

# Geometry and topology of the quasi-plane Szekeres model

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(Dated:)

This paper is a revised version of arXiv:0805.0529 and *Phys.Rev.* **D78**, 064038 (2008), taking into account the erratum published in *Phys.Rev.* **D85**, 069903(E) (2012). Geometrical and topological properties of the quasi-plane Szekeres model and of the plane symmetric dust model are discussed. Some related comments on the quasi-hyperbolic model are made. These properties include: (1) The pattern of expansion in the plane symmetric case, and the Newtonian model that imitates it; (2) The possibility of toroidal topology of the  $t = \text{const}$  sections in the plane symmetric model; (3) The absence of apparent horizons in the quasi-plane and quasi-hyperbolic models (they are globally trapped); (4) Description of the toroidal topology in the Szekeres coordinates; (5) Interpretation of the mass function in the quasi-plane model.

PACS numbers:

Keywords:

## I. MOTIVATION

The quasi-spherical Szekeres model [1] – [17] is rather well-understood by now. In spite of its nontrivial geometry, its basic defining features are not too difficult to grasp intuitively. In a simple-minded way one may say that it is obtained when the spherical symmetry orbits in the Lemaître – Tolman model [16, 17] are made nonconcentric to destroy the symmetry, but the energy-momentum tensor is still that of dust. Recently, that model even found application to solving problems directly related to observational cosmology [13, 14]. In contrast to this, the first serious attempt to interpret the quasi-plane and quasi-hyperbolic models [18] revealed that even the corresponding plane- and hyperbolically symmetric models are not really understood and require more investigation. Some properties of those models were established in Ref. [18], the present paper is a continuation of that research.

The aim of the present paper is to clarify some of the basic geometrical features of the quasi-plane Szekeres model, and of the plane symmetric dust model. The following topics are investigated here: (1) The pattern of expansion in the plane symmetric model, and the Newtonian model that imitates it; (2) The possibility of toroidal topology of the  $t = \text{const}$  sections in the plane symmetric model; (3) The absence of apparent horizons in the quasi-plane and quasi-hyperbolic models (they are globally trapped); (4) Description of the toroidal topology in the Szekeres coordinates; (5) Interpretation of the mass function in the quasi-plane model. For the most part, the paper is devoted to showing that the space of constant time in the plane symmetric dust models can be interpreted as a family of flat tori, with the ones of

smaller diameter enclosed inside those of larger diameter. Such a topology explains several properties of the models, among them the pattern of decelerated expansion and the finiteness of the mass function. It turns out that these models are of lesser use in astrophysical cosmology than the quasi-spherical ones. Because of being globally trapped, they cannot be used for modeling dynamical black holes. Because they expand by the same law as the positive-energy Lemaître – Tolman model, they cannot model the formation of structures that collapse to very dense states. They might be applicable for the description of formation of moderate condensations, like galaxy clusters, and of voids.

Mena, Natário and Tod also considered the quasi-plane and quasi-hyperbolic Szekeres models with toroidal and higher-genus topologies [19]. They considered the matching of those solutions, with nonzero cosmological constant (corresponding to  $\Lambda > 0$  in the notation adopted here), to the plane- and hyperbolically symmetric counterparts of the Schwarzschild solution, also allowed to have nontrivial topologies of the symmetry orbits. However, there is no overlap between their results and those of the present paper, as they mainly considered the global geometry of the resulting black hole, while here the emphasis is put on local geometry of the topologically nontrivial Szekeres spacetime.

The present text is a corrected version of Ref. [20], taking into account the erratum [21]. The error hereby corrected was revealed by Charles Hellaby. By this opportunity, typos and style were corrected as well.

## II. INTRODUCING THE SZEKERES SOLUTIONS

The metric of the Szekeres solutions is

$$ds^2 = dt^2 - e^{2\alpha} dz^2 - e^{2\beta} (dx^2 + dy^2), \quad (2.1)$$

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where  $\alpha$  and  $\beta$  are functions of  $(t, x, y, z)$  to be determined from the Einstein equations with a dust source. The coordinates of (2.1) are comoving so the velocity field of the dust is  $u^\mu = \delta^\mu_0$  and  $\dot{u}^\mu = 0$ .

There are two families of Szekeres solutions, depending on whether  $\beta_{,z} = 0$  or  $\beta_{,z} \neq 0$ . The first family is a simultaneous generalisation of the Friedmann and Kantowski – Sachs [22] models. So far it has found no useful application in astrophysical cosmology, and we shall not discuss it here (see Ref. [16]); we shall deal only with the second family. After the Einstein equations are solved, the metric functions in (2.1) become

$$\begin{aligned} e^\beta &= \Phi(t, z)/e^{\nu(z, x, y)}, \\ e^\alpha &= h(z)\Phi(t, z)\beta_{,z} \equiv h(z)(\Phi_{,z} + \Phi\nu_{,z}), \\ e^{-\nu} &= A(z)(x^2 + y^2) + 2B_1(z)x + 2B_2(z)y + C(z), \end{aligned} \quad (2.2)$$

where the function  $\Phi(t, z)$  is a solution of the equation

$$\Phi_{,t}{}^2 = -k(z) + \frac{2M(z)}{\Phi} + \frac{1}{3}\Lambda\Phi^2; \quad (2.3)$$

while  $h(z)$ ,  $k(z)$ ,  $M(z)$ ,  $A(z)$ ,  $B_1(z)$ ,  $B_2(z)$  and  $C(z)$  are arbitrary functions obeying

$$g(z) \stackrel{\text{def}}{=} 4(AC - B_1^2 - B_2^2) = 1/h^2(z) + k(z). \quad (2.4)$$

The mass-density is

$$\kappa\rho = \frac{(2Me^{3\nu})_{,z}}{e^{2\beta}(e^\beta)_{,z}}; \quad \kappa = 8\pi G/c^2. \quad (2.5)$$

In the present paper we will mostly consider the case  $\Lambda = 0$ .

These solutions have in general no symmetry, and acquire a 3-dimensional symmetry group with 2-dimensional orbits when  $A$ ,  $B_1$ ,  $B_2$  and  $C$  are all constant (that is, when  $\nu_{,z} = 0$ ). The sign of  $g(z)$  determines the geometry of the surfaces ( $t = \text{const}$ ,  $z = \text{const}$ ), and the symmetry of the limiting solution. The geometry is spherical, plane or hyperbolic when  $g > 0$ ,  $g = 0$  or  $g < 0$ , respectively. With  $A$ ,  $B_1$ ,  $B_2$  and  $C$  being functions of  $z$ , the surfaces  $z = \text{const}$  within a single space  $t = \text{const}$  may have different geometries (i.e. they can be spheres in one part of the space and surfaces of constant negative curvature elsewhere, the curvature being zero at the boundary). The sign of  $k(z)$  determines the type of evolution; with  $k > 0 = \Lambda$  the model expands away from an initial singularity and then recollapses to a final singularity, with  $k < 0 = \Lambda$  the model is either ever-expanding or ever-collapsing, depending on the initial conditions;  $k = 0 = \Lambda$  is the intermediate case corresponding to the ‘flat’ Friedmann model.

The Robertson–Walker limit follows when  $z = r$ ,  $\Phi(t, z) = rR(t)$ ,  $k = k_0r^2$  where  $k_0 = \text{const}$  and  $B_1 = B_2 = 0$ ,  $C = 4A = 1$ . This definition of the R–W limit includes the definition of the limiting radial coordinate (the Szekeres model is covariant with the transformations  $z = f(z')$ , where  $f(z')$  is an arbitrary function).

The Szekeres models are subdivided according to the sign of  $g(z)$  into quasi-spherical (with  $g > 0$ ), quasi-plane ( $g = 0$ ) and quasi-hyperbolic ( $g < 0$ ).<sup>1</sup> Despite suggestions to the contrary made in the literature, the geometry of the latter two classes has, until very recently, not been investigated at all and is not really understood; work on their interpretation has only been begun by Helalby and Krasinski [18]. The sign of  $g(z)$  is independent of the sign of  $k(z)$ , but limitations are imposed on  $k(z)$  by the signature of the spacetime: for it to be the physical  $(+---)$ , the function  $h^2$  must be non-negative (possibly zero at isolated points, but not on open subsets), which, via (2.4) means that  $g(z) - k(z) \geq 0$  everywhere. Thus, with  $g > 0$  (in the quasi-spherical case) all three possibilities for  $k$  are allowed; with  $g = 0$  only the two  $k \leq 0$  evolutions are admissible, and with  $g < 0$ , only the  $k < 0$  evolution is allowed.

Only the quasi-spherical model is rather well investigated, and found useful application in astrophysical cosmology. We recall now its basic properties.

It may be imagined as such a generalisation of the Lemaître–Tolman (L–T) model in which the spheres of constant mass were made non-concentric. The functions  $A(z)$ ,  $B_1(z)$  and  $B_2(z)$  determine how the center of a sphere changes its position in a space  $t = \text{const}$  when the radius of the sphere is increased or decreased (see a discussion of this in Ref. [15]). Still, this is a rather simple geometry because all the arbitrary functions depend on just one variable,  $z$ . They give us some limited possibility to model real structures in the Universe (see elegant examples in Refs. [13, 14]), but a fully satisfactory model should involve arbitrary functions of all three spatial variables, to allow modelling of arbitrary structures. Such models are still nonexistent, so the Szekeres models are so far the best devices that exist.

