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Editor's Note: An Example of a New Type of Cosmological Solutions of Einstein's Field Equations of Gravitation.

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Rotating Universes in General Relativity Theory

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Kurt Gödel became interested in general relativity theory while he and Einstein were both on staff of the Institute for Advanced Studies in Princeton, and saw a lot of each other (Ref. 25, p.7, Ref. 19, p.157). Gödel's resultant two technical papers were highly original, and had a major impact. Indeed (see Ref. 38, p.111-112, Refs. 29,8) the beginning of the modern studies of singularities in general relativity in many ways had its seeds in the presentation by Gödel in 1949 [10] of an exact solution of Einstein's equations for pressure-free matter, which could be thought of as a singularity-free rotating but non-expanding cosmological model. This was one of the papers presented in a special issue of *Reviews of Modern Physics* dedicated to Einstein on his 70th birthday. Gödel used this space-time as an example helping to clarify the nature of time in general relativity, for it is an exact solution of the Einstein equations in which there are closed timelike lines. He shortly thereafter published a further paper [12] discussing a family of exact solutions of Einstein's equations representing rotating and expanding spatially homogeneous universe models. As these permit non-zero redshifts, they could include realistic models of the observed universe. These papers stimulated many investigations leading to fruitful developments. This may partly have been due to the enigmatic style in which they were written: for decades after, much effort was invested in giving proofs for results stated without proof by Gödel.

Gödel's first paper [10] gave an exact rotating fluid-filled cosmological solution of Einstein's gravitational field equations. It is uniquely characterized by its symmetry properties: it is the only perfect-fluid filled universe invariant under a G_5 of isometries multiply transitive on space-time; it is therefore locally rotationally symmetric [6]. The density μ and pressure p are the same everywhere, and hence [5,7] it does not expand ($\theta = 0$) and matter moves geodesically ($\dot{u}_a = 0$). It also has zero shear ($\sigma = 0$), so the matter velocity vector is a Killing vector field but is not hypersurface orthogonal, and the only non-zero kinematic quantity is the vorticity. The vorticity vector is covariantly constant. This kinematic description uniquely characterizes these space-times (Ref. 5, Theorem 1.5.2 and 2.5.4). The matter source in the original solution is pressure-free matter, but there is a cosmological constant λ of negative sign (the opposite sign to that usually encountered). More generally one can regard the matter source as being a perfect fluid. The non-trivial covariant field equations are then

$$\lambda + 2\omega^2 = \tfrac{1}{2}\kappa(\mu + 3p), \qquad \lambda = \tfrac{1}{2}\kappa(-\mu + p). \tag{1}$$

In the pressure-free case (considered in Gödel's original paper)

$$\lambda = -\tfrac{1}{2}\kappa\mu = -\omega^2 < 0\,. \tag{2}$$

One can alternatively represent the matter as a fluid or scalar field without cosmological constant: then

$$\lambda = 0 \;\Rightarrow\; p = \mu = \omega^2/\kappa\,. \tag{3}$$

In the introduction to this paper, Gödel says that no solution with rotating matter source had been known up to that time. This is not quite correct. A cylindrically symmetric stationary dust solution was found by Lanczos in 1924, and this is the oldest solution with rotating source that the editors are aware of [21]. Lanczos never mentioned the word "rotation" in his paper, and he may not have been aware that the matter in his model was rotating. Van Stockum [39] rediscovered this solution in 1937, see the remarks on his paper in Ehlers [5], and was aware it was rotating. Further, the notion of rotation/vorticity in relativity had been investigated in some depth by Synge [35]. Thus solutions with nonzero rotation had existed since 1924, even though they were not properly understood and were not generally known; certainly they were not mentioned by Gödel.

Gödel used his exact solution of the Einstein equations to examine properties of time and causality in general relativity. The essential point he demonstrated is that the Einstein field equations for a fluid matter source are compatible with causal violation. Using axially symmetric comoving coordinates, the light cones tip over more and more the further one moves out from the origin of coordinates (Figure 31 in Ref. 13), so that for large enough radial distance from the origin, there are closed circular timelike lines.

