

Geometry and topology of the quasilane Szekeres model

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(Received 7 May 2008; published 15 September 2008)

Geometrical and topological properties of the quasilane Szekeres model and of the plane-symmetric dust model are discussed. Some related comments on the quasihyperbolic model are made. These properties include: (1) The pattern of expansion in the plane-symmetric case, and the Newtonian model that imitates it; (2) The possibility of toroidal topology of the $t = \text{const}$ sections in the plane-symmetric model; (3) The absence of apparent horizons in the quasilane and quasihyperbolic models (they are globally trapped); (4) Description of the toroidal topology in the Szekeres coordinates; (5) Consequences of toroidal topology in the nonsymmetric quasilane model; (6) Avoidance of shell crossings in the toroidal model; (7) Interpretation of the mass function in the quasilane model, with the toroidal and with the infinite space.

DOI: [10.1103/PhysRevD.78.064038](https://doi.org/10.1103/PhysRevD.78.064038)

PACS numbers: 04.20.Jb

I. MOTIVATION

The quasispherical Szekeres model [1–17] is rather well understood by now. In spite of its nontrivial geometry, its basic defining features are not too difficult to grasp intuitively. In a simple-minded way one may say that it is obtained when the spherical symmetry orbits in the Lemaître-Tolman (L-T) model [16,17] are made nonconcentric to destroy the symmetry, but the energy-momentum tensor is still that of dust. Recently, that model even found application to solving problems directly related to observational cosmology [13,14]. In contrast to this, the first serious attempt to interpret the quasilane and quasihyperbolic models [18] revealed that even the corresponding plane- and hyperbolically symmetric models are not really understood and require more investigation. Some properties of those models were established in Ref. [18]; the present paper is a continuation of that research.

The aim of the present paper is to clarify some of the basic geometrical features of the quasilane Szekeres model, and of the plane-symmetric dust model. The following topics are investigated here: (1) The pattern of expansion in the plane-symmetric model, and the Newtonian model that imitates it; (2) The possibility of toroidal topology of the $t = \text{const}$ sections in the plane-symmetric model; (3) The absence of apparent horizons in the quasilane and quasihyperbolic models (they are globally trapped); (4) Description of the toroidal topology in the Szekeres coordinates; (5) Consequences of toroidal topology in the nonsymmetric quasilane model—it turns out that in the nonsymmetric case the orbits of quasisymmetry must be projective planes rather than tori; (6) Avoidance of shell crossings—in the toroidal model they can be avoided, unlike in the model with infinite orbits; (7) Interpretation of the mass function in the quasilane model. For the most part, the paper is devoted to

showing that the space of constant time in the plane-symmetric dust and the quasilane Szekeres models can be interpreted as a family of flat tori, with the ones of smaller diameter enclosed inside those of larger diameter. Such a topology explains several properties of the models, among them the pattern of accelerated expansion and the finiteness of the mass function, and also it allows to construct models that are free of shell crossings. In the end it turns out that these models are of lesser use in astrophysical cosmology than the quasispherical ones. Because of being globally trapped, they cannot be used for modeling dynamical black holes. Because they expand by the same law as the positive-energy Lemaître-Tolman model, they cannot model the formation of structures that collapse to very dense states. They might be applicable for the description of formation of moderate condensations, like galaxy clusters, and of voids.

Mena, Natário, and Tod have recently also considered the quasilane and quasihyperbolic Szekeres models with toroidal and higher-genus topologies [19]. They considered the matching of those solutions, with nonzero cosmological constant (corresponding to $\Lambda > 0$ in the notation adopted here), to the plane- and hyperbolically symmetric counterparts of the Schwarzschild solution, also allowed to have nontrivial topologies of the symmetry orbits. However, there is no overlap between their results and those of the present paper, as they mainly considered the global geometry of the resulting black hole, while here the emphasis is put on local geometry of the topologically nontrivial Szekeres spacetime.

In the next section, basic facts about the Szekeres solutions are recalled.

II. INTRODUCING THE SZEKERES SOLUTIONS

The metric of the Szekeres solutions is

$$ds^2 = dt^2 - e^{2\alpha} dz^2 - e^{2\beta}(dx^2 + dy^2), \quad (2.1)$$

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where α and β are functions of (t, x, y, z) to be determined from the Einstein equations with a dust source. The coordinates of (2.1) are comoving so the velocity field of the dust is $u^\mu = \delta^\mu_0$ and $\dot{u}^\mu = 0$.

There are two families of Szekeres solutions, depending on whether $\beta_{,z} = 0$ or $\beta_{,z} \neq 0$. The first family is a simultaneous generalization of the Friedmann and Kantowski-Sachs [20] models. So far it has found no useful application in astrophysical cosmology, and we shall not discuss it here (see Ref. [16]); we shall deal only with the second family. After the Einstein equations are solved, the metric functions in (2.1) become

$$\begin{aligned} e^\beta &= \Phi(t, z)/e^{\nu(z,x,y)}, \\ e^\alpha &= h(z)\Phi(t, z)\beta_{,z} \equiv h(z)(\Phi_{,z} + \Phi\nu_{,z}), \\ e^{-\nu} &= A(z)(x^2 + y^2) + 2B_1(z)x + 2B_2(z)y + C(z), \end{aligned} \quad (2.2)$$

where the function $\Phi(t, z)$ is a solution of the equation

$$\Phi_{,t}{}^2 = -k(z) + \frac{2M(z)}{\Phi} + \frac{1}{3}\Lambda\Phi^2; \quad (2.3)$$

while $h(z)$, $k(z)$, $M(z)$, $A(z)$, $B_1(z)$, $B_2(z)$, and $C(z)$ are arbitrary functions obeying

$$g(z) \stackrel{\text{def}}{=} AC - B_1^2 - B_2^2 = \frac{1}{4}[1/h^2(z) + k(z)]. \quad (2.4)$$

The mass-density is

$$\kappa\rho = \frac{(2Me^{3\nu})_{,z}}{e^{2\beta}(e^\beta)_{,z}}; \quad \kappa = 8\pi G/c^2. \quad (2.5)$$

In the present paper we will mostly consider the case $\Lambda = 0$.

This solution has in general no symmetry, and acquires a three-dimensional symmetry group with two-dimensional orbits when A , B_1 , B_2 , and C are all constant (that is, when $\nu_{,z} = 0$). The sign of $g(z)$ determines the geometry of the surfaces ($t = \text{const}$, $z = \text{const}$), and the symmetry of the limiting solution. The geometry is spherical, plane, or hyperbolic when $g > 0$, $g = 0$, or $g < 0$, respectively. With A , B_1 , B_2 , and C being functions of z , the surfaces $z = \text{const}$ within a single space $t = \text{const}$ may have different geometries (i.e., they can be spheres in one part of the space and the surfaces of constant negative curvature elsewhere, the curvature being zero at the boundary). The sign of $k(z)$ determines the type of evolution; with $k > 0 = \Lambda$ the model expands away from an initial singularity and then recollapses to a final singularity, with $k < 0 = \Lambda$ the model is either ever-expanding or ever-collapsing, depending on the initial conditions; $k = 0 = