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GOLDEN OLDIE EDITORIAL

Editorial note to: T. Levi-Civita, The physical reality of some normal spaces of Bianchi

and to: Einsteinian ds^2 in Newtonian fields. IX: The analog of the logarithmic potential

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As mentioned in the accompanying biography, Levi-Civita and his teacher Ricci,[1] prompted by Felix Klein, wrote an important review of tensor calculus (under the name of 'absolute differential calculus') in 1901 [1]. Einstein, advised by Marcel Grossmann, began to study this in 1912 and corresponded with Levi-Civita about it in the period leading up to the publication of the first paper on general relativity.[2]

Levi-Civita was thus well informed of Einstein's work and had the mathematical tools to contribute to the field. He published many papers in the early years of general relativity (according to [2], about 40 in all) and engaged in popularization of the theory. While many of his papers are of interest, we have chosen here to present the two in which he gives the first publication of two very widely-used (and rediscovered) exact solutions in general relativity. The only other exact solutions papers before 1920 which are quoted in [3] are those of Schwarzschild, Droste and Reissner from 1916, of Weyl from 1917 and of Kottler and Nordström from 1918. Fuller discussions of both solutions can be found in [4], Sections 7.1 and 10.2.

[1] Most of whose works appeared under his full name, Ricci-Curbastro.

[2] For some information about this and other correspondence of Levi-Civita's, as well as many references to other information about him, see [2].

The republications of the original Levi-Civita papers can be found in this issue following the editorial note and online via doi:10.1007/s10714-011-1188-4 (The physical reality of some normal spaces of Bianchi) and via doi:10.1007/s10714-011-1189-3 (Einsteinian ds^2 in Newtonian fields. IX: The analog of the logarithmic potential).

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In both derivations Levi-Civita considers a line element

$$\mathrm{d}s^2 = V^2\mathrm{d}t^2 - \mathrm{d}\ell^2 \tag{1}$$

where $\mathrm{d}\ell^2$ is a three-dimensional line element, and it is this latter his discussion focusses on. V is referred to as the speed of light. Also in both cases, the spatial metric has the form of Bianchi's "normal spaces" [5]. These are three-dimensional spaces in which the coordinates can be chosen to be mutually orthogonal.

Each paper opens with a short summary, analogous to an abstract. Section 1 of the earlier of the two papers introduces the energy-momentum tensor for a stationary electromagnetic field, assumed either electric or magnetic (in fact, one could apply a duality rotation giving a mixture of the two). Section 2 discusses the geometry of metrics (1) in which the principal curvatures, i.e. the eigenvalues of the Ricci curvature, are taken to be constant but not equal, and the mean curvature is assumed positive, giving Bianchi's "spaces B)". In Section 3, Levi-Civita finds and solves the Einstein field equations for this geometry and matter content, which leads to the four-dimensional metric, in Levi-Civita's notation,[3]

$$\mathrm{d}s^2 = -\mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \sin^2\frac{x_2}{R}\,\mathrm{d}x_3^2 + (c_1 e^{x_1/R} + c_2 e^{-x_1/R})^2\mathrm{d}t^2.$$

Shifting the origin of x_1, rescaling all the coordinates, and writing R as k, this can readily be identified as the well known "Bertotti-Robinson" metric [6,7]

$$\mathrm{d}s^2 = k^2(-\mathrm{d}\vartheta^2 - \sin^2\vartheta\,\mathrm{d}\varphi^2 - \mathrm{d}x^2 + \sinh^2 x\,\mathrm{d}t^2)\,, \tag{2}$$

given here in the form (apart from reversed signature) in which it appears in [3]. Bertotti in fact gave the more general form with a cosmological constant, which is not in Levi-Civita's paper and whose vacuum limit was given by Nariai (see [8]). The form (2) is the unique Einstein-Maxwell field which is homogeneous and has a homogeneous non-null electromagnetic field: Levi-Civita correspondingly shows in his Section 3 that under his physical assumptions the solution must have the geometry he had earlier assumed.

It may be worth noting that Levi-Civita's description talks of a field being created and reaching equilibrium: without a proper treatment of the non-static period such a description cannot be justified, especially as the resulting manifold is not globally $\mathbb{R}^4$. Instead it is the product of two two-dimensional spaces of constant curvature, and has been rediscovered as both a spherically-symmetric and a spatially-homogeneous solution. It is conformally flat and has been given in various coordinate systems (see [3]). It appears as one of the families characterized by conformally flat orbits and static Einstein-Maxwell fields without sources [9].