Various paradoxes ensue.[1] Furthermore by traveling far enough away, any observer can reach an arbitrarily distant event in the past on her own world line, and so influence events in her own past history at an arbitrary early proper time in that history. Because the universe is space-time homogeneous, there are closed timelike curves through every event (the causal violation is not localized to some small region). It must be emphasized that this breakdown of causality does not occur because of multiple-connectivity of the space-time. The Gödel universe is simply connected (indeed it is homeomorphic to R^4), so the closed timelike lines are essential in that they cannot be removed by going to a covering space.

A necessary condition that causal violation can occur is that there exist no cosmic time, that is, no time function which increases in the future direction along every (timelike) world line. Gödel demonstrated that no such time function exists in these models, indeed he showed there are no inextendible spacelike surfaces at all in this space-time (on attempting to extend them, they necessarily become null and then timelike). This is possible because of the cosmic rotation signaled by the non-zero vorticity (for if the vorticity were zero, there would be a potential function for the fluid flow vector field that would provide a cosmic time function). However not all rotating universes admit causal violation; it occurs here because of the uniform extent of the rotation (it does not die away at infinity).

Gödel did not describe the geodesic properties of this space-time, but may have investigated them (see pp.560-1 and footnote 11 in Ref. 11). Later investigations by Kundt [20] and Chandrasekhar and Wright [2] explicitly showed that there are no closed timelike geodesics in the Gödel universe. This is compatible with Gödel's results because the closed timelike lines he found are non-geodesic. The past null cone of each point on the co-

[1] See The Matricide Paradox in Ref. 37, pp.508–9, and the related but simpler billiard ball paradoxes, pp.509–515.

ordinate axis, generated by the null geodesics through that point, diverges out from there to a maximum radius r_m where closed (non-geodesic) null lines occur and it experiences self-intersections, and then reconverges to the axis [13]. No timelike or null geodesic ever reaches further from its starting point than r_m. This study of geodesics also showed that these space-times are geodesically complete (and so singularity-free). This means that this universe is an example of an Anti-Mach metric (see Refs. 17,24,32,1).

At the end of his paper, Gödel related his solution to the rotation of galaxies, comparing observed rotation rates in a paper with the vorticity in his solution. He acknowledged that his solution was not a realistic universe model, in that it does not expand (and so cannot explain the observed galactic redshifts). Nevertheless it is interesting that he made some attempt to relate it to astrophysical observations of galactic rotation by E. Hubble, estimating ω and a value of 10^{-30}gm/cc for the density of matter, presumably also obtained from Hubble's data. This section clearly shows Gödel functioning in the mode of an applied mathematician (comparing observational data with model parameters to check the validity of a universe model) rather than logician.

Some while after the publication of this solution, Heckmann and Schücking showed there is an exact Newtonian analogue of the solution [15], characterized by rigid rotation, provided one drops the usual Newtonian boundary conditions for the gravitational potential. Clearly there is in this case no implication of causal violation, which is not possible in Newtonian space-time; but this does give a Newtonian example of an anti-Mach cosmology.

Gödel's stationary rotating universe is not a viable model of the real universe because in it the galaxies show no systematic redshifts [10]. Apparently Gödel must now have put a great deal of effort into examining properties of more realistic universe models that both rotate and expand. The results were presented at an International Congress of Mathematics held at Cambridge (Massachusetts) from 30th August to 5th September 1950 [12]. This represents the first explicit construction of spatially homogeneous expanding and rotating cosmological models. They are invariant under a non-abelian G_3 of isometries simply transitive on spacelike surfaces. These are now called *Bianchi universes* [17,9,22], because the classification of the 3-dimensional symmetry group transitive on the homogeneous 3-spaces is derived from that introduced much earlier by L. Bianchi, based on the structure constants of the symmetry group.