Often, it is more practical to reparametrise the arbitrary functions in the Szekeres metric as follows (this parametrisation was invented by Hellaby [23]). Even if  $A = 0$  initially, a transformation of the  $(x, y)$ -coordinates can restore  $A \neq 0$ , so we may assume  $A \neq 0$  with no loss of generality (see Ref. [16]). Then let  $g \neq 0$ . Writing

$$\begin{aligned} (A, B_1, B_2) &= \frac{\sqrt{|g|}}{2S}(1, -P, -Q), \quad \varepsilon \stackrel{\text{def}}{=} g/|g|, \quad (2.6) \\ k &= -|g| \times 2E, \quad M = |g|^{3/2}\widetilde{M}, \quad \Phi = R\sqrt{|g|}, \end{aligned}$$

we can represent the metric (2.1) as

$$e^{-\nu}/\sqrt{|g|} \stackrel{\text{def}}{=} \mathcal{E} \stackrel{\text{def}}{=} \frac{S}{2} \left[ \left( \frac{x-P}{S} \right)^2 + \left( \frac{y-Q}{S} \right)^2 + \varepsilon \right],$$

<sup>1</sup> We stress once again that *the same* Szekeres model may be quasi-spherical in one part of the spacetime, and quasi-hyperbolic elsewhere, with the boundary between these two regions being quasi-plane, see an explicit simple example in Ref. [18]. In most of the literature published so far, these models have been considered separately, but this was only for purposes of systematic research.

$$ds^2 = dt^2 - \frac{(R_{,z} - R\mathcal{E}_{,z}/\mathcal{E})^2}{\varepsilon + 2E(z)} dz^2 - \frac{R^2}{\mathcal{E}^2} (dx^2 + dy^2). \quad (2.7)$$

When  $g = 0$ , the transition from (2.1) to (2.7) is  $A = 1/(2S)$ ,  $B_1 = -P/(2S)$ ,  $B_2 = -Q/(2S)$ ,  $k = -2E$ ,  $\widetilde{M} = M$  and  $\Phi = R$ . Then (2.7) applies with  $\varepsilon = 0$ , and the resulting model is quasi-plane.

For further reference, the evolution equation (2.3), in the variables of (2.7) becomes

$$R_{,t}{}^2 = 2E(z) + \frac{2\widetilde{M}(z)}{R} + \frac{1}{3}\Lambda R^2; \quad (2.8)$$

From now on, we will use this representation; the tilde over  $M$  will be dropped, but it must be remembered that the  $M$  in (2.8) is not the same as the one in (2.3).

The representation (2.7) makes the calculations simpler because the arbitrary functions in it are independent (the condition (2.4) has been incorporated in this form). However, it obscures the fact that the cases  $\varepsilon = +1, 0, -1$  can be parts of the same spacetime.

Rotation and acceleration of the dust source are zero, the expansion is

$$\Theta = 3\frac{R_{,t}}{R} + \frac{R_{,tz} - R_{,t}R_{,z}/R}{R_{,z} - R\mathcal{E}_{,z}/\mathcal{E}}, \quad (2.9)$$

and the shear tensor is

$$\sigma^\alpha{}_\beta = \frac{1}{3}\Sigma \text{diag}(0, 2, -1, -1), \quad \text{where} \\ \Sigma = \frac{\Phi_{,tz} - \Phi_{,t}\Phi_{,z}/\Phi}{\Phi_{,z} - \Phi\nu_{,z}} \equiv \frac{R_{,tz} - R_{,t}R_{,z}/R}{R_{,z} - R\mathcal{E}_{,z}/\mathcal{E}}. \quad (2.10)$$

Definitions of the Szekeres solutions by invariant properties can be found in Ref. [16].

When  $\Lambda \neq 0$ , the solutions of (2.8) involve elliptic functions. A general formal integral of (2.8) was presented by Barrow and Stein-Schabes [24]. Any solution of (2.8) will contain one more arbitrary function of  $z$  that will be denoted  $t_B(z)$ , and will enter the solution in the combination  $(t - t_B(z))$ . The instant  $t = t_B(z)$  defines the initial moment of evolution; when  $\Lambda = 0$  it is necessarily a singularity corresponding to  $\Phi = 0$ , and it goes over into the Big Bang singularity in the Friedmann limit. When  $t_{B,z} \neq 0$  (that is, in general) the instant of singularity is position-dependent, as in the L-T model.

Just as in the L-T model, another singularity may occur where  $(e^\beta)_{,z} = 0$  (if this equation has solutions). This is a shell crossing, but it is qualitatively different from that in the L-T model. As can be seen from (2.2), in the quasi-spherical case, when a shell crossing exists, its intersection with a  $t = \text{const}$  space will be a circle, or, in exceptional cases, a single point, not a sphere. In the quasi-spherical models shell crossings can be avoided altogether if the arbitrary functions are chosen appropriately, see the complete list and derivation in Ref. [15]. In the quasi-hyperbolic models, shell crossings can be

avoided in one sheet of each hyperboloid, but are unavoidable in the other, see Ref. [18]. In the quasi-plane model, if the flat surfaces existing in it are interpreted as infinite planes, shell crossings are unavoidable [18].

Equation (2.8) is identical with the Friedmann equation, but, just like in the L-T limit, with  $k$  and  $M$  depending on  $z$ , each surface  $z = \text{const}$  evolves independently of the others.

The models defined by (2.1)–(2.5) contain 8 functions of  $z$ , but only 5 of them correspond to independent physical degrees of freedom. One of the 8 functions is determined by (2.4),  $g(z)$  was made constant by the reparametrisation (2.6), and one can be specified by a choice of  $z$ , for example by defining  $z' = M$ , or  $M = z'^3 \times \{\text{a constant}\}$ .

A quasi-spherical Szekeres region can be matched to the Schwarzschild solution across a  $z = \text{const}$  hypersurface [6]. The other two Szekeres regions can be matched to the plane- and hyperbolically symmetric counterparts of the Schwarzschild solution (see Ref. [25] for the solutions and [18] for the matching).

In the following, we will represent the Szekeres solutions with  $\beta_{,z} \neq 0$  in the parametrisation introduced in (2.7). The formula for density in these variables is

$$\kappa\rho = \frac{2(M_{,z} - 3M\mathcal{E}_{,z}/\mathcal{E})}{R^2(R_{,z} - R\mathcal{E}_{,z}/\mathcal{E})}, \quad (2.11)$$

where, let it be recalled, the  $M$  above is the  $\widetilde{M}$  of (2.6).

### III. THE PLANE SYMMETRIC MODELS

The plane symmetric dust models (first found by Ellis [26]) result from (2.7) when  $\varepsilon = 0$  and  $(P, Q, S)$  are independent of  $z$ . The constant  $S$  can then be scaled to 1 by appropriate redefinitions of  $R$ ,  $E$  and  $M$ . Then, with constant  $P$  and  $Q$ , the coordinate transformation

$$x = P + \frac{2p}{p^2 + q^2}, \quad y = Q + \frac{2q}{p^2 + q^2} \quad (3.1)$$

changes the metric to

$$ds^2 = dt^2 - \frac{R_{,z}{}^2}{2E(z)} dz^2 - R^2 (dp^2 + dq^2), \quad (3.2)$$

while the energy-density simplifies to

$$\frac{8\pi G}{c^2}\rho = \frac{2M_{,z}}{R^2 R_{,z}}. \quad (3.3)$$

These models are called plane symmetric because their symmetries are the same as those of the Euclidean plane; in the coordinates of (3.2) they are:

$$p' = p + A_1, \quad (3.4a)$$

$$q' = q + A_2, \quad (3.4b)$$

$$(p', q') = (p \cos \alpha + q \sin \alpha, -p \sin \alpha + q \cos \alpha), \quad (3.4c)$$

where  $A_1$ ,  $A_2$  and  $\alpha$  are arbitrary constants – the group parameters.

Note that eqs. (2.8) and (3.3) are identical to their counterparts in the spherically symmetric models. In particular, the function  $M(z)$  enters in the same way as the active gravitational mass did in spherical models. However, if we wish to interpret  $M(z)$  as a mass contained in a volume, we encounter a problem – see below.

Examples of plane symmetric spaces are the Euclidean plane and the Euclidean space  $E_3$  with the metric  $ds_3^2 = dx^2 + dy^2 + dz^2$ . However, the space of constant  $t$  in (3.2) can never become flat; its curvature tensor is [18]:

$${}^3R^{zp}{}_{zp} = {}^3R^{zq}{}_{zq} = -\frac{E_{,z}}{RR_{,z}}, \quad {}^3R^{pq}{}_{pq} = -\frac{2E}{R^2}. \quad (3.5)$$

Nevertheless, the surfaces  $P_2$  of constant  $t$  and  $z$  in (3.2) are flat. Thus, there is some mystery in the geometry of the spacetimes (3.2). One component of the mystery is this: In the quasi-spherical case, and in the associated spherically symmetric model, the surfaces of constant  $t$  and  $z$  were spheres, and  $M(z)$  was a mass inside a sphere of coordinate radius  $z$ . In the plane symmetric case, if the  $P_2$  surfaces are infinite planes, they do not enclose any finite volume, so where does the mass  $M(z)$  reside?

With  $M$  being a constant, the metric (3.2) becomes vacuum – the plane symmetric analogue of the Schwarzschild spacetime.

In the quasi-spherical Szekeres, and in spherically symmetric solutions, analogies exist between the relativistic and the Newtonian models. We will now compare the plane symmetric model with its possible Newtonian counterparts. For this purpose, let us note the pattern of expansion in (3.2) and (2.8) with  $\Lambda = 0$ . When  $R_{,t} \neq 0$ , eq. (2.8) implies

$$R_{,tt} = -M/R^2. \quad (3.6)$$

Note that  $R_{,tt} = 0$  implies  $M = 0$ , which is the Minkowski metric. Now take a pair of dust particles, located at  $(t, z_1, p_0, q_0)$  and at  $(t, z_2, p_0, q_0)$ , and consider the affine distance between them:

$$\ell_{12}(t) = \int_{z_1}^{z_2} \frac{R_{,z} dz}{\sqrt{2E}} \implies \frac{d^2\ell_{12}}{dt^2} = \int_{z_1}^{z_2} \frac{R_{,ttz} dz}{\sqrt{2E}}. \quad (3.7)$$

Thus, the two particles will be receding from each other (or approaching each other if collapse is considered) with acceleration that can never be zero.<sup>2</sup>

Take another pair of dust particles, located at  $(t, z_0, p_1, q_0)$  and at  $(t, z_0, p_2, q_0)$ . The distance between them, measured within the symmetry orbit, is

$$\ell_{34}(t) = \int_{p_1}^{p_2} R dp \equiv R(p_2 - p_1)$$

<sup>2</sup> The acceleration would be zero if  $R_{,ttz} = 0$ , which leads to a contradiction in the Einstein equations.

$$\implies \frac{d^2\ell_{34}}{dt^2} = R_{,tt}(p_2 - p_1), \quad (3.8)$$

i.e. the acceleration of the expansion can never vanish in this direction, either, unless  $M = 0$ . The same result will follow for any direction in the  $(p, q)$  surface. Thus, the expansion or collapse in this model proceeds with acceleration in every spatial direction.<sup>3</sup> We will compare this result with the Newtonian situation.

