical Goldberg–Sachs theorem.

Consequently, employing the Einstein equations and also the field equations describing the sources of gravitational field one obtains a significant information about the spacetime geometry (connection form) [7–12] which in many cases simplifies the analysis of the equations and enables one to get explicitly the spacetime metric. In particular, there are distinguished examples of this procedure in the cases of the Einstein–Maxwell equations or the Einstein equations with pure radiation field [3].

(b) *Complex spacetime* Complex spacetime, which is the main object of *complex relativity*, is a 4-dimensional complex analytic manifold endowed with a holomorphic metric.

Complex relativity has attracted a great deal of interest for many years. First, a complex spacetime was found by Newman [13,14] as a space of "good cuts" for asymptotically flat Lorentzian spacetimes. Such a complex spacetime has been called

$\mathcal{H}$*-space* or *heavenly space.* Then, complex spacetimes emerge in a natural way in the twistor program of Penrose [15,16]. $\mathcal{H}$-space appears here as a non-linear graviton. Finally, the program has been proposed by J. Plebański in which the "physical" spacetime metrics could be obtained from the complex metrics by taking the real slices.

This promising program was justified by two outstanding results:

(i) the reduction of the $\mathcal{H}$-space equations ($R_{ab} = 0$, $*C_{abcd} = \pm C_{abcd}$; where $*$ denotes the Hodge duality operation) to a single second order non-linear partial differential equation (the *first* or the *second heavenly equation* ($\mathcal{H}$*-equation*)) for one holomorphic function [17],

(ii) the reduction of the $\mathcal{HH}$-space equations ($R_{ab} = 0$, $C_{abcd} + *C_{abcd}$ or $C_{abcd} - *C_{abcd}$ is algebraically special) to a single second order non-linear partial differential equation ($\mathcal{HH}$*-equation*) for one holomorphic function [18,19].

[*Remark* Recently we learned that the first heavenly equation was found in 1935 by two Japanese mathematicians from the Hiroshima University, Sibata and Moringa [20]. Of course, J. Plebański was not aware of this fact].

Looking at the considerations leading to the $\mathcal{H}$ or $\mathcal{HH}$-equations one quickly realizes that the crucial point lies in a complex version of the Goldberg–Sachs theorem given by Plebański and Hacyan [21]

Before we cite that version some remarks are needed.

The Weyl tensor can be split into its self-dual and anti-self-dual parts as follows

$$C_{abcd} = \frac{1}{2}(C_{abcd} + *C_{abcd}) + \frac{1}{2}(C_{abcd} - *C_{abcd}).$$

Unlike in the Lorentzian spacetimes, where the self-dual part $\frac{1}{2}(C_{abcd} + *C_{abcd})$ is complex conjugated to the anti-self-dual part $\frac{1}{2}(C_{abcd} - *C_{abcd})$, in the case of complex spacetimes these two objects are independent. Consequently one can consider independently the Petrov algebraic classification of the self-dual part and the anti-self-dual part of the Weyl tensor.

Now we need also the notion of a *self-dual* (*anti-self-dual*) *null string* or α (β, resp.) *surface* according to the Penrose twistorial approach.

The self-dual (anti-self-dual) null string is a totally null 2-dimensional complex surface Σ in the complex spacetime $V_4^{\mathbb{C}}$ such that the null bivector V_{ab}, i.e., the simple 2-form V_{ab}, orthogonal to Σ is self-dual (anti-self-dual, resp.). One can quickly show that Σ is also a totally geodesic complex 2-surface.

Then, the complex version of the Goldberg–Sachs theorem can be stated as follows:

THEOREM (Plebański–Hacyan) *The self-dual* (*anti-self-dual*) *part of the Weyl tensor of a vacuum complex spacetime* $V_4^{\mathbb{C}}$, $R_{ab} = 0$, *is algebraically special with* k^a *being the multiple principal null vector if and only if for each point* $p \in V_4^{\mathbb{C}}$ *there exist an open neighbourhood* U *of* p *and a congruence of self-dual* (*anti-self-dual, resp.*) *null strings on* U. □

Thus, in the complex case a congruence of null strings plays the role of the shear-free null geodesic congruence of the Lorentzian spacetime.