The models examined by Gödel belong to the Bianchi IX family, invariant under the group $SO(3)$, and consequently with compact spacelike surfaces of homogeneity. The matter content is taken to be pressure-free matter ('dust'). The space-times are rotating solutions ($\omega \neq 0$) with the usual space-time signature, satisfying the further conditions I–III in his paper. The last condition implies that the models are expanding. In order that vorticity be non-zero, the models are *tilted*, i.e. the matter flow lines are not orthogonal to the surfaces of homogeneity [18]. The paper argues that these conditions allow only the Type IX group as the group of isometries, and introduces a decomposition of the metric tensor into projection tensors along and perpendicular to the fluid flow lines, that has become fundamental in later work, as well as the idea of an expansion quadric (what is now called the expansion tensor). Gödel stated, mainly without proof, a number of interesting properties of these space-times, which remain interesting cosmological models today.

On the one hand, he developed relations between vorticity and the local existence of time functions determining simultaneity for a family of observers: $\omega = 0$ implies the local existence of a time function defining simultaneity for all fundamental observers [5,7], and so $\omega \neq 0$ implies tilt [18] and an anisotropy in source number counts. He estimated the size of this anisotropy and went on to develop vorticity conservation relations,[2] and gave the condition for the vorticity vector to be parallel propagated along the matter flow lines (it must be an eigenvector of the shear tensor, cf. Ref. 5), relating this to the axes of rotation of galaxies. Further, he linked these local studies to the global topology and the existence of closed timelike lines: provided the matter flowlines themselves do not close up, spatial homogeneity precludes closed timelike lines, but if the surfaces of homogeneity are timelike then closed timelike lines will occur (because these surfaces are compact).

On the other hand, he gave some dynamical results that are deeper in that they involve a detailed study of the Einstein field equations (rather than just the kinematic identities that are the basis of the vorticity conservation results, see Ref. 5). First, he considered the locally rotationally symmetric ('LRS') cases, showing there exist no LRS cases satisfying the stated conditions. Second, he stated that there are no expanding and rotating spatially homogeneous type IX universes with vanishing shear. Third, he stated existence of stationary homogeneous rotating solutions with finite space, no closed timelike lines, and positive cosmological constant ($\lambda > 0$), in particular such as differ arbitrarily little from Einstein's static universe; but that there exist no stationary homogeneous solutions with $\lambda = 0$. These results however are almost an afterthought; the reason is that such models are unrealistic, for they cannot expand on average.

[2] Partly implied in previous work by Synge [35].

Gödel gave only the briefest of hints as to how he proved the dynamic results. Because of the symmetry of these space-times, the Einstein field equations reduce to a system of ordinary differential equations. He did not give those equations, but he gave a Lagrangean function from which they could be derived, and stated an existence theorem.

This paper by Gödel is enigmatic, because the proofs of some of the major results are only sketched in the briefest manner; the material is presented in a somewhat random order; and it is sparse on references.[3] Nevertheless it was a profound contribution to theoretical cosmology.

These papers lead to an in-depth reconsideration of the nature of time and causality in relativity theory, developed particularly by Penrose, Carter, Geroch, and Hawking, that were crucial in the later studies of causality and singularities: specifically, the following emerged:

(i) the idea of causal domains,

(ii) a series of causality conditions of increasing strength generalizing and completing Gödel's statement on the relation between time functions and causality,

(iii) the broad idea of null boundaries of causal domains, and an understanding of their properties.

These ideas are discussed in broad outline in [38]; they are presented in technical detail in [27,28] and [13].

The papers also resulted in a series of studies that greatly expanded our understanding of the dynamics of universe models, extending and in many cases completing the work initiated by Gödel. They initiated systematic analysis of the family of Bianchi universe models. Taub [36] developed the equations for empty Bianchi universes with arbitrary group type, and gave an enlightening study of their properties. Heckmann and Schücking [17] extended the equations to a study of fluid-filled Bianchi models, initiating the systematic study of this class of models. This has become an important topic of study in terms of providing a parametrized set of alternative models to the standard Friedmann–Lemaître models of cosmology [40]. These analyses were extended to the case of Newtonian cosmology by Heckmann and Schücking [15,16], see also Raychaudhuri [31]. An interesting aspect is that the particular family of models investigated by Gödel (Bianchi Type IX) were shown by Misner [23] to exhibit strongly oscillatory behaviour at early times; an ongoing debate has considered if these models are truly 'chaotic' in the mathematical sense as understood today (see Hobill in Ref. 40 and Ref. 4).