#### IV. A NEWTONIAN ANALOGUE OF THE PLANE SYMMETRIC DUST SPACETIME

At first sight, it seems that the Newtonian model analogous to the plane symmetric dust model should be dust whose density is constant on parallel  $(x, y)$ -planes, and depends only on  $z$ . Let us follow this idea.

If the potential is plane symmetric, then, in the adapted coordinates, it depends only on  $z$ . Thus, the Poisson equation simplifies to

$$\frac{d^2V}{dz^2} = 4\pi G\rho(z). \quad (4.1)$$

The general solution of this is

$$V = 4\pi G \int_{z_0}^z dz' \int_{z_0}^{z'} d\tilde{z} \rho(\tilde{z}) + Az + B, \quad (4.2)$$

where  $A$  and  $B$  are integration constants;  $z_0$  is a reference value of  $z$  at which we can specify an initial condition. If we wish to have  $V = \text{const}$  (i.e. zero force) when  $\rho \equiv 0$ , we must take  $A = 0$ , and then  $V(z_0) = B$ .<sup>4</sup> The equations of motion in this potential are

$$\frac{dv^i}{dt} = -\frac{\partial V}{\partial x^i}, \quad (4.3)$$

where  $v^i$  are components of the velocity field of matter, so

$$\frac{dv^x}{dt} = \frac{dv^y}{dt} = 0, \quad \frac{dv^z}{dt} = -\frac{dV}{dz} = -4\pi G \int_{z_0}^z dz' \rho(z'). \quad (4.4)$$

This, however, gives a pattern of expansion different from that in the relativistic plane symmetric model. In (4.4), expansion with acceleration proceeds only in the  $z$ -direction, while in the directions orthogonal to  $z$  there is no acceleration, or, in a special case, not even any expansion. Consequently, no obvious Newtonian analogue exists for the relativistic plane symmetric model.<sup>5</sup>

<sup>3</sup> Since  $M \geq 0$ , this is fact deceleration.

<sup>4</sup> An  $A \neq 0$  would be qualitatively similar to the cosmological constant in relativity.

<sup>5</sup> Incidentally, there will be no Newtonian analogue for the hyperbolic model, since the orbits of hyperbolic symmetry cannot be embedded in a Euclidean space at all. They can be embedded in a flat 3-dimensional space, but the space then must have the signature  $(-++)$ .

Equation (4.4) shares one property with the relativistic evolution equation (2.8). If  $\rho$  is bounded in the range of integration, then the force that drives the motion of the fluid is finite, giving the illusion that the potential is generated by some finite mass. However, if we wanted to calculate  $V(z)$  by summing up contributions to it from all the volume elements of the fluid, like is done in calculating the gravitational potentials of finite portions of matter, then the result would be an infinite value of  $V$ , in consequence of the source having infinite extent in the  $(x, y)$ -plane. Thus, if we want to interpret the r.h.s. in (4.3) as being generated by a mass, then the mass that drives the evolution is not the total mass in the source, but the mass of a finite portion of the source.

We now provide a solution of the Poisson equation that qualitatively mimics the pattern of expansion of the plane symmetric relativistic model. Its equipotential surfaces will be locally plane symmetric, but their symmetries will not be symmetries of the whole space.

Consider two families of cones given by the equations (see Fig. 1)

$$u = z - \alpha r, \quad v = z + r/\alpha, \quad r \stackrel{\text{def}}{=} \sqrt{x^2 + y^2}, \quad (4.5)$$

where  $\alpha$  is a constant and  $(x, y, z)$  are Cartesian coordinates. The cones of constant  $u$  are orthogonal to the cones of constant  $v$ , and the two families are co-axial. We choose  $u$  and  $v$  as two coordinates in space; the third coordinate will be the angle  $\varphi$  around the axis of symmetry. We begin with the Euclidean metric in the cylindrical coordinates,  $ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$ , and transform this to the  $(u, v, \varphi)$  coordinates by

$$r = \frac{\alpha(v - u)}{1 + \alpha^2}, \quad z = \frac{u + \alpha^2 v}{1 + \alpha^2}. \quad (4.6)$$

The transformed metric is

$$ds^2 = \frac{du^2 + \alpha^2 dv^2}{1 + \alpha^2} + \frac{\alpha^2(v - u)^2 d\varphi^2}{(1 + \alpha^2)^2}. \quad (4.7)$$

The Laplace operator, which in the cylindrical coordinates is

$$\Delta V = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}, \quad (4.8)$$

in the  $(u, v, \varphi)$ -coordinates becomes

$$\Delta V = (1 + \alpha^2) \left[ -\frac{1}{v - u} \frac{\partial V}{\partial u} + \frac{\partial^2 V}{\partial u^2} + \frac{1}{\alpha^2(v - u)} \frac{\partial V}{\partial v} + \frac{1}{\alpha^2} \frac{\partial^2 V}{\partial v^2} + \frac{1 + \alpha^2}{\alpha^2(v - u)^2} \frac{\partial^2 V}{\partial \varphi^2} \right]. \quad (4.9)$$

Thus, if  $V$  depends only on  $u$ , then the Poisson equation says:

$$(1 + \alpha^2) \left( -\frac{1}{v - u} \frac{\partial V}{\partial u} + \frac{\partial^2 V}{\partial u^2} \right) = -4\pi G\rho. \quad (4.10)$$

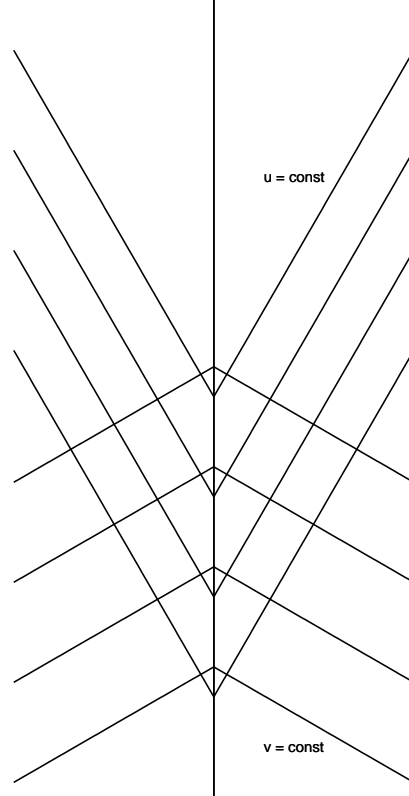


FIG. 1: The cones  $u = \text{constant}$  are orthogonal to the cones  $v = \text{constant}$ . The coordinates in space are  $u, v$  and the angle around the axis of symmetry. The figure shows an axial cross-section through the setup. A gravitational potential which depends only on  $u$  in these coordinates gives an expansion pattern that is qualitatively similar to the one in a plane symmetric dust space-time.

The gradient of  $V(u)$  has nonzero components in all directions, and so will create expansion decelerated in all directions. The expansion will be isotropic with respect to the  $u = v$  axis, and the anisotropy between the  $(x, y)$  and the  $z$ -directions is controlled by  $\alpha$ .

This potential was introduced here for illustrative purposes. In order to make it credible, one should solve the continuity equation and the Euler equations of motion in it. We do not quote here the appropriate calculations because they lead to an intransparent tangle of differential equations. For dust, that set is overdetermined, so probably has no solutions.

## V. PLANE SYMMETRIC 3-SPACES INTERPRETED AS TORI

Although known for a long time (see Ref. [26]), the plane symmetric model has not been investigated for its geometrical and physical properties.

Since a flat spatial geometry is not possible in it (see eq. (3.5)), we now consider other possible 3-geometries with planar symmetry. The next simplest is a space of constant curvature. From (3.5), the space of constant  $t = t_0$  will have constant curvature when

$$2E = \pm C^2 R^2, \quad (5.1)$$

where  $C$  is a constant. The curvature is positive when  $E < 0$  and negative when  $E > 0$ . Since the signature of spacetime requires  $E \geq 0$ , we follow only the + case. Choosing  $R(t_0, z) = R$  as the spatial coordinate in this space, we get:

$$ds_3^2 = \mathcal{S}^2 \left[ \frac{dR^2}{C^2 R^2} + R^2 (dp^2 + dq^2) \right]. \quad (5.2)$$

Note that only one hypersurface can have the property (5.1) (since  $E$  is independent of  $t$  while  $R$  depends on  $t$ ). Thus, the 3-geometry of a space of constant  $t$  can evolve away from or toward (5.2), or through (5.2), but cannot preserve this geometry over a finite time.