There exist some generalizations of the Plebański–Hacyan theorem to the non-vacuum complex spacetime, $R_{ab} \neq 0$, which are analogous to the Kundt–Thompson theorem [12]

The Plebański–Hacyan theorem and its non-vacuum generalizations can be immediately carried over to the case of a real 4-dimensional Riemannian manifold of the signature $(+ + --)$. The only difference lies in the fact that in this last case the geometrical objects are real.

(c) *Euclidean spacetime* It is worthwhile to note that there exists a counterpart of the classical Goldberg–Sachs theorem in the case of 4-dimensional Riemannian manifolds of Euclidean signature $(+ + ++)$ (*Euclidean spacetimes*). Such spaces have attracted considerable interest in quantum gravity where they appear as the so called *gravitational instantons* [22].

It was shown that also here the Goldberg–Sachs theorem enables us to reduce the Einstein field equations [23–25]. This theorem reads now [23,24]:

THEOREM *The self-dual or anti-self-dual part of the Weyl tensor of the vacuum Euclidean spacetime V_4^E, $R_{ab} = 0$, is algebraically special if and only if V_4^E is a locally Hermitian manifold i.e., for each point $p \in V_4^E$ there exist an open neighbourhood U of p and two complex coordinates (z^1, z^2) on U such that the metric ds^2 of V_4^E takes the following form on U*

$$ds^2 = g_{\alpha\bar{\beta}}(dz^\alpha \otimes dz^{\bar{\beta}} + dz^{\bar{\beta}} \otimes dz^\alpha)$$

$\overline{g_{\alpha\bar{\beta}}} = g_{\beta\bar{\alpha}}$, $\alpha, \beta = 1, 2$ *and* $dz^{\bar{\beta}} := \overline{dz^\beta}$, *where the overbar stands for the complex conjugation.* □

Last but not least, one meets an application of the Goldberg–Sachs theorem when the so-called *Cauchy–Riemann structure* on a Riemannian manifold is considered. It has been found that there exists a close relation between the Cauchy–Riemann structure on a 4-dimensional Riemannian manifold and the Goldberg–Sachs theorem [26,27].

Concluding our note we would like to point out the universality of the Goldberg–Sachs theorem. We can see that this theorem plays an important role not only in standard relativity, but also in the case of any 4-dimensional Riemannian manifold giving a deep insight into its geometry.

Acknowledgments One of us (MP) is indebted to Sebastian Formański for his assistance and interest in this note.

Joshua N. Goldberg: a brief biography

By Josh Goldberg

I was born on May 30, 1925 in Rochester, NY and had my early education there. After serving for 2 years in the US Navy during WWII, I returned to the University of

Rochester and received a BS in 1947. At Syracuse University, I received an MS (1950) for a thesis on a problem in molecular beams and a PhD (1952) for a dissertation on conservation laws, invariance, and equations of motion. Both were supervised by Peter Bergmann. Following this, I worked for four years at a non-profit research institute on various problems of interest to the Defense Department. While there, I wrote a paper showing that gravitational radiation shows up in the 11th order in v/c (3 1/2 PN). By coincidence, my next position was with the US Airforce at Wright–Patterson Airforce Base. There I was able to have a small group doing research on general relativity as well as a fund for support of research at various universities in the US and abroad. In 1960–1961, I was an NSF Fellow with Hermann Bondi at King's College. Ray Sachs was there at the time and we collaborated on the paper establishing the necessary and sufficient conditions that a solution of the Einstein equations be algebraically special. In 1963, I left the Airforce and became Professor of Physics at Syracuse University. My principal research has been in the areas of conservation laws, equations of motion, gravitational radiation, and topics related to quantum gravity. Since retirement in 1995, I have kept an office in the Physics Department so that I keep up with major developments in general relativity, cosmology, and astrophysics. However, I also am the grant seeker for the Syracuse Friends of Chamber Music as well as a member of the board of the local chapter of the New York Civil Liberties Union.