Additionally, the local covariant analysis of dynamics of cosmological

[3] Indeed the only reference is to his own paper, Ref. 11.

models developed from Gödel's second paper, utilising and extending his use of the projection tensors and his analyses of vorticity and the expansion tensor [5,7]. A proof of his theorem on shear-free motion was given for the general homogeneous case by Schücking [33] and then extended to the general inhomogeneous dust case by Ellis [6]. Extension to various perfect fluid cases followed, see Collins [3] for a summary.

Perhaps most significant of all, Gödel's paper seems to have been influential in the formulation of Raychaudhuri's fundamentally important equation, giving the rate of change of the volume expansion along fluid flow lines in terms of the fluid shear, rotation, and matter content [30,5]. This and its null analogue, together with topological methods embodying a study of causal boundaries, became a crucial component in the Hawking-Penrose singularity theorems [26,14,13]. A detailed discussion of these various influences is given in Ellis [8]. Gödel's work also led to a reconsideration of the nature of time in relativity from a more philosophical viewpoint; see particularly his interchange with Einstein in Ref. 34, pp. 27–29,65–67,687–688. The discussion continues today in many recent works on causal violation in relation to wormholes.

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Short biography

Kurt Friedrich Gödel was born on 28th April 1906 in Brno (then called Brünn), Czech Republic (then part of the Austro-Hungarian empire), the younger son of an Austrian-German couple. He graduated from the Deutsches Staats-Realgymnasium in Brno in 1924, and then studied at the University of Vienna. He initially hesitated between mathematics and physics, but soon chose logic as his main area of activity. He finished his PhD Thesis in 1929, at the age of 23. In the period 1929–39 he created his most important works that established his position as the No 1 logician of the 20th century. His famous incompleteness theorem became the subject of his Habilitationsschrift in 1932.

In the 1930s Gödel visited the newly-established Institute for Advanced Study in Princeton three times as a postdoctoral fellow, and in 1940 he moved there to take up permanent residence. His decision to leave Austria was provoked by the Nazis who had found him politically unreliable and had put his position at the University under review, but, nevertheless, found him fit for military service and would very likely have drafted him. The journey across the Atlantic was too risky at that time, so Gödel and his wife travelled by railway all through Siberia to Manchuria, and then by ship to Yokohama and to San Francisco. His position in Vienna was approved three months later, but he never returned to Europe and he consistently refused to accept any Austrian honours later in his life ("sometimes for mindboggling reasons", as one of his biographers remarked; Ref. 1). From 1940 he was an Ordinary Member of the Institute for Advanced Study, and from 1946 a Permanent Member. While there, he became a personal friend of Einstein's and briefly engaged in research on relativity that resulted in the two papers reprinted here. He became Professor in 1953.

The honours he received include the Einstein Award (1951) and memberships of the National Academy of Sciences of the USA (1955), the American Academy of Arts and Sciences (1957) and of the Royal Society (1968).

He retired in 1976 and died on 14th January 1978 in Princeton Hospital.

Gödel is best known for his proof of incompleteness of arithmetic formulated as an axiomatic system: there exists a statement A such that neither A nor its negation can be proven to be true by working from the axioms. Impressed by this, John von Neumann once called Gödel "the greatest logician since Aristotle" [1]. Gödel's other main subjects of work were axiomatic set theory, philosophy and metaphysics. His research in relativity (1947–1951) was just a brief excursion, but the results have proven to be of durable importance.

Gödel's life and work are extensively described in Refs. 1–3; in particular, Ref. 3 is a complete edition of all his works, including the unpublished notes.

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based on Ref. 1