The surface of constant  $q$  in (5.2) has the metric  $ds_2^2 = dR^2/(C^2 R^2) + R^2 dp^2$ . To visualise it, we embed it now in a 3-dimensional Euclidean space with the metric

$$ds_3^2 = dZ^2 + dR^2 + R^2 dp^2. \quad (5.3)$$

Our  $ds_2^2$  is the metric of the surface  $Z = Z_0(R)$ , where  $Z_{0,R}^2 + 1 = 1/(CR)^2$ , thus

$$\begin{aligned} Z_0 &= \pm \int \frac{\sqrt{1 - C^2 R^2}}{CR} dR \\ &= \pm \frac{1}{C} \left[ \ln \left( \frac{CR}{1 + \sqrt{1 - C^2 R^2}} \right) + \sqrt{1 - C^2 R^2} \right]. \end{aligned} \quad (5.4)$$

This embedding is possible only in the range  $R \leq 1/C$ . The  $R > 1/C$  part of the surface can be embedded in a flat space of signature  $(- + +)$ .<sup>6</sup> The metric is then

$$ds_3^2 = -dZ_1^2 + dR^2 + R^2 dp^2, \quad (5.5)$$

and the embedding equation is

$$\begin{aligned} Z_1 &= \pm \int \frac{\sqrt{C^2 R^2 - 1}}{CR} dR = \pm \frac{1}{C} \left[ \sqrt{C^2 R^2 - 1} \right. \\ &\quad \left. - 2 \arctan \left( CR + \sqrt{C^2 R^2 - 1} \right) + \frac{\pi}{2} \right] \end{aligned} \quad (5.6)$$

(the constant of integration was chosen so that  $Z_0(1/C) = Z_1(1/C)$ ). The functions  $Z(R)$  and  $Z_1(R)$  are shown in Fig. 2. Note that in both embeddings, (5.3) and (5.5),  $p$  appears as the polar angle in the plane  $(R, p)$ . If  $p$  is to be interpreted as actually being a polar angle, with the period  $2\pi$ , then all points with the coordinates  $(t, z, p + 2\pi n, q)$ , where  $n$  is any integer, should be identical with the point of coordinates  $(t, z, p, q)$ . Since  $p \rightarrow (p + \text{constant})$  are symmetry transformations of the spacetime (3.1), there is no problem with such an identification. Thus we should imagine the  $(R, p)$  surface as being created by rotating the curve from Fig. 2 around the  $Z$  axis.

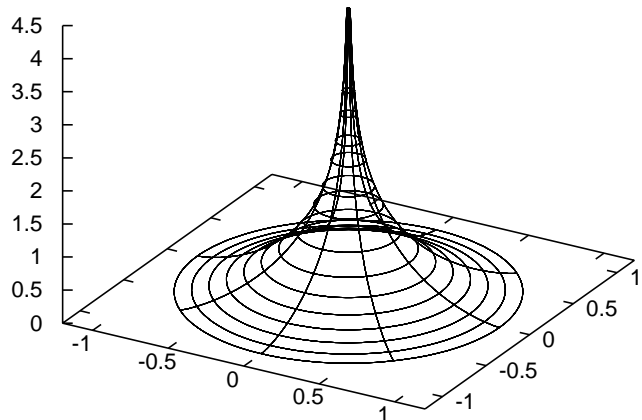
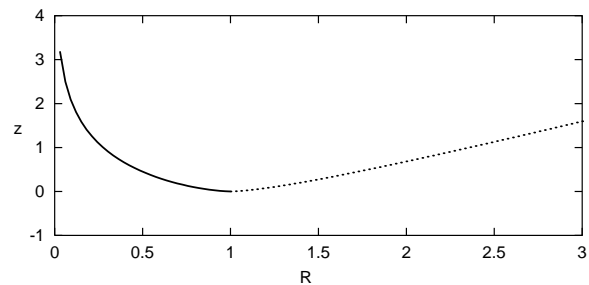


FIG. 2: **Top panel:** The function  $Z_0(R)$  (from 0 to 1, solid line) and the function  $Z_1(R)$  (from 1 to 3, dotted line), given by eqs. (5.4) and (5.6), respectively. In the graph we chose the  $-$  sign for  $Z_0(R)$  and the  $+$  sign for  $Z_1(R)$ . The graph is the cross-section of the  $(R, p)$ -surface in the spaces (5.3) (left part of the curve) and (5.5) (right part of the curve). **Bottom panel:** The  $(R, p)$  (or  $(R, q)$ ) surface obtained by rotating the graph from the top panel around the  $R = 0$  axis. The lower end of the funnel is where the embedding in the Euclidean space breaks down, i.e. where the solid line meets the dotted line in the top panel. Upwards, the funnel goes infinitely far and becomes infinitely thin.

However, the same picture would be obtained for an  $(R, q)$  surface in (5.2), given by  $p = \text{const}$ . We would find that in that surface,  $q$  is the angular coordinate of the polar coordinates  $(R, q)$ , and points of coordinates  $(t, z, p, q + 2\pi m)$  can be identified with the point of coordinates  $(t, z, p, q)$ . We are thus led to conclude that  $(p, q)$  are both angular coordinates with the period  $2\pi$ , and that the points of coordinates  $(p, q)$  have to be identified with the points of coordinates  $(p + 2n\pi, q + 2m\pi)$ , where  $n$  and  $m$  are arbitrary integers. The tentative conclusion is that the  $(p, q)$ -surface is a flat torus.

The conclusion is tentative in the sense that, while we identify the set  $p = p_0$  with the set  $p = p_0 + 2\pi$ , we are still free to carry out the symmetry transformations within the set  $p = p_0$ . Thus, the identification can possibly be done with a twist, that will turn a square into a Möbius strip, or with a two-way twist, that will turn it into a projective plane [28]. We will use the term “toroidal topology” that will be meant to include an or-

<sup>6</sup> A similar phenomenon is known from the maximally extended Reissner – Nordström spacetime, when the region inside the interior horizon is depicted, see Ref. [16].

dinary torus, and also the identifications with twists.

The conclusions drawn from an embedding can be misleading. As an example, consider the hyperbolically symmetric counterpart of (3.2):

$$ds^2 = dt^2 - \frac{R_{,z}^2}{2E(z) - 1} dz^2 - R^2 (d\vartheta^2 + \sinh^2 \vartheta d\varphi^2). \quad (5.7)$$

The surface of constant  $t$  and constant  $\varphi$  has the metric  $ds_2^2 = R_{,z}^2 dz^2 / (2E(z) - 1) - R^2 d\vartheta^2$ , and embedding it in a Euclidean space we would conclude that  $\vartheta$  is the polar coordinate with the period  $2\pi$ . However, in this case  $\vartheta \rightarrow (\vartheta + \text{constant})$  are not symmetry transformations of the spacetime (or of a space of constant  $t$ ), and so identifications of points with different values of  $\vartheta$  are not permitted. Thus, the embedding in this case is not a one-to-one mapping. Consequently, the toroidal interpretation of the plane symmetric case must be treated as one possibility, and not as a definitive conclusion.

Note from (5.2) that the length of any segment of a curve given by  $p = \text{const}$  and  $q = \text{const}$  that goes into the point  $R = 0$  is infinite. The length of such a ‘‘radial’’ line between the values  $R_1$  and  $R_2$  is

$$\ell_{12} = \left| \int_{R_1}^{R_2} \frac{dR}{CR} \right| = \left| \frac{1}{C} \ln \left( \frac{R_2}{R_1} \right) \right| \xrightarrow{R_2 \rightarrow 0} \infty$$

Thus, Fig. 2 correctly suggests that a surface of constant  $p$  or constant  $q$  in the space (5.2) has the shape of an infinite funnel, and the point with coordinate  $R = 0$  is not accessible (does not in fact belong to this surface). This conclusion is consistent with the observation made in Ref. [18] that in the planar Szekeres metric ‘‘there is no real origin, but  $R$ ,  $M$  and  $E$  can asymptotically approach zero’’<sup>7</sup>

It can be concluded from (3.2) that the  $(p, q)$  surface should have a toroidal topology with any form of  $E$ , as will now be shown. The 3-metric of a  $t = t_0$  space is:

$$ds_3^2 = \frac{dR_0^2}{2E} + R_0^2 (dp^2 + dq^2), \quad (5.8)$$

where  $R_0(z) \stackrel{\text{def}}{=} R(t_0, z)$ . We can now embed a surface of constant  $p$  or a surface of constant  $q$  in a 3-dimensional flat space by the same method that we used for (5.2), only the equation of embedding will not be explicit:

$$\pm Z_{,R}^2 + 1 = 1/(2E) > 0, \quad (5.9)$$

the upper sign being for embedding in the Euclidean space, the lower sign for the embedding in the pseudoeuclidean space. In each case the coordinates  $p$  and  $q$  turn out to be the azimuthal coordinates. As argued in Ref. [18], if a nonsingular origin (where  $R = E = 0$ ) is to exist,

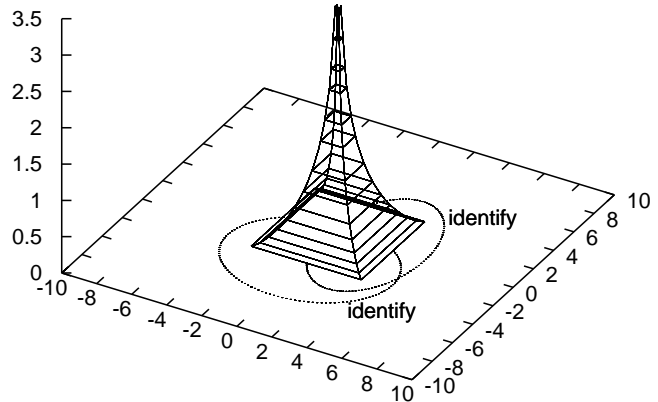


FIG. 3: A sketch of the 3-space of the plane symmetric toroidal model (a faithful picture cannot be drawn because the 2-dimensional flat torus cannot be embedded in a Euclidean space, and the 3-space of a planar Szekeres model cannot be made flat). Each square section of the funnel represents a 2-dimensional flat torus, so its front edge coincides with the back edge, and the left edge coincides with the right edge. Each torus is an orbit of the symmetry group of the model. Each smaller torus is contained within all the larger ones, but the inclusion relation cannot be depicted in the Euclidean space of this graph. Also, in the curved 3-space, the 2-tori surround the asymptotic origin, which is the tip of the funnel, situated infinitely high above the plane shown.

then it will be infinitely far from every point of the  $t = \text{const}$  space. This implies the infinite funnel geometry of Fig. 2. A sketch of such a space is shown in Fig. 3.

