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Repeatable light paths in the conformally flat cosmological models

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This is a supplement to an earlier paper [Phys. Rev. D **84**, 023510 (2011)], where those shearfree normal cosmological models were identified, in which all light rays have repeatable paths. All of them are conformally flat, but less general than the Stephani model and more general than Robertson-Walker. In this paper, their defining feature is identified: in each of them, in comoving coordinates, the time dependence factors out so that the cofactor is a static metric. An example is given of a congruence of test observers and sources in the Minkowski spacetime that displays nonrepeatable light paths.

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I. MOTIVATION

This paper is a supplement to Ref. [1], which in turn was a continuation of Ref. [2]. In Ref. [2] null geodesics in the $\beta' \neq 0$ Szekeres models [3–5] were investigated, and it turned out that, in general, they have nonrepeatable paths. This means, given a fixed comoving light source S and a fixed comoving observer O, two light rays emitted from S at different instants that hit O, intersect different sequences of matter world lines on the way. The observer will thus see the image of the source drift across the sky. This is a potentially observable effect. It exists also for nonradial rays in the spherically symmetric Lemaître–Tolman [6,7] model, but is identically zero in the Robertson-Walker (RW) models. Thus, it could be used as an observational test of homogeneity of the Universe—see an astronomy-oriented discussion in Refs. [8,9], where the drift was termed ‘‘cosmic parallax.’’

In Ref. [2] it was shown that the drift vanishes for *all* null geodesics only when the Szekeres model reduces to the Friedmann limit. The condition for this is the same as for zero shear in the flow of the cosmic medium. This gave rise to the question whether the drift is caused by shear or rather by the inhomogeneity.

To clarify this question, in Ref. [1] the condition for zero drift for all null geodesics was investigated in the shearfree normal (SFN) models [10–12]. They are the solutions of Einstein's equations with a perfect fluid source that have zero shear, zero rotation, and nonzero expansion in the cosmic fluid. If shear were indeed the cause of the drift, then in the SFN models the drift should vanish. It turned out that, in general, this is not the case. These models consist of 3 Petrov type D metrics that are spherically, plane and hyperbolically symmetric, and of the conformally flat Stephani solution [12,13] that, in general, has no symmetry. It was found that in each of these solutions, a drift-free subcase exists [1] that has zero conformal curvature, but is less general than the zero-Weyl-tensor limit of the relevant case. At the same time, each subcase is more general than the RW limit, having nonzero acceleration. This gives rise to one more problem: what is the underlying cause of the repeatability of all light paths when these models are non-RW.

This is the question answered here. It is shown that each of the drift-free cases, in the comoving coordinates, is a conformal image of a static spacetime. The key point is not just conformal equivalence (all these models are conformally flat), but the form of the conformal factor. This will be explained in Sec. III.

The repeatability of light paths (RLP) is defined relative to a family of observers and light sources. In a cosmological spacetime, such as Szekeres or Lemaître–Tolman or SFN or RW, it is natural to assume the light sources and observers comoving with the cosmic medium, as in Refs. [1,2]. But one can as well assume the light sources and observers moving along a congruence of timelike curves unrelated to the flow lines of the cosmic matter, and investigate the RLP property for them. It is shown in the Appendix that even in the Minkowski spacetime, a timelike congruence can be devised that displays the non-RLP property.

II. THE DRIFT-FREE SFN MODELS

In Ref. [1] the following drift-free SFN models were identified; all are subcases of the Stephani [12,13] solution.

A. The spherically symmetric model

The metric of this model is

$$ds^2 = \left(\frac{FV,_t}{V}\right)^2 dt^2 - \frac{1}{V^2}(dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2), \tag{2.1}$$

where $F(t)$ is an arbitrary function, related by $\theta = 3/F$ to the expansion scalar $\theta = u^\rho{}_{;\rho}$ of the velocity field $u^\alpha = (FV,_t)^{-1} V \delta^\alpha_0$. The function V is

$$V = B_1 + B_2 r^2 + (A_1 + A_2 r^2) S(t), \tag{2.2}$$

where (A_1, A_2, B_1, B_2) are arbitrary constants and $S(t)$ is an arbitrary function. This model is conformally flat, but is

