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GOLDEN OLDIE EDITORIAL

Editorial note to: Brandon Carter, Black hole equilibrium states Part I. Analytic and geometric properties of the Kerr solutions

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Part 1: General remarks

By Niky Kamran

Part 1 of Carter's "Black hole equilibrium states" gives a masterly account by one of the leading researchers in classical general relativity of the analytic and geometric properties of the Kerr solutions. These solutions include not only the two parameter family of Kerr solutions of the Einstein vacuum equations, but also the Kerr–Newman and Kerr–de Sitter solutions, which are the generalizations of the vacuum Kerr solutions to the cases in which either a non-singular electromagnetic field or a cosmological constant is present. The derivation of the Kerr solution and its generalizations given by Carter in Section 5 of the paper is anchored around the requirement that the metric should be stationary and axisymmetric, and that both the Hamilton–Jacobi equation for

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the non-null geodesics and the massive Klein–Gordon equation should be separable. This leads to a canonical form for the admissible metric for which Carter had solved the Einstein–Maxwell equations with cosmological constant in an earlier paper [1], yielding a six-parameter family of Petrov type D solutions which includes not only the Kerr–Newman and Kerr–de Sitter solutions, but also the Demiański–Newman solutions and their special cases (Taub-NUT, Robinson–Bertotti, etc...). In the coordinates chosen by Carter, the components of the metric are rational functions, and the action of the local isometry group is manifestly orthogonally transitive. In particular, the expression of the Kerr–de Sitter metric in a de Sitter-based Kerr–Schild form was not made explicit in [1], but this is done in Section 5 Part 1 of Carter's "Black hole equilibrium states". The maximal analytic extensions for the Kerr, Kerr–Newman and Kerr–de Sitter solutions and the corresponding conformal diagrams are motivated and constructed with great care in Sections 4 and 6 of the paper, which remains in our opinion one of the most readable expositions of these global techniques in general relativity.

Carter's paper had a significant impact on the development of the theory of exact solutions, particularly on the search for all Petrov type D solutions of the Einstein–Maxwell equations with cosmological constant. Soon after the publication of [1], Debever [2] showed that Carter's six-parameter family could be embedded in a more general class of type D solutions by weakening the separability hypotheses made by Carter to the conformally invariant Hamilton–Jacobi equation for the null geodesics. However, it is Plebański and Demiański [11] who, starting from an ansatz for the metric equivalent to the one obtained by Debever, were the first to explicitly construct a seven-parameter family of exact solutions of the Einstein–Maxwell equations with cosmological constant extending the Carter class and thereby the Kerr family of solutions to the case in which an additional acceleration parameter is present. This class contains the C metric of Levi Civita.

On a different front, shortly after the publication of Carter's work, Kinnersley [10] was able to directly integrate the Einstein vacuum equations in type D without making any a-priori assumptions about the existence of a pair of commuting Killing vectors, or any kind of separability property for the Hamilton–Jacobi equation. Kinnersley was able to prove by explicit integration that every type D vacuum metric admits a pair of commuting Killing vectors. In his integration procedure, Kinnersley chose a radial coordinate adapted to one of the repeated principal null directions of the Weyl tensor, and this led to a complicated expression for the most general type D vacuum metric involving Jacobi elliptic functions. Kinnersley's explicit integration was generalized to the case of the Einstein–Maxwell equations with cosmological constant by Debever [3], who also showed the existence of two commuting Killing vectors in that case. The question of whether the Plebański–Demiański solutions contain all the Petrov type D solutions of the Einstein–Maxwell equations with cosmological constant (with a non-singular aligned Maxwell field) remained open and called for a geometric synthesis in which an a-priori proof would be given that the type D solutions of the Einstein–Maxwell equations with cosmological constant admit a certain number of key geometric properties which had been already noted by Carter in the case of the Kerr metric, namely the existence of a two-parameter Abelian group of isometries acting orthogonally transitively, and a separable Hamilton–Jacobi equation.

This synthesis was carried out in a series of papers by Debever and McLenaghan, culminating in [7] and [6]. The metric obtained in [6] constitutes the starting point of the integration of the Einstein–Maxwell equations with cosmological constant (with a non-singular aligned Maxwell field), in type D, which was carried out by Debever et al. [5] and also by Garcia and Salazar [8,9]. It is noteworthy that in [5], a single expression is presented for the general solution in which the components of the metric are rational functions of the local coordinates. Most of the solutions thereby obtained are special cases of the seven-parameter family of Plebański and Demiański [11] and Debever [2], with the exception of a family of solutions having the property that all the orbits of the two-parameter group of local isometries are null. The explicit change of variables connecting the Kinnersley [10] and Debever [3] forms of the solutions, involving Jacobi elliptic functions, to the alternate form in which the components of the metric are rational functions was obtained by Debever and Kamran [4]. Finally, we remark that Section 8 of Carter's paper contains a very clear account of the integration of the geodesic and charged particle orbit equations by the Hamilton–Jacobi method. The existence of a fourth first integral, quadratic in the momenta and arising from an irreducible Killing tensor independent of the isometry group, is shown explicitly. The remarkable separability properties discovered by Carter for the Kerr solution have greatly stimulated the research activity in the area of separation of variables, particularly in the non-orthogonal case.

Part 2: Remarks and corrections to the text

By Andrzej Krasiński

The article reprinted here was of outstanding importance and was highly appreciated by the readers, even though the publication medium (a volume of summer school proceedings) was not quite typical. Unfortunately, the text was infested with several print-errors. Thus, whoever wanted to make use of the results reported in it, had to carefully verify all the calculations by him/herself. In this reprinting, we have corrected all the typos that we captured. By this opportunity, we also explain some details of the article that might be difficult to grasp for a first-time reader. The corrected errors are marked by editorial footnotes in the text of the paper. A few obvious typos were corrected without marking. This note provides explanations of some calculation details.