The toroidal geometry and topology of the  $(p, q)$  surfaces neatly explains the pattern of expansion. The circumference of the torus along the  $p$ - or  $q$ -direction is  $2\pi R$  in the coordinates of (5.8). Thus, as  $R$  increases with time, the circumference of the torus increases in proportion to  $R$ , which causes transversal expansion.

The toroidal topology also solves the problem of where the mass generating the gravitational field resides. As observed in Ref. [18], the regularity conditions at an origin  $z = z_c$  are independent of  $\varepsilon$  (in some cases they cannot be fulfilled, for example with  $\varepsilon < 0$ ). Thus,  $E/M^{2/3}$  must tend to a nonzero constant as  $z \rightarrow z_c$ . Knowing this, let us calculate the amount of rest mass in an arbitrary volume  $\mathcal{V}$ , from (3.3) and (3.2). That amount is  $\mathcal{M} \stackrel{\text{def}}{=} \int_{\mathcal{V}} \rho \sqrt{g_3} d_3x$ , where  $g_3$  is the determinant of the 3-metric of a  $t = \text{const}$  subspace of (3.2). Thus

$$\mathcal{M} = \frac{c^2}{4\pi G} \int_{\mathcal{V}} \frac{M_{,z}}{\sqrt{2E}} dpdqdz \equiv \frac{c^2}{4\pi G} \int_{\mathcal{V}} \frac{1}{\sqrt{2E}} dpdq dM. \quad (5.10)$$

With  $0 < E \propto M^{2/3}$  in the vicinity of  $M = 0$ , the integral with respect to  $M$  is finite. With a toroidal topology, the ranges of  $p$  and  $q$  are finite, so the integrals over  $p$  and  $q$  also give final values. Thus, the total amount of mass in each space  $t = \text{const}$  is finite.

<sup>7</sup> In this quote, notation has been adapted to that used here.

The relation  $\mathcal{M}_{,z} = M_{,z}/\sqrt{2E}$  that follows from (5.10) is similar to  $\mathcal{M}_{,r} = M_{,r}/\sqrt{1+2E}$ , which held in the spherically symmetric and quasi-spherical models. By analogy, we conclude that in a plane symmetric spacetime the factor  $1/\sqrt{2E}$  measures the relativistic mass defect/excess, i.e. the discrepancy between the active gravitational mass  $M$  and the sum of rest masses  $\mathcal{M}$ .

## VI. NO APPARENT HORIZONS IN THE QUASI-PLANE AND QUASI-HYPERBOLIC MODELS

An apparent horizon is the envelope of the region of trapped surfaces. A (past or future) trapped surface is such, on which both the ingoing and outgoing (past- or future-directed respectively) families of null geodesics converge. A future AH always forms in spherically symmetric or quasi-spherical-Szekeres collapse before the Big Crunch singularity is achieved, a past AH always exists after the Big Bang singularity.

It turns out that the AH-s do not exist in the quasi-plane and quasi-hyperbolic Szekeres models, and, consequently, neither do they exist in the plane- and hyperbolically symmetric dust models. Actually, a stronger result holds: these spacetimes remain trapped for all the time. This follows by the method invented by Szekeres [2], which applies here almost unchanged – only the final conclusion is radically different in consequence of the different sign of  $\varepsilon$ . To avoid getting into complicated details, we begin by using the general form (2.1) of the metric. Suppose a trapped surface exists, and call it  $\Sigma$ .

We assume  $\Sigma$  to be one of the orbits of the quasi-symmetry, i.e. to have its equation of the form  $\{t = \text{constant}, z = \text{constant}\}$ . It will be explained later (see after (6.9)) why it is sufficient to consider such surfaces to prove the conclusion. The traditional definition of a trapped surface requires that it be compact. With the toroidal topology in the planar model, our  $\Sigma$  will be compact indeed. With the infinite topology, and in the quasi-hyperbolic model,  $\Sigma$  will be infinite. In view of the final result of our consideration, this fact will turn out to be unimportant. We choose these infinite surfaces because of their simple geometry.

Consider any family of null geodesics intersecting  $\Sigma$  orthogonally, and let the tangent vector field of those geodesics be  $k^\mu$ . Let  $(t, z, x, y) = (x^0, x^1, x^2, x^3)$ . We have then

$$k_\mu k^\mu = 0, \quad k^\nu k^\mu{}_{;\nu} = 0 \quad \text{everywhere} \quad (6.1)$$

because  $k^\mu$  is tangent to null geodesics, and

$$k^2 = k^3 = 0, \quad (k^0)^2 - e^{2\alpha} (k^1)^2 = 0 \quad \text{on } \Sigma \quad (6.2)$$

because  $k^\mu$  is assumed orthogonal to  $\Sigma$ , so at the points of  $\Sigma$  it must be spanned on the vector fields normal to  $\Sigma$ , which are  $(1, 0, 0, 0)$  and  $(0, e^{-\alpha}, 0, 0)$ . The affine parameter along each null geodesic may be chosen so that

$$k^0 = e^\alpha, \quad k^1 = e = \pm 1 \quad \text{on } \Sigma, \quad (6.3)$$

where we will call the geodesics with  $e = -1$  “ingoing”, and those with  $e = +1$  “outgoing”.<sup>8</sup> A surface  $\Sigma$  is trapped when the expansion  $k^\mu{}_{;\mu}$  calculated on  $\Sigma$  is negative for both families. We have on  $\Sigma$ , using (6.2):

$$k^\mu{}_{;\mu} = k^0{}_{,t} + k^1{}_{,z} + e^\alpha (\alpha_{,t} + 2\beta_{,t}) + e (\alpha_{,z} + 2\beta_{,z}). \quad (6.4)$$

In order to simplify this, we now differentiate the first of (6.1) by  $t$ , and write out the second of (6.1) for  $\mu = 1$ , in both cases taking the result on  $\Sigma$ , i.e. making use of the simplifications given in (6.2):

$$\begin{aligned} k^0{}_{,t} - e e^\alpha k^1{}_{,t} - e^\alpha \alpha_{,t} &= 0, \\ e^\alpha k^1{}_{,t} + e (k^1{}_{,z} + 2e^\alpha \alpha_{,t}) + \alpha_{,z} &= 0. \end{aligned} \quad (6.5)$$

Eliminating  $k^1{}_{,t}$  from (6.5), and using the result to substitute for  $k^0{}_{,t} + k^1{}_{,z}$  in (6.4) we get

$$k^\mu{}_{;\mu} = 2(e^\alpha \beta_{,t} + e \beta_{,z}). \quad (6.6)$$

Using now the expressions for  $e^\alpha$  and  $e^\beta$  in the notation of (2.7), i.e.

$$e^\alpha = \frac{R_{,z} - R\mathcal{E}_{,z}/\mathcal{E}}{\sqrt{\varepsilon + 2E(z)}}, \quad e^\beta = \frac{R}{\mathcal{E}} \quad (6.7)$$

we get in (6.6)

$$k^\mu{}_{;\mu} = 2 \left( \frac{R_{,z}}{R} - \frac{\mathcal{E}_{,z}}{\mathcal{E}} \right) \left( \frac{R_{,t}}{\sqrt{\varepsilon + 2E}} + e \right). \quad (6.8)$$

The first factor changes sign only at shell crossings (see Ref. [15]), so we take it to be positive. Consider collapse,  $R_{,t} < 0$ . For the ingoing family,  $e = -1$ , we have  $k^\mu{}_{;\mu} < 0$ , without further conditions. For the outgoing family,  $e = +1$ ,  $k^\mu{}_{;\mu}$  will be negative when  $R_{,t}/\sqrt{\varepsilon + 2E} < -1$ , which, with negative  $R_{,t}$ , means that  $R_{,t}^2 > \varepsilon + 2E$ . Using (2.8) with  $\Lambda = 0$  for  $R_{,t}^2$ , we then obtain

$$2M/R > \varepsilon. \quad (6.9)$$

With  $\varepsilon = 0$  and  $\varepsilon = -1$ , this is always fulfilled,<sup>9</sup> with the only exception of the ‘asymptotic origin’ in the planar model, where  $M/R = \varepsilon = 0$ .

A surface given by  $\{t = \text{constant}, z = \text{constant}\}$  passes through every point of the spacetime. Since each such surface has now been shown to be trapped at all of its points, this means that all points of the whole spacetime are trapped.

Thus, the quasi-hyperbolic and quasi-plane model, along with their hyperbolically- and plane-symmetric

<sup>8</sup> When the surface of constant  $t$  and  $z$  is infinite, it cannot be closed, therefore the labelling “ingoing” and “outgoing” is only conventional.

<sup>9</sup> Note that  $M$  must be positive, or else (3.6) would imply that collapse is retarded and expansion accelerated. This would be gravitational repulsion.