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more general than RW because the pressure in it is spatially inhomogeneous. The RW limit results when

$$A_1 \neq 0 \quad \text{and} \quad B_2 = A_2 B_1 / A_1. \tag{2.3}$$

B. The plane symmetric model

The metric is here

$$\mathrm{d}s^2 = \left(\frac{FV,_t}{V}\right)^2 \mathrm{d}t^2 - \frac{1}{V^2}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \tag{2.4}$$

$$V = B_1 + B_2 z + (A_1 + A_2 z) S(t), \tag{2.5}$$

the meaning of all symbols being the same as before. Again, this model is conformally flat, less general than Stephani [12,13], but more general than RW, and the prescription for the RW limit (here having $k \leq 0$ necessarily) is given by (2.3). The resulting RW metric is represented in untypical coordinates; see Eqs. (7.8) in Ref. [1] for a transformation to a familiar form.

C. The hyperbolically symmetric model

The metric is

$$\mathrm{d}s^2 = \left(\frac{FV,_t}{V}\right)^2 \mathrm{d}t^2 - \frac{1}{V^2}(\mathrm{d}r^2 + \mathrm{d}\vartheta^2 + \sinh^2\vartheta \mathrm{d}\varphi^2), \tag{2.6}$$

(note the missing factor r^2 compared to case A), where

$$V = B_1 \sin r + B_2 \cos r + (A_1 \sin r + A_2 \cos r) S(t), \tag{2.7}$$

the meaning of all symbols being again the same as in case A. As in both cases above, this model is conformally flat, less general than Stephani [12,13], and more general than RW (this time with $k < 0$ necessarily). As in case B, the RW limit, resulting via (2.3), is represented in untypical coordinates; see Ref. [1].

D. The axially symmetric model

The general form of the metric is (2.4), but this time

$$V = \frac{C_5 - C_4 x_0 - \frac{1}{2}x_0^2 + \frac{1}{2}[(x - x_0)^2 + y^2 + z^2]}{D_1 x_0 + D_2}, \tag{2.8}$$

(C_4, C_5, D_1, D_2) being arbitrary constants and $F(t)$, $x_0(t)$ being arbitrary functions.[1] To calculate the RW limit, the following reparametrization is applied to (2.8):

$$x_0 = \delta U(t), \qquad D_1 = d_1/(\delta k), \qquad D_2 = d_2/k, \qquad C_5 = 2/k, \tag{2.9}$$

where (δ, k) are constants. Then $\delta \longrightarrow 0$ in (2.8) gives

$$V = \frac{2}{d_1 U + d_2}\left[1 + \frac{1}{4}k(x^2 + y^2 + z^2)\right], \tag{2.10}$$

which clearly corresponds to the RW metric, the scale factor being $R(t) = 2/(d_1 U + d_2)$.

III. THE CHARACTERISTIC PROPERTY OF THE DRIFT-FREE CASES

In all the four cases presented in Sec. II the whole time dependence is contained in $(1/V)^2$. Namely, in case A

$$\mathrm{d}s^2 = \frac{1}{V^2}\{[F(A_1 + A_2 r^2)S,_t]^2 \mathrm{d}t^2 - \mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + r^2 \sin^2\vartheta \mathrm{d}\varphi^2)\}, \tag{3.1}$$

and the metric in braces is seen to be static. In case B and case C the factoring out of time dependence occurs in similarly simple ways. In case D the transformation

$$t' = \int \frac{F x_{0,t}}{(D_1 x_0 + D_2)^2} \mathrm{d}t, \tag{3.2}$$

makes explicitly static the cofactor of $(1/V)^2$. The RW models have the same property.

In the general Stephani solution the conformal mapping to the Minkowski metric involves mixing t with spatial coordinates.[2] The congruence of curves in the Minkowski spacetime, to which the world lines of matter of a general Stephani solution are thereby mapped, must thus also display the non-RLP property.

In all cases listed in Sec. II, the time dependence factors out as in (3.1), and the world lines of cosmic medium are mapped into the world lines of static observers. Relative to the congruence of static observers, all light paths are evidently repeatable.

IV. CONCLUSION

The result of Sec. III is the following:

Corollary 1.

In the Szekeres and SFN families of cosmological models, the subcases, in which all null geodesics have repeatable paths, are characterized by the following properties:

(1) The conformal curvature is zero.

(2) The time dependence of the metric represented in comoving coordinates factors out as in (3.1).

In the Szekeres models, condition (1) is at the same time sufficient—it reduces the Szekeres models directly to the Friedmann limit.