As with the related Nariai solution [8], direct physical applications of this metric have been few. In Section 4 of the paper Levi-Civita considers possible physical values of R (although the geometry can hardly be considered realistic) and concludes that

[3] In the translated papers, the translator has helpfully provided footnotes giving the correspondence with modern notation where it may not be obvious.

the effects are very small. Section 5 of the paper deals with cases containing spaces of constant spatial curvature, finding the general expression for V but concluding that the matter required is unphysical. In Section 6 the cases with a cosmological constant (which have the same geometry as the second case of Section 5) are discussed.

The metric (2) includes the AIII and BIII metrics of [10] and appears as a limit of various families of solutions (see e.g. §21.1 of [3] and [11]). Classical and quantum field equations have been solved on it, e.g. [12,13], and it, and its rotating generalization found by Carter [14,15, page 100], have been used, for example, as a background in which to study the Dirac equation [16,17].

Like other simple metrics solving the Einstein equations (such as constant curvature spaces, plane waves and the Schwarzschild solution), the metric has been generalized to other cases, for example to higher-dimensional products of spaces of constant curvature [18–20], generalized gravity theories [21] and string theory [22]. It appears as a "near-horizon" approximation to black holes in generalized or higher-dimensional theories [23,24]. The gluing of (2) to a Reissner-Nordström solution has been used, one way round, with (2) as the interior, to model an elementary particle [25], and the other way round, sometimes in a generalized theory, as a wormhole [26,27].

Although it would be reasonable to rename (2) the Levi-Civita metric, this would confusingly duplicate the name given to the static cylindrically symmetric vacuum metric which is the main result of the second paper reproduced here. As the title of that paper reveals, it is one of a long series which treats more general static axisymmetric cases which began in volume 26, page 307 (1917) of the same proceedings, and it treats the cylindrical case.[4] The spatial metric is shown to be one of Bianchi's normal spaces. The closed Killing vector orbits of the axisymmetry are called isometrics, and their curvature (in the usual Frenet-Serret sense for curves) and other properties are discussed in Section 1. (Incidentally, the fact that they are geodesics of r =const follows trivially from the reflection symmetries and uniqueness of geodesics.)

In Section 2 the logarithmic potential is introduced and the other metric coefficients found from the vacuum Einstein equations. After rescaling coordinates to remove extraneous constants, the resulting metric is

$$\mathrm{d}s^2 = -r^{-2h}\left[r^{2h^2}\left(\mathrm{d}r^2+\mathrm{d}z^2\right)+\frac{r^2}{C^2}\,\mathrm{d}\varphi^2\right]+r^{2h}\mathrm{d}t^2, \tag{3}$$

where $\varphi \in [0,\ 2\pi]$ parametrizes the circles round the axis, $r \in [0,\ \infty)$, and t and $z \in (-\infty,\ \infty)$. Locally, C can be transformed to 1; its global meaning is considered below. The end of Section 2, and Section 3, discuss the geometry of this solution, Section 4 considers the "gravitational force" (in the sense of the acceleration of the congruence of reference timelike lines), and Section 5 the Newtonian approximation. Finally Section 6 discusses the gravitational effect of a homogeneous cylinder, compared with the Newtonian case. The mass per unit length is $\sigma = h/2$, as confirmed by Israel [31].

[4] Due to a formatting error, only another paper of this series, whose correct date is 1918, is cited as the original reference in [3]. Paper VIII of the same series prompted a paper by Weyl which will also appear in this series [28].

The solution (3) is flat for $h = 0, 1$, and of Petrov type D for $h = \frac{1}{2}, 2, -1$; the non-flat cases are singular on the axis $r = 0$ (Levi-Civita assumes $h < 1$). For $h = 0$, with φ cyclic, the solution represents a cosmic string. For $h = 1$ one cannot sensibly consider φ to be an angular coordinate, and Herrera et al. [32] note that at this value z and φ exchange their roles. At the limiting value the source appears to become a plane. Swapping over the names of z and φ gives the same metrics for $h \in (0, 1)$ and $h \in (1, \infty)$, after coordinate transformations (see [4]). For $h = \frac{1}{2}$ the solution is equivalent to the BIII solution of [10]. Circular timelike geodesics exist if $0 \le h < \frac{1}{2}$: these become null at $h = \frac{1}{2}$. This suggests that another interpretation is needed if $h \in \left(\frac{1}{2}, 1\right)$.

The solution is one of the two classes (for spacetimes with signature ± 2) of the Kasner family [29], which was originally presented with the signature $+4$ and the comment that all similar solutions had been found. (The name "Kasner solution" is sometimes wrongly restricted to the cosmological class, where the essential variable is timelike.) It is a member of the Weyl class [30] given by

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{e}^{-2U}[\mathrm{e}^{2k}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2 \mathrm{d}\varphi^2] - \mathrm{e}^{2U}\mathrm{d}t^2, \\ \Delta U &= \rho^{-1}(\rho U_{,M})_{,M} = 0, \quad k_{,\rho} = \rho(U_{,\rho}^2 - U_{,z}^2), \quad k_{,z} = 2\rho U_{,\rho} U_{,z}. \end{aligned} \tag{4}$$

(This corrects Equations (20.3) of [3].)