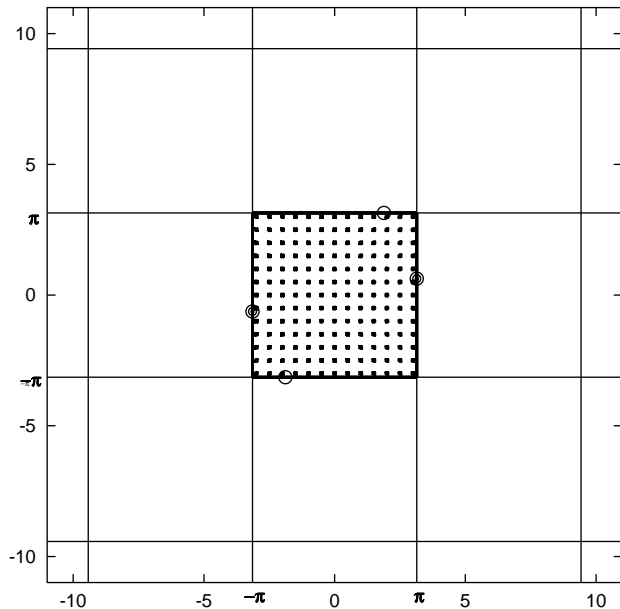


FIG. 4: A map of the elementary torus in the  $(p, q)$  coordinates (the central dotted square). Its left edge coincides in space with the right edge, the lower edge coincides with the upper edge. Sometimes it is convenient to consider the torus as a subset of the  $\mathbb{R}^2$  plane, in which case the plane should be imagined as covered with an infinite number of copies of the elementary square. The identifications may be done with a twist, i.e. with reflections in  $p = 0$  and  $q = 0$ , so that a projective plane results instead of a torus. The open circles and the full circles show the pairs of points to be identified in the latter situation.

limits, are globally future-trapped (when collapsing), and no apparent horizon exists for them. It follows now easily that the corresponding expanding models are globally past-trapped.

This is consistent with the fact that the corresponding vacuum solutions have no event horizons (see eq. (6.22) in Ref. [18]) and are globally nonstatic.

With no apparent horizons, no black holes may form (more precisely, the whole Universe is one black hole). This excludes the quasi-plane and quasi-hyperbolic models from an important area of application of the quasi-spherical and spherically symmetric dust models.

## VII. THE TOROIDAL PLANE SYMMETRIC MODEL IN THE SZEKERES COORDINATES

The metric of a torus given by constant  $t$  and  $z$  in (3.2) is  $ds^2 = R^2 (dp^2 + dq^2)$ , where  $2\pi R$  is the circumference of the torus in the  $p$ -direction and in the  $q$ -direction. In the following, we will consider the torus with  $R = 1$ , and we will call it the 'elementary square' or 'elementary torus'. It will be more convenient to assume that, in the coordinates of (3.2), the elementary torus is the square  $\{p, q\} \in [-\pi, \pi] \times [-\pi, \pi]$ , shown in Fig. 4, rather than  $\{p, q\} \in [0, 2\pi] \times [0, 2\pi]$ . The  $\mathbb{R}^2$ -space of the  $(p, q)$  coordinates can be imagined as filled with infinitely many copies of this square.

As observed in Ref. [18], the function  $S$  in the quasi-plane model can be absorbed into the other functions by the redefinition

$$(R, E, M) = (\tilde{R}/S, \tilde{E}/S^2, \tilde{M}/S^3), \quad (7.1)$$

so we can assume  $S = 1$  with no loss of generality. We do this in the following.

The coordinates of points to be identified are related in a more complicated way in the Szekeres coordinates of (2.1), in which the plane symmetric model is given by:

$$ds^2 = dt^2 - \frac{R_{,z}^2}{2E(z)} dz^2 - R^2 \frac{4(dx^2 + dy^2)}{[(x-P)^2 + (y-Q)^2]^2}, \quad (7.2)$$

with  $P$  and  $Q$  being arbitrary constants. A line  $q = q_0$  corresponds, in the  $(x, y)$ -coordinates, to

$$(x-P)^2 + (y-Q-1/q_0)^2 = 1/q_0^2, \quad (7.3)$$

which is in general a circle of radius  $1/q_0$  and the center at  $(x, y) = (P, Q + 1/q_0)$ . In the special case  $q_0 = 0$  the image becomes the straight line  $y = Q$ . Consequently, the lines  $q = \pm\pi$  in the  $(p, q)$ -coordinates go over into the circles

$$(x-P)^2 + (y-Q \mp 1/\pi)^2 = 1/\pi^2, \quad (7.4)$$

while the lines  $p = \pm\pi$  go over into the circles

$$(x-P \mp 1/\pi)^2 + (y-Q)^2 = 1/\pi^2. \quad (7.5)$$

The image of the central point  $(p, q) = (0, 0)$  is the infinity of the  $(x, y)$ -plane. Conversely, the point  $(x, y) = (P, Q)$  is the image of the infinity of the  $(p, q)$ -coordinates.

Moreover, from (7.3) follows that the image of the area  $\{q^2 < q_0^2\}$  (an infinite strip of the  $(p, q)$  plane contained between  $q = -q_0$  and  $q = q_0 > 0$ ) is the area *outside* the circles  $(x-P)^2 + (y-Q \mp 1/q_0)^2 = 1/q_0^2$ . Similarly, the image of the area  $\{p^2 < p_0^2\}$  is the area outside the circles  $(x-P \mp 1/p_0)^2 + (y-Q)^2 = 1/p_0^2$ . Consequently, the image of the elementary torus in the  $(x, y)$  coordinates will be the infinite subset of the  $\mathbb{R}^2$  plane lying outside all four circles, see Fig. 5.

This is the explanation to Fig. 5. The image of the line  $p = 0$  in the  $(x, y)$ -coordinates is the vertical line  $x = P$  in the figure, with  $(x, y) = (P, Q)$  being the image of infinity of the  $(p, q)$  coordinates. Similarly, the image of  $q = 0$  is the horizontal line  $y = Q$ . In the  $(p, q)$ -coordinates of (3.2), the torus is the area encircled by the straight lines  $p = -\pi, p = \pi, q = -\pi$  and  $q = \pi$ . The image of the line  $q = -\pi$  is the circle C1, of radius  $1/\pi$ . The image of the line  $q = \pi$  is the circle C2, of the same radius. The image of the torus must be contained outside these two circles – in the area covered with vertical strokes. Then, the image of  $p = -\pi$  is the circle C3, and the image of  $p = \pi$  is the circle C4, both of the same radius  $1/\pi$ . Consequently, the image of the torus must be contained outside these circles – in the area covered by horizontal strokes. Thus, the

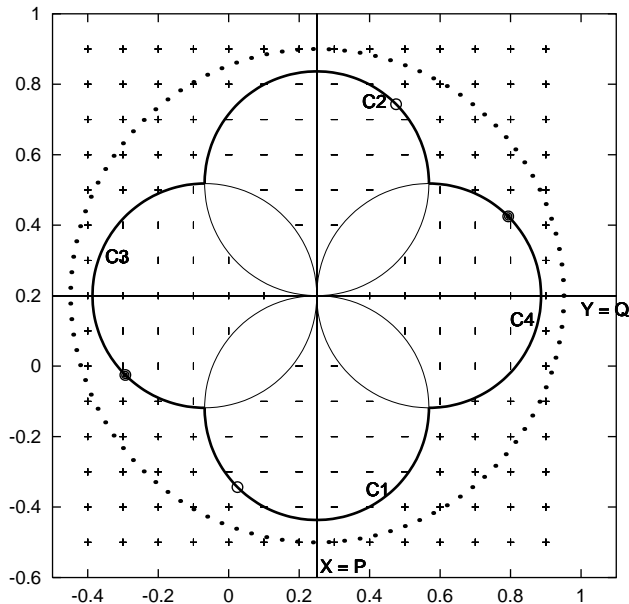


FIG. 5: The image of the torus  $\{p, q\} \in [-\pi, \pi] \times [-\pi, \pi]$  in the Szekeres coordinates. The values of  $P$  and  $Q$  were chosen arbitrarily, but other elements of the figure are drawn to scale. The small empty circles and the small solid circles mark pairs of points to be identified when we consider a projective plane instead of a torus. More explanation in the text.

image of the whole torus is the common subset of these two areas – the area in the figure outside the thick line and covered with crosses. The area inside the thick closed curve contains an infinite number of images of copies of the elementary torus.

Each circle in the figure with the center at  $(x, y) = (P, Q)$  and with radius  $a > 2/\pi$  is an image of a circle of radius  $2/a < \pi$  centered at  $(p, q) = (0, 0)$  that lies all within a single copy of the elementary torus (one such circle is shown in the figure with a dotted line). In particular, this applies to a circle of radius 1.

### VIII. A NONSYMMETRIC PLANAR MODEL

The intention of the foregoing considerations was to prepare the ground for carrying out similar identifications in the full nonsymmetric quasi-plane model. Unfortunately, no such identifications were proven possible. The claim made in Ref. [20] turned out to be erroneous, in consequence of a computational error. That error invalidates the statements made in Ref. [20] in sections VIII, IX and XI, which are therefore not included in the present text. Section VIII contained the incorrect identification, Section IX – the prescription to avoid shell crossings with a toroidal topology, and Section XI – a prescription for constructing a two-sided (orientable) compact surface out of four copies of a projective plane, which is nonorientable. The surface thus constructed was supposed to contain the active gravitational mass in a similar way to the one described in Sec. V for tori.

It has not been proved that nontrivial topologies for the surfaces of constant  $(t, z)$  are possible or impossible in the metric (2.7) with  $\varepsilon = 0$ , so the problem is still open.

### IX. FORMATION OF STRUCTURES IN THE PLANAR MODEL

This was Section X in Ref. [20].

As shown for the Lemaître – Tolman models (see Refs. [27] and [16], Sec. 18.19), in the ever-expanding case  $E > 0$  an increasing density perturbation,  $\rho_{,z}/\rho$ , freezes asymptotically into the background – i.e. it tends to a finite value determined uniquely by the initial conditions. Consequently, it is impossible in these models to describe the formation of condensations that collapse to a very high density, such as a galaxy with a central black hole. Since the evolution of the quasi-plane and quasi-hyperbolic models is described by the same equations, they will suffer from the same problem. (And we have already found in Sec. VI that these models cannot describe black holes.)

Thus, these models can be used for considering the formation of moderate-amplitude condensations and voids.

### X. INTERPRETATION OF $M(z)$ FOR THE QUASI-PLANE MODEL WITH INFINITE SPACES

With the toroidal interpretation in the plane symmetric limit, the proof that  $M(z)$  is a measure of the active gravitational mass was rather simple. The same may be shown also when the quasi-plane model is interpreted as infinite in extent, but in a more complicated way. We do show it in this section – however, this is only for mathematical completeness. As demonstrated in Ref. [18], with the infinite spaces the quasi-plane Szekeres solutions have irremovable shell crossings, and so are in fact not acceptable as cosmological models.