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[1]Equation (2.8) corrects a typo in (A141) of Ref. [1], where the whole term containing ${x_0}^2$ should be multiplied by 1/2. In (A140) of Ref. [1] the second $(1/R),_{tt}$ should be $(1/R),_t$.

[2]Because of the 5 arbitrary functions of t in the Stephani metric, this conformal mapping cannot be calculated explicitly; we just know it exists, since the Weyl tensor is zero.

APPENDIX: AN EXAMPLE OF A TIMELIKE CONGRUENCE IN THE MINKOWSKI SPACETIME THAT DISPLAYS THE NON-RLP PROPERTY

The motivation for this example is explained in Sec. I. Take the Minkowski metric in the spherical coordinates

$$\mathrm{d}s^2 = \mathrm{d}t'^2 - \mathrm{d}r'^2 - r'^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2), \tag{A1}$$

and carry out the following transformation on it:

$$\begin{aligned} t' &= (r-t)^2 + 1/(r+t)^2, \\ r' &= (r-t)^2 - 1/(r+t)^2. \end{aligned} \tag{A2}$$

The result is the metric

$$\mathrm{d}s^2 = \frac{1}{(r+t)^4}\mathrm{d}\tilde{s}^2, \tag{A3}$$

where

$$\mathrm{d}\tilde{s}^2 = 16u(\mathrm{d}t^2 - \mathrm{d}r^2) - (u^2-1)^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2), \tag{A4}$$

$$u \stackrel{\text{def}}{=} r^2 - t^2. \tag{A5}$$

Now we assume that the curves with the unit tangent vector field $u^\alpha = [(r+t)^2/(4\sqrt{u})]\delta^\alpha_0$ are world lines of test observers and test light sources. Relative to this congruence, generic light rays have nonrepeatable paths.

It suffices to consider (A4) with the unit tangent vector of the timelike congruence being $u^\alpha = [1/(4\sqrt{u})]\delta^\alpha_0$ instead of (A3). Since conformal images of null geodesics are null geodesics, the only RLPs in (A3) will be the images of the RLPs in (A4), where the mapping is defined by the same coordinates being used in both manifolds.

We investigate the conditions of repeatability by the method used in Refs. [1,2]. We first observe that r can be chosen as a (nonaffine) parameter along open segments of null geodesics. Now consider two light rays sent from the same source S at different instants toward the same observer O. When the earlier ray arrives at a hypersurface $r = r_0$ at the point with the coordinates (t, ϑ, φ), the later ray will arrive at $r = r_0$ at the point $(t+\tau, \vartheta+\zeta, \varphi+\psi)$. The equations of propagation of (τ, ζ, ψ) are obtained from the geodesic equations by subtracting the equation for the earlier ray from the corresponding equation for the later ray, and linearizing the result in (τ, ζ, ψ). The condition for a repeatable path is that $\zeta = \psi = 0$ is a solution of the propagation equations.

Applying this operation and this condition to the geodesic equations parametrized by r, we obtain

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}r}\chi = 0, \tag{A6}$$

where

$$\chi \stackrel{\text{def}}{=} \frac{3u^4 + 6u^2 - 1}{u(u^2-1)} t\tau\left[r\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 - 2t\frac{\mathrm{d}t}{\mathrm{d}r} + r\right] + (3u^2+1)\left(r\frac{\mathrm{d}t}{\mathrm{d}r}\frac{\mathrm{d}\tau}{\mathrm{d}r} - \tau\frac{\mathrm{d}t}{\mathrm{d}r} - t\frac{\mathrm{d}\tau}{\mathrm{d}r}\right). \tag{A7}$$

One solution of (A6) is $\mathrm{d}\vartheta/\mathrm{d}r = 0$, which defines null geodesics that are radial in the coordinates of (A4).

To find whether $\chi = 0$ has any solutions we proceed by the method described in Refs. [1,2]. After a lengthy calculation (much simpler, though, than in Refs. [1,2]), we obtain a contradiction, which means that no other RLPs than the radial null geodesics $\mathrm{d}\vartheta/\mathrm{d}r = \mathrm{d}\varphi/\mathrm{d}r = 0$ exist for "comoving" observers in the metric (A4). This implies

Corollary 2.

With a suitably chosen timelike congruence of test observers and test light sources, nonrepeatable light paths exist even in the Minkowski spacetime. ■

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