General Relativity and Gravitation, Vol. 31, No. 10, 1999

Editor's Note: On a Class of Solutions of the Gravitation Equations of Relativity

by B. Datt

Zeitschrift für Physik **108, 314 (1938).**

Presented in author's own way, the results of this paper do not seem conspicuous. However, one of them was at least 30 years ahead of its time, and the physics community of the 1930s (including the author himself) was totally unprepared to recognize its importance. This is the solution from Section 7 (the one "with little physical significance"). Rewritten in a notation that brings out its geometrical interpretation and relation to solutions found later, the metric is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \lambda^2(t,r)\mathrm{d}r^2 - \Phi^2(t)(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2), \tag{1}$$

where λ stands for Datt's $e^{\lambda/2} = y$ and Φ stands for Datt's $e^{g/2} = x^2$. The source in the Einstein equations is dust, and the function $\Phi(t)$ is determined by the equation

$$\Phi_{,t}{}^2 = -1 + 2M/\Phi, \tag{2}$$

where M is an arbitrary constant. The solution of eq. (2) above is given by Datt's eq. (19) (where $\alpha^2 = 2M$). The function λ is given by Datt's equation (21), which, in the notation of (1)-(2) above, is

$$\lambda = 2X(r)(1 - Z\cot Z) + Y(r)\cot Z, \tag{3}$$

where $Z := \arcsin[\Phi/(2M)]^{1/2}$, and $X(r)$ and $Y(r)$ are arbitrary functions. The mass-density is given by eq. (9') in the paper, which, in the notation of (1)-(3) above, simplifies to

$$\frac{8\pi G}{c^2}\rho = \frac{2X}{\Phi^2\lambda}. \tag{4}$$

0001-7701/99/1000-1615$16.00/0

This solution is an inhomogeneous (but spherically symmetric) generalization of the now-famous Kantowski-Sachs solution [1]. The latter results from (1)-(4) above when Y/X = constant. With $X = 0$, Datt's solution goes over into the Schwarzschild metric extended into the inside of the event horizon (to see this, Φ should be chosen as the time coordinate, and $r' = \int Y \mathrm{d}r$ as the radial coordinate).

Datt's solution was generalized for nonzero cosmological constant by Ruban [2] in 1969. The $\beta' = 0$ subfamily of the Szekeres solutions [3] reduces to (1)-(4) in the limit of spherical symmetry. This solution is related to the Lemaître-Tolman (LT) solution [4]. The latter follows as a solution of the Einstein equations with a dust source if the function Φ in the metric (1) above depends also on r. Then it follows that $\lambda = \Phi,_r / (1 + 2E)$, where $E(r)$ is an arbitrary function, and the limit $\Phi,_r \longrightarrow 0$ cannot be taken in this solution because it leads to a singular metric. However, if $\Phi,_r = 0$ is assumed from the beginning, then (1)-(4) above results. There exists a reparametrization of the LT solution after which the Datt solution becomes a regular limit — see Ref. 5 (in Ref. 5 the reparametrization is done for the Szekeres solutions [3], but both the LT and the Datt solution are spherically symmetric subcases of the Szekeres solutions). For more on the relationship between (1)-(4) and various other solutions see Ref. 6.

The geometrical and physical meaning of Datt's solution was explained by Ruban in Refs. 2,7 and 8 — see also Ref. 6. The solution can be matched to the Schwarzschild solution, and then the constant M from eq. (2) above is equal to the Schwarzschild mass parameter M. However, this active gravitational mass does not depend on the amount of rest-mass contained in the source, which is seen from (1) and (4) above to be equal to $\mathcal{M}(r) = 4\pi \int_{r_1}^{r_2} \rho \Phi^2 \lambda \mathrm{d}r = (c^2/G) \int_{r_1}^{r_2} X \mathrm{d}r$, and is an increasing function of r. Hence, Datt's solution matched to the Schwarzschild solution is an "ideal gravitational machine" that converts the entire rest-mass of accreted matter into radiation energy and leaves the active gravitational mass unchanged; the latter is determined by initial conditions. As also observed by Ruban in Refs. 2 and 7, the Datt solution has no analogues in Newtonian theory and does not emerge in linear approximations to Einstein's theory.

Most of the other solutions from the paper are not interesting and none of them were even new in 1938. The solution from Section 5 is a coordinate transform of the LT solution. This solution gave rise to a very large number of important and interesting papers on cosmology, singularities and several other aspects of relativity (see Ref. 6), but Datt's contribution would not count among them. Of the five solutions from Section 6, solution (a) is the Minkowski metric in spherical coordinates, solution (b) is also the Minkowski metric in a more elaborate disguise, solution (c) is the spatially flat Friedmann dust, solution (d) is a simple coordinate transform of (b) and solution (e) is the spatially flat limit of the self-similar subcase of the LT solution — see Ref. 6 for details.

As can be seen from author's Ref. 5, activity aimed at modelling the Universe in relativity was well under way in those days. In the papers by de Sitter, by Einstein and de Sitter, and in the second paper by Narlikar and Moghe, models of the Robertson-Walker class were considered. In the paper by Delsarte, properties of manifolds were discussed whose Ricci tensor obeys $R_{,ij} = \varphi_{,ij}/\varphi$, where φ is a scalar. In the first paper by Narlikar and Moghe, five spherically symmetric inhomogeneous perfect fluid solutions were found, one of which is a generalization of a spatially flat Robertson-Walker metric with an unknown equation of state — see Ref. 6 for more details. Datt's list in his Ref. 5 is by far incomplete. Ref. 6 contains a more complete account, but it does not include solutions with no Robertson-Walker limit.

From the description above it follows that the author has chanced upon a gem and has not appreciated it himself. The moral seems to be that it is easier to find a solution of Einstein's equations than to interpret it, and that the latter is more important than the former. The undersigned wishes to bring this maxim to the attention of authors who contribute their papers to this journal.

REFERENCES

1. Kantowski, R., and Sachs, R. K. (1966). *J. Math. Phys.* **7**, 443.
2. Ruban, V. A. (1969). *Zhurn. Eksper. Teor. Fiz.* **56**, 1914 [(1969). *Sov. Phys. JETP* **29**, 1027].
3. Szekeres, P. (1975). *Commun. Math. Phys.* **41**, 55.
4. Lemaître, G. (1933). *Ann. Soc. Sci. Bruxelles*, **A53**, 51; English translation (1997). *Gen. Rel. Grav.* **29**, 641.
5. Hellaby, C. (1996). *Class. Quantum Grav.* **13**, 2357.
6. Krasiński, A. (1997). *Inhomogeneous Cosmological Models* (Cambridge University Press, Cambridge).
7. Ruban, V. A. (1968). *Pisma v Red. Zhurn. Eksper. Teor. Fiz.* **8**, 669 [(1968). *Sov. Phys. JETP Lett.* **8**, 414].
8. Ruban, V. A. (1983). *Zhurn. Eksper. Teor. Fiz.* **85**, 801 [(1983). *Sov. Phys. JETP* **58**, 463].

*****

The editors regret to inform the readers that no biographical note about the author could be obtained from any source available to us. We promise to keep this case open indefinitely, and we will print the biographical note at any time if we receive it. Readers who might be able to help are kindly asked to contact the editors of this journal.

— *Andrzej Krasiński*