Allowing complex values of h and complex transformations of the variables, (3) can be cast [33] into the Lewis form [34], and then [35] contains as a special case the Petrov solution [36]

$$k^2 \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{e}^{-2x}\mathrm{d}y^2 + \mathrm{e}^{x}[\cos\sqrt{3}x(\mathrm{d}z^2 - \mathrm{d}t^2) - 2\sin\sqrt{3}x\, \mathrm{d}z\, \mathrm{d}t], \tag{5}$$

the only vacuum solution admitting a simply-transitive G_4 as its maximal group of motions. The analogous transformations for the cosmological Kasner case were considered by McIntosh [37].

The generalization of (3) to add a cosmological constant has been discussed by several authors [38–40], and electromagnetic generalizations have also been considered [41,42].

The metric (3) naturally appears as a limit of more complicated cylindrical or axisymmetric metrics. One interesting case is that of the γ or Zipoy-Voorhees metric, where the potential U is that of a finite rod and the Levi-Civita metric arises as its length tends to infinity [43]: oddly, the flat case $h = 1$ is the limit of finite rods giving Schwarzschild solutions. Another interesting limit is that the special case $h = \frac{1}{2}$ is the (real) static limit of the Lewis class [44].

Surprisingly, stationary cylindrical vacua in the Weyl class can all be locally transformed to the Levi-Civita form [45] (and if complex transformations were allowed, the same is true for those in the Lewis class). This can be understood by considering constructing stationary metrics by taking the solution (3) for $\varphi \in (-\infty, \infty)$ and making identifications to roll it up into a cylinder [46]. There are then two parameters of the identification which are invariantly defined by the global holonomy of the solution, one of which is the C above which gives the angular deficit round the axis. These constants and their relation to interior solutions have been discussed in considerable detail in a series of papers by Herrera, Santos and others, e.g. [32,47]. Making the z coordinate cyclic adds a third parameter [4,48].

Studies have been made of the nature of the singularity [49] and of the Dirac equation on the Levi-Civita background [50]. Many papers have considered the matching of the Levi-Civita solution to cylindrical sources, whether fluid bodies, shells or more general sources, e.g. [51–56], often with a view to physical interpretation of the metric's parameters. However, it has not occurred extensively in discussions of alternative classical gravity theories or higher-dimensional theories, possibly because cylindrically symmetric metrics are not asymptotically flat and do not represent bounded bodies, although it is generally believed they give good approximations for very long but finite cylinders: some exceptions to this remark can be found in [57–59].

The number and variety of modern references to these two exact solutions readily illustrates the reasons for including these papers in this series.

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Comment by the Golden Oldie editor: To avoid confusion, the numbering of footnotes in both papers was changed in this reprinting. In the original versions, footnote numbers started from 1 on each page; here they are numbered continuously throughout the paper. Translator's footnotes are numbered separately: their footnotemarks are preceded by an asterisk. Also, some of the notations were modernized. For example, Levi-Civita's notation for a sum, $\sum_{j=1}^{3}$, was $\sum_{1}^{3}{}_{j}$. Some peculiarities of Levi-Civita's notation and terminology that were not modernized are explained in translator's footnotes.

Tullio Levi-Civita—a brief biography

By A. Krasiński, compiled and abstracted from Ref. [60] and Ref. [61]

Tullio Levi-Civita was born on 29 March 1873 in Padua, Italy, into a Jewish family. He attended a secondary school in Padua. He enrolled in 1890 in the Faculty of Mathematics of the University of Padua. Two of his teachers were Giuseppe Veronese and Gregorio Ricci-Curbastro; Levi-Civita later collaborated with the latter. He wrote a dissertation, supervised by Ricci-Curbastro, on absolute invariants, and this also marks the beginning of his use of the tensor calculus. By including some results from Lie's theory of transformation groups, Levi-Civita extended the theory of absolute invariants to more general cases than those considered by Ricci-Curbastro.

He graduated in 1892 and his dissertation was published in the following year, with some minor changes. He was awarded his teaching diploma in 1894 and in 1895 he was appointed to the teacher's college which was attached to the Faculty of Science at Padua.

Levi-Civita was appointed to the Chair of Rational Mechanics at Padua in 1898, a post which he was to hold for 20 years. Several times during these years attempts had been made to have him move to Rome, but Levi-Civita was happy to remain in Padua. Levi-Civita was a pacifist with firm socialist ideas and it may have been that he felt Padua suited his personality better than Rome at the time.