The biography of Rainer Sachs was published together with the Sachs–Wolfe Golden Oldie in *Gen. Relativ. Gravit.* **39**, 1941 (2007), article DOI 10.1007/s10714-007-0448-9.

References

1. Ellis, G.F.R.: Relativistic cosmology, In: Sachs, R.K. (ed.) Proceedings of the International School of Physics 'Enrico Fermi'. Course 47: General Relativity and Cosmology, pp. 104–18. Academic Press, New York (1971) (To be reprinted in Gen. Relativ. Gravit. as another Golden Oldie in due time)
2. Kerr, R.P., Schild, A.: A new class of vacuum solutions of the Einstein field equations. In: Atti del Convegno sulla Relatività Generale: Problemi dell'Energia e Onde Gravitazionali. G. Barbèra, Firenze (1965) (To be reprinted in Gen. Relativ. Gravit. as another Golden Oldie in due time)
3. Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C., Herlt, E.: Exact Solutions of Einstein's Field Equations. Cambridge University Press, Cambridge (2003)
4. Plebański, J., Krasiński, A.: An Introduction to General Relativity and Cosmology. Cambridge University Press, Cambridge (2006)
5. Kundt, W., Thompson, A.: C. R. Acad. Sci. (Paris) **254**, 4257 (1962)
6. Robinson, I., Schild, A.: J. Math Phys. **4**, 484 (1963)
7. Mariot, L.: C. R. Acad. Sci. (Paris) **238**, 2055 (1954)
8. Mariot, L.: C. R. Acad. Sci. (Paris) **239**, 1189 (1954)
9. Robinson, I.: J. Math Phys. **2**, 290 (1961)
10. Szekeres, P.: J. Math. Phys. **7**, 751 (1966)
11. Plebański, J.F.: Spinors, Tetrads and Forms. Unpublished monograph of CINVESTAV, Mexico City (1974)
12. Przanowski, M., Plebański, J.F.: Acta Phys. Pol. B **10**, 573 (1997)
13. Newman, E.T.: In: General Relativity and Gravitation. Shaviv, G., Rosen, J. (eds.) Wiley, New York (1975)
14. Ko, M., Ludvigsen, M., Newman, E.T., Tod, K.P.: Phys. Rep. **71**, 51 (1981)
15. Penrose, R.: Gen. Relativ. Gravit. **7**, 31 (1976)

16. Penrose, R., Ward, R.S.: Twistors for flat and curved space-time. In: Held, A. (ed.) General Relativity and Gravitation, vol. 2, pp. 283–328. Plenum Press, New York (1980)
17. Plebański, J.F.: J. Math. Phys. **16**, 2395 (1975)
18. Plebański, J.F., Robinson, I.: Phys. Rev. Lett. **37**, 493 (1976)
19. Boyer, C.P., Finley, J.D. III, Plebański, J.F.: Complex general relativity $\mathcal{H}$ and $\mathcal{HH}$-spaces—a survey of one approach. In: Held, A. (ed.) General Relativity and Gravitation, vol. 2, pp. 241–281. Plenum Press, New York (1980)
20. Sibata, T., Moringa, K.: J. Sci. Hiroshima Univ. **5**, 173 (1935)
21. Plebański, J.F., Hacyan, S.: J. Math. Phys. **16**, 2403 (1975)
22. Gibbons, G.W., Hawking, S.W. (eds.): Euclidean Quantum Gravity. World Scientific, Singapore (1993)
23. Hacyan, S.: Phys. Lett. A **75**, 23 (1979)
24. Przanowski, M., Broda, B.: Acta Phys. Pol. B **14**, 637 (1983)
25. Przanowski, M., Baka, B.: Gen. Relativ. Gravit. **16**, 797 (1984)
26. Trautman, A.: On complex structures in physics, In: On Einstein's Path, Essays in Honor of Engelbert Schucking. Harvey, A. (ed.) Springer, New York, pp. 487–501 (1999)
27. Nurowski, P., Trautman, A.: Diff. Geom. Appl. **17**, 175 (2002)