This section was Sec. XII in Ref. [20].

Recall that with  $\varepsilon = 0$  we are free to rename the functions  $R$ ,  $E$  and  $M$  as in (7.1), and the result will be the same as if  $S \equiv 1$ . Thus, we assume  $S \equiv 1$  throughout this section.

For the beginning let us consider the plane symmetric subcase of the  $\varepsilon = 0$  model, which has  $P_{,z} = Q_{,z} = 0$ . Let us choose a circle of radius 1 centred at  $(x, y) = (P, Q)$  (both  $P$  and  $Q$  being now constant) in every surface of constant  $t$  and  $z$ . Let  $d_2xy$  be the surface element in the  $(x, y)$  plane, and let  $U$  be the outside of the unit circle. This region has finite surface area. Then, introducing  $(u, \varphi)$  by

$$x = P + u \cos \varphi, \quad y = Q + u \sin \varphi, \quad (10.1)$$

we get

$$\int_U d_2xy \frac{1}{\mathcal{E}^2} = \int_0^{2\pi} d\varphi \int_1^\infty \frac{4}{u^3} du = 4\pi, \quad (10.2)$$

in every ( $t = \text{const}, z = \text{const}$ ) surface. Now let  $\mathcal{V}$  be a 3-dimensional set in a  $t = \text{const}$  space, extending from  $z = z_c$  to a running value of  $z$ , whose every section of constant  $z$  is  $U$  – the outside of the unit circle  $(x-P)^2 + (y-Q)^2 = 1$ . Calculating  $\mathcal{M} = \int_{\mathcal{V}} \rho \sqrt{|g_3|} d_3x$  with  $\varepsilon = 0$ ,  $S = 1$  and  $P, Q$  constant we get  $\mathcal{E}_{,z} = 0$  and

$$\mathcal{M} = \frac{1}{4\pi} \int_U d_2xy \int_{z_c}^z du \frac{M_{,u}}{\sqrt{2E}\mathcal{E}^2} = \int_{z_c}^z \frac{M_{,u}}{\sqrt{2E}}(x) du, \quad (10.3)$$

Thus, in this case,  $M$  behaves as the active gravitational mass contained outside a tube of coordinate height  $(z - z_c)$  which has radius equal to 1 at every  $z$  value, while  $1/\sqrt{2E}$  plays the role of the mass defect/excess factor. We recall that the  $(x, y)$  coordinates of (2.7), in the plane symmetric case  $\varepsilon = 0 = \mathcal{E}_{,z}$ ,  $S = 1$  are related by the inversion (3.1) to the Cartesian coordinates  $(p, q)$  in a plane, so the outside of the tube in the  $(x, y)$  coordinates is in reality the inside of the same tube in the Cartesian coordinates. Thus, physically,  $M$  is the active mass within a tube  $\mathcal{V}_2$ .

We calculated the integrals in (10.2) and (10.3) around the central point  $(x, y) = (P, Q)$ . However, with plane symmetry, the origin of the Cartesian coordinates can be transferred to any other point by a symmetry transformation.<sup>10</sup>

Let us consider the transformation (3.1), after which the metric becomes (3.2), which is formally (2.7) with  $\mathcal{E} = 1$  and  $(x, y)$  renamed to  $(p, q)$ . In this form, the transformation

$$p = p' + A_p, \quad q = q' + A_q \quad (10.4)$$

(with  $A_p$  and  $A_q$  being arbitrary constants) is an isometry of (3.2). Thus, the transformation (10.4) does not change either the metric (3.2) or the value of the integral (10.2), which, in the variables  $(p, q)$ , becomes simply  $4 \int_{S_1} dpdq = 4\pi$ , independently of where the centre of the circle  $S_1$  is located.

Now let us consider the planar metric  $\varepsilon = 0$  that is not plane symmetric, i.e. with  $P_{,z}$  and  $Q_{,z}$  not vanishing simultaneously. Let  $U(z)$  be the outside of a unit circle in an  $z = \text{const}$  surface, with the centre at

$(x, y) = (P(z), Q(z))$ . Within each single such surface, applying the transformation of variables (10.1), we get

$$\begin{aligned} \int_U d_2xy \frac{1}{\mathcal{E}^2} &= \int_0^{2\pi} d\varphi \int_1^\infty \frac{4}{u^3} du = 4\pi, \\ \int_U d_2xy \frac{\mathcal{E}_{,z}}{\mathcal{E}^3} & \\ &= \int_0^{2\pi} d\varphi \int_1^\infty \frac{-4u \cos \varphi P_{,z} - 4u \sin \varphi Q_{,z}}{u^5} du = 0. \end{aligned} \quad (10.5)$$

These integrals do not depend on  $P_{,z}$  or  $Q_{,z}$ , but the centres of the circles no longer have the same  $(x, y)$  coordinates at each  $z$ . Thus, to use (10.5) in an analogue of (10.3), we have to take a volume  $\mathcal{V}$  which is a wiggly tube: its every cross-section with a constant  $z$  surface is a unit circle, but the centres of the circles do not lie on a line orthogonal to the  $z = \text{const}$  surfaces. Instead, they lie on the curve in the  $t = \text{const}$  space given by the parametric equations  $x = P(z), y = Q(z)$ . Because of the second of (10.5), (10.3) still follows for this single tube.

The whole 3-space  $t = \text{const}$  is now no longer homogeneous with respect to the group of plane symmetries. However, each single  $z = \text{const}$  surface in that space is homogeneous. In particular, the surface containing the base of the tube,  $z = z_0$ , is homogeneous. Thus, we can apply the inversion (3.1) with  $P = P(z_0), Q = Q(z_0)$ . The inside and outside of the unit circle in the  $z = z_0$  surface will thereby simply interchange, but the resulting transformations in other  $z = \text{const}$  surfaces will be more complicated, and the wiggly tube will deform substantially. Still, in the inverted coordinates we are now free to move the centre of the base circle (within the  $z = z_0$  surface) to any other point.

We now carry out this plan. Let us denote:

$$\begin{aligned} P(z_0) &\stackrel{\text{def}}{=} P_0, & Q(z_0) &\stackrel{\text{def}}{=} Q_0, \\ V &\stackrel{\text{def}}{=} (P_0 - P)^2 + (Q_0 - Q)^2. \end{aligned} \quad (10.6)$$

To the metric (2.7) with  $\varepsilon = 0$  and  $S = 1$  we apply the inversion adapted to the surface  $\{t = \text{const}, z = z_0\}$ :

$$x = P_0 + \frac{p}{p^2 + q^2}, \quad y = Q_0 + \frac{q}{p^2 + q^2}. \quad (10.7)$$

After this, the 2-metric  $R^2 (dx^2 + dy^2) / \mathcal{E}^2$  becomes:

$$\begin{aligned} ds_2^2 &= \frac{1}{\mathcal{E}^2} (dx^2 + dy^2), \\ 2\tilde{\mathcal{E}} &= V (p^2 + q^2) + 2(P_0 - P)p + 2(Q_0 - Q)q + 1. \end{aligned} \quad (10.8)$$

In these coordinates, the surface  $\{t = \text{const}, z = z_0\}$  is explicitly homogeneous, so we are now free to shift the origin of coordinates to any other point by

$$p = p' + A_1, \quad q = q' + A_2, \quad (10.9)$$

with  $A_1$  and  $A_2$  being constants. After the shift, the metric is still Szekeres with  $\varepsilon = 0$ , but with complicated expressions for the new  $P, Q$  and  $S$ .

<sup>10</sup> The circle of unit radius in the Cartesian coordinates, when moved to another point of the  $(p, q)$  plane, will not have a unit coordinate radius in the  $(x, y)$  coordinates, and the image of the center of the circle will not be the center of the image circle. However, the surface area of the circle and the invariant distances between points are not changed.

After the transformations (10.7) and (10.9) the region  $U$  of integration in (10.5) (which was the outside of a tube extending out to infinity) goes over into a finite region – the inside of a certain tube whose edge is the image of the family of unit circles in  $(x, y)$ . In the integrals (10.5), the two transformations are just changes of integration variables, so the values of the integrals do not change, and thus (10.3) still applies. This shows that over each point of the surface of constant  $t$  and of  $z = z_0$  in the Szekeres  $\varepsilon = 0$  metric, we can find a region of finite volume (a wiggly tube) such that the function  $M$  can be interpreted as the active gravitational mass within that tube.

As an illustration we now consider a special case of the transformation (10.9) with  $A_1 \stackrel{\text{def}}{=} \lambda$  and  $A_2 = 0$ . But first we give the complete transformation that will take us back to the coordinates of (2.7) in the surface  $z = z_0$ .

To the variables  $(p', q')$  of (10.9) we apply the inversion in a circle of radius 1 centred at  $(p', q') = (0, 0)$ , and the shift by  $(P_0, Q_0)$  to the resulting  $(x', y')$  coordinates. Calling the final coordinates  $(x_2, y_2)$ , we calculate the effect of (10.9) on the variables  $(x, y)$ . The complete transformation from  $(x, y)$  to  $(x_2, y_2)$  is

$$\begin{aligned} x_2 &= P_0 + \frac{1}{W} (x - P_0 + A_1 \mathcal{U}), \\ y_2 &= Q_0 + \frac{1}{W} (y - Q_0 + A_2 \mathcal{U}), \end{aligned} \quad (10.10)$$

where

$$\mathcal{U} \stackrel{\text{def}}{=} (x - P_0)^2 + (y - Q_0)^2, \quad (10.11)$$

$$W \stackrel{\text{def}}{=} 1 + 2[A_1(x - P_0) + A_2(y - Q_0)] + (A_1^2 + A_2^2)\mathcal{U}.$$

This set of transformations does not change the metric (2.7) in the hypersurface  $z = z_0$ , but after the transformations the unit circle  $\mathcal{U} = 1$  goes over into the circle

$$(x_2 - P_0 - A_1/\gamma)^2 + (y_2 - Q_0 - A_2/\gamma)^2 = 1/\gamma^2, \quad (10.12)$$

where  $\gamma \stackrel{\text{def}}{=} A_1^2 + A_2^2 - 1$ .