After World War I ended, the University of Rome made strenuous efforts to strengthen both its teaching and research and many leading scientists were attracted there. Levi-Civita was always very international in his outlook and the ability of Rome to attract top quality students from abroad must have figured in his reasons to now want to move there. In 1918 he was appointed to the Chair of Higher Analysis at Rome, and 2 years later he was appointed to the Chair of Mechanics there.

The year 1922 marked the start of the International Congresses of Applied Mechanics, the first of which took place in Innsbruck, albeit with rather limited attendance. The first full congress took place in Delft in 1924. Levi-Civita was one of the leading figures in the creation of these congresses, and remained an active member of the Congress committee to the end of his life.

The effect of totalitarianism and anti-Semitism on scientific and university life gave Levi-Civita problems. In 1931 all Italian professors were required to sign an oath to Fascism. Volterra refused to take the oath and was dismissed. Although he was deeply opposed to such ideas, Levi-Civita felt that for the sake of his family and his research school in Rome he had to sign despite his strong moral objections.

He lectured in the United States in 1933 and in Moscow and Kiev in 1935. In 1936 he returned to the United States, lecturing at Harvard, Princeton and the Rice Institute. While in Houston he gave an interview which was seen as critical of Italy and the Italian consul asked for clarification. He was recalled to Italy, but because of his leading international status the Italian government felt that it should not react too strongly. In 1936, the International Mathematical Congress was held in Oslo, but Levi-Civita, and all other Italian mathematicians, were forbidden to attend by their government. Despite this Levi-Civita was appointed as a member of the Commission for awarding Fields Medals.

On 5 September 1938 the Racial Laws were passed which excluded all those of Jewish background from universities, schools, academies and other institutions. Levi-Civita was dismissed from his professorship, forced to leave the editorial board of *Zentralblatt für Mathematik* (which stimulated the founding of the Mathematical Reviews), and prevented from attending the Fifth International Congress of Applied Mechanics in the United States.

Levi-Civita had great command of pure mathematics, with particularly strong geometric intuition which he applied to a variety of problems of applied mathematics. One of his papers in 1895 improved on Riemann's contour integral formula for the number of primes in a given interval. He is best known, however, for his work on the absolute differential calculus and its applications to the theory of relativity. In 1886 he published a famous paper in which he developed the calculus of tensors, following on the work of Christoffel, including covariant differentiation. In 1900 he published, jointly with Ricci-Curbastro, the theory of tensors in "Méthodes de calcul differential absolu et leures applications", in a form which was used by Einstein 15 years later. The paper was requested by Klein when he met Levi-Civita in Padua in 1899 and, following Klein's wishes, it appeared in *Mathematische Annalen*. In 1917, inspired by Einstein's theory of relativity, Levi-Civita made his most important contribution to differential geometry—he introduced the concept of parallel displacement in curved spaces.

Weyl took up Levi-Civita's ideas and made them into a unified theory of gravitation and electromagnetism. Levi-Civita's work was of great importance in the theory of relativity, and he produced a series of papers elegantly treating the problem of a static gravitational field. This topic was discussed in a correspondence between him and Einstein.

Analytic dynamics was another topic studied by Levi-Civita, many of his papers examining special cases of the three-body problem. He began publishing papers on the subject in 1903, with another important paper appearing in 1906 which strengthened his earlier results. In 1920 he published a compendium on the three-body problem in *Acta Mathematica*. Then, near the end of his career, he became interested in the n-body problem. In 1950 (9 years after his death) a book by Levi-Civita entitled "Le probleme des n corps en relativité générale" was published.

He also wrote on the theory of systems of ordinary and partial differential equations. He added to the theory of Cauchy and Kovalevskaya and wrote up this work in an excellent book written in 1931.

Levi-Civita's interest in hydrodynamics began early in his career with his paper "Note on the resistance of fluids", appearing in 1901. He worked later on waves in a canal and his proof of the existence of irrotational waves was a major contribution to a long-standing open question.

In 1933 Levi-Civita contributed to Dirac's equations of quantum theory.

Three of Levi-Civita's books became well-known; they are "Questioni di meccanica classica e relativistica" (1924), "Lezioni di calcolo differenziale assoluto" (1925), and in particular the three-volume "Lezioni di meccanica razionale" (1923–1927). His collected works, "Opere matematiche: memorie e note", were published in four volumes in 1954.

The Royal Society conferred the Sylvester medal on Levi-Civita in 1922, while in 1930 he was elected a foreign member. He was also an honorary member of the London Mathematical Society, the Royal Society of Edinburgh, and the Edinburgh Mathematical Society. He attended the 1930 Colloquium of the Edinburgh Mathematical Society in St Andrews.

After he was dismissed from his post, the blow soon caused his health to deteriorate, and he developed severe heart problems. He died of a stroke on 29 Dec 1941 in Rome.

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