1. The fact that (5.15) implies either both (Z_r, Z_μ) or both (Q_r, Q_μ) being constant can be verified as follows. Suppose that $Z_{r,r} \neq 0$. Then (5.15) implies $Q_{\mu,\mu} = Q_{r,r} Z_{\mu,\mu} / Z_{r,r}$. Differentiating this by r we find that either $Q_{r,r}/Z_{r,r} = C =$ constant or $Z_{\mu,\mu} = 0$. In the first case we get $Q_r = CZ_r + D$ and then from (5.15) $Q_\mu = CZ_\mu + E$, with D and E being constants. This means $Z = EZ_r - DZ_\mu$, which is equivalent to Q_r and Q_μ being constant. In the second case, with $Z_{\mu,\mu} = 0$ we get $Q_{\mu,\mu} = 0$ from (5.15). By virtue of (5.13) this means that Z depends only on r, which is equivalent to $\{Z_\mu =$ constant, $Z_r = 0\}$ – a subcase of Z_μ and Z_r being constant.

Then suppose that $Z_{r,r} = 0$. Then (5.15) says that either Z_μ or Q_r is constant. With Z_μ being constant the thesis follows, with Q_r being constant Z depends only on μ, which is equivalent to $\{Z_r = \text{constant},\ Z_\mu = 0\}$.

2. To achieve (5.23) and (5.24), we carry out the following operations in Eq. (5.18):
   (1) Transform $(t, \varphi) = (\alpha t', \beta\varphi')$, where α and β are constants;
   (2) Rescale the constants and functions $(C_\mu, C_r, Z_r) = (\beta C'_\mu/\alpha, C'_r/\alpha, Z'_r/\beta)$;
   (3) Transform r and redefine the functions Δ_r and Δ_μ by $(r, \Delta_r, \Delta_\mu) = (r'/\beta, \Delta'_r/(\alpha\beta^2), \Delta'_\mu/\alpha)$.

   After this, the metric (5.18) regains the old form, but with the new constants (C'_r, C'_μ) in place of the old (C_r, C_μ). Choosing $\beta = \alpha C_\mu$ we achieve (5.23), while (5.24) is just a new name given to C_r. Note that we have not given any value to α, thus after achieving (5.23) we are still free to do the above transformations and rescalings with $\beta = 1$ and arbitrary α.
3. In spite of the relatively simple appearance of Eqs. (5.46)–(5.58), their derivation from the Einstein–Maxwell equations is a lengthy and complicated calculation. This derivation is presented in detail in the textbook by Plebański and Krasiński [12].
4. Contrary to what the author says, the metric (5.59) seems to result quite neatly from (5.54), if the following transformations are done:

$$\begin{aligned} M &= k, \quad r = c\lambda + k, \quad Q = cQ'/k, \\ P^2 &= nc^2 - a^2 + k^2 - c^2 Q'^2/k^2, \\ t &= (a^2 + k^2)\tau/c, \quad \varphi = \varphi' + a\tau/c, \end{aligned}$$

   and then the limit $c \to 0$ is taken. However, the same limit applied to the electromagnetic potential (5.55) gives a result slightly different from the original (5.60): the first term in the A_0 component comes out being Q rather than $Q\lambda$. As given in the original text, the A_μ does not obey the Maxwell equations unless $Q = 0$. The correction was done in our reprinted text (see footnote 55).

Brandon Carter: a brief autobiography

By Brandon Carter

Until the age of 12, I lived near Sydney, N.S.W. (Australia) where I was born on 26 May 1942, attending the primary school at Castle Hill from 1948 to 1954.

During the next 12 years my home was at Penicuik, near Edinburgh (Scotland) where I attended high school at George Watson's College from 1954 to 1959. I began my university education at St Andrews (Scotland) where I studied "natural philosophy" (meaning physics) as an undergraduate from 1959 to 1961, before going to Pembroke College, Cambridge (England) where I started as an undergraduate, reading for the "mathematics tripos", from 1961 to 1964, and where I continued as a research student under the direction of Dennis Sciama (in the Department of Applied Mathematics and Theoretical Physics) and under the influence of Roger Penrose (at the University of London) from 1964 to 1967.

After spending the first six months of 1968 as a post-doc—working on 'black holes'—at the Universities of Princeton (with Professor J.A. Wheeler) and Maryland (with Professor C.W. Misner) in the United States, I settled at Cambridge (England), where I took up various topics including the mechanics of neutron star crusts and the formulation of the 'anthropic principle'. I worked there first as a member of the research staff of the Institute of Astronomy from 1968 to 1973 (apart from another short period in the United States at the University of Chicago with Professor S. Chandrasekhar in 1970) and then continued in a lecturing position on the University teaching staff (again in the Department of Applied Mathematics and Theoretical Physics) from 1973 to 1975.

In 1975 I moved to France, where I joined Silvano Bonazzola as co-director of a C.N.R.S team (the Groupe d'Astrophysique Relativiste) in the Paris Observatory at Meudon for the next 12 years. After another six month visit to the United States, this time at the University of California in Santa Barbara (with Professor J.B. Hartle) in 1987, I returned to my present position as a C.N.R.S. research director at Meudon, where my more recent activities have included work on the mechanics of the (classical) brane configurations that may occur as topological defects (such as superfluid vortices in neutron stars, or various kinds of cosmic strings or domain walls that may conceivably have been formed in the early universe).

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