Now we specialize this to the 1-parameter subgroup  $A_1 = \lambda$ ,  $A_2 = 0$ , i.e. to the shift along the  $p$ -direction in

(10.9). The transformation (10.10) – (10.11) becomes:

$$\begin{aligned} x_2 &= P_0 + \frac{x - P_0 + \lambda\{(x - P_0)^2 + (y - Q_0)^2\}}{W_0}, \\ y_2 &= Q_0 + \frac{y - Q_0}{W_0}, \end{aligned} \quad (10.13)$$

$$W_0 \stackrel{\text{def}}{=} \lambda^2\{(x - P_0)^2 + (y - Q_0)^2\} + 2\lambda(x - P_0) + 1,$$

and its inverse is obtained by replacing  $\lambda$  with  $(-\lambda)$ , i.e.

$$\begin{aligned} x &= P_0 + \frac{x_2 - P_0 - \lambda\{(x_2 - P_0)^2 + (y_2 - Q_0)^2\}}{\widetilde{W}_0}, \\ y &= Q_0 + \frac{y_2 - Q_0}{\widetilde{W}_0}, \end{aligned} \quad (10.14)$$

$$\widetilde{W}_0 \stackrel{\text{def}}{=} \lambda^2\{(x_2 - P_0)^2 + (y_2 - Q_0)^2\} - 2\lambda(x_2 - P_0) + 1.$$

The Jacobians of the two transformations are, respectively

$$\begin{aligned} \widetilde{J} &= \left| \frac{\partial(x_2, y_2)}{\partial(x, y)} \right| = \frac{1}{W_0^2}, \\ J &= \left| \frac{\partial(x, y)}{\partial(x_2, y_2)} \right| = \frac{1}{\widetilde{W}_0^2}. \end{aligned} \quad (10.15)$$

The following identities are useful in calculations:

$$\begin{aligned} (x_2 - P_0)^2 + (y_2 - Q_0)^2 &\equiv \frac{(x - P_0)^2 + (y - Q_0)^2}{W_0}, \\ (x - P_0)^2 + (y - Q_0)^2 &\equiv \frac{(x_2 - P_0)^2 + (y_2 - Q_0)^2}{\widetilde{W}_0}. \end{aligned} \quad (10.16)$$

The transformation (10.13) takes the circle  $(x - P)^2 + (y - Q)^2 = u^2$  to a shifted circle with a different radius, namely

$$(x_2 - A)^2 + (y_2 - B)^2 = (u/p_0)^2, \quad (10.17)$$

where

$$\begin{aligned} A &\stackrel{\text{def}}{=} \frac{\lambda^2 P_0 [(P_0 - P)^2 + (Q_0 - Q)^2 - u^2] - \lambda [u^2 + P_0^2 - P^2 - (Q_0 - Q)^2] + P}{p_0}, \\ B &\stackrel{\text{def}}{=} \frac{\lambda^2 Q_0 [(P_0 - P)^2 + (Q_0 - Q)^2 - u^2] - 2\lambda Q_0 (P_0 - P) + Q}{p_0}, \\ p_0 &\stackrel{\text{def}}{=} \lambda^2 [(P_0 - P)^2 + (Q_0 - Q)^2 - u^2] - 2\lambda(P_0 - P) + 1 \end{aligned} \quad (10.18)$$

(in fact, these will be applied with  $u = 1$ .)

Now using (10.14) – (10.16) we find:

$$(x - P)^2 + (y - Q)^2 = \frac{S_0}{\widetilde{W}_0} [(\alpha_2 + \alpha)^2 + (y_2 + \beta)^2], \quad (10.19)$$

where

$$\begin{aligned} S_0 &\stackrel{\text{def}}{=} 1 - 2\lambda(P_0 - P) + \lambda^2 [(P_0 - P)^2 + (Q_0 - Q)^2], \\ \alpha &\stackrel{\text{def}}{=} -P_0 + \frac{P_0 - P - \lambda [(P_0 - P)^2 + (Q_0 - Q)^2]}{S_0}, \\ \beta &\stackrel{\text{def}}{=} -Q_0 + \frac{Q_0 - Q}{S_0}. \end{aligned} \quad (10.20)$$

Note that  $S_0$ ,  $\alpha$  and  $\beta$  are functions only of  $z$ , they do not depend on  $x_2$  and  $y_2$ . We can see that the  $\widetilde{W}_0^2$  that will appear in the transformed integral in (10.5),  $\int_U d_2xy/\mathcal{E}^2$ , will be canceled by the  $\widetilde{W}_0^2$  from the Jacobian of the transformation, and what remains will be

$$\int \frac{1}{\mathcal{E}^2} dx dy = \frac{4}{S_0(z)^2} \int \frac{1}{[(x_2 + \alpha)^2 + (y_2 + \beta)^2]^2} dx_2 dy_2, \quad (10.21)$$

an integral of exactly the form (10.5), except for the additional factor  $1/S_0^2$ .

The transformation (10.13) affects also the metric. Under (10.13),  $\mathcal{E}$  changes as follows:

$$\mathcal{E} = (x - P)^2 + (y - Q)^2 = \frac{S_0}{W_0} [(x_2 + \alpha)^2 + (y_2 + \beta)^2]. \quad (10.22)$$

The expression  $dx^2 + dy^2$ , after the transformation (10.13) goes over into

$$(dx_2^2 + dy_2^2) / \widetilde{W}_0^2, \quad (10.23)$$

and so the two equations above imply that after the transformation:

$$\frac{dx^2 + dy^2}{\mathcal{E}^2} = \frac{dx_2^2 + dy_2^2}{S_0^2 [(x_2 + \alpha)^2 + (y_2 + \beta)^2]^2}. \quad (10.24)$$

We are in the same Szekeres model as at the beginning, but at a different location. The alien form of the Szekeres metric in these coordinates results from the fact that the transformation (10.13) is not a symmetry, apart from the single surface  $z = z_0$ . Since the quantities in this equation ( $\alpha$ ,  $\beta$  and  $S_0$ ) depend on the continuous parameter  $\lambda$ , the equation above describes the result of a shift to any location (with  $\lambda = 0$  corresponding to identity). Equations (10.24) and (10.21) show that the transformation (10.13) is area-preserving.

The second integral in (10.5) does not change its zero value under the transformation (10.13) – since that transformation is simply a change of variables under a definite integral, applied both to the integrand and to the area of integration. Thus, the second integral in (10.5) does not contribute to calculating  $\mathcal{M}$  in (10.3).

## XI. SUMMARY

Continuing the research begun in Ref. [18], geometrical properties of the quasi-plane Szekeres model were

investigated here, along with the corresponding properties of the plane symmetric model. The following results were achieved:

1. The pattern of decelerated expansion in the plane symmetric model was analysed and shown to be in complete disagreement with the Newtonian analogues and intuitions (Sections III and IV). An example of a Newtonian potential that gives a similar pattern of expansion has co-axial parallel cones as its equipotential surfaces; it has not been investigated whether such a potential can be generated by any realistic matter distribution.

2. Embeddings of the constant  $t$  and constant  $z$  surfaces in the Euclidean space suggest that the flat surfaces contained in the plane symmetric model can be interpreted as flat tori whose circumferences are proportional to the function  $R(t, z)$ , and thus vary with time (Sec. V). Such a topology immediately explains the pattern of expansion and implies that the total mass contained within a  $z = \text{const}$  surface is finite.

3. The quasi-plane and quasi-hyperbolic models are permanently trapped (Sec. VI), so no apparent horizons exist in them. Consequently, these models cannot be used to describe the formation of black holes (the whole Universe is one black hole all the time).

4. The quasi-plane model cannot describe the formation of structures that collapse to very high densities (Sec. IX), since the density perturbations tend to finite values in the asymptotic future.

5. In the full (nonsymmetric) case the mass function is proportional to the active gravitational mass within a 'wiggly tube' of finite radius (Sec. X).

Whether the toroidal interpretation is a necessity is still unknown. However, this paper demonstrated that with the toroidal topology the plane symmetric model becomes in several respects simpler, and, however paradoxical this may sound, more realistic.

The plane symmetric model with toroidal spaces may be a testing ground for the idea of a 'small Universe', proposed by Ellis [29]. A small Universe is one with compact spatial sections, in which thus a present observer has already seen several times around the space. Several papers were devoted to checking this idea against the observational data (see, for example, Refs. [30] – [35]; a conclusive proof or disproof of any nontrivial topology is, unfortunately, still lacking). However, the background geometry has always been a homogeneous isotropic Robertson – Walker metric with identifications in the underlying manifold. The plane symmetric toroidal Szekeres (Ellis) model has a less general topology (identifications in it occur only in two-dimensional surfaces, in the  $z$ -direction the space is infinite), but is inhomogeneous, so might be useful for considering light propagation and comparing the mass distribution in the model with the observed images.

**Acknowledgements** The research for this paper was inspired by a collaboration with Charles Hellaby, initiated in 2006 at the Department of Mathematics and Applied Mathematics in Cape Town. It was supported by

the Polish Ministry of Science and Education grant no 1 P03B 075 29. Discussions with Ch. Hellaby helped in clarifying several points. I am grateful to Dr. Stanisław Bajtlik for directing me to the relevant references on the observational aspects of the topology of space. Members of the relativity seminar at the Institute of Theoretical

Physics, Warsaw University, are gratefully acknowledged for their valuable comments on the geometry of projective planes that led to an improvement in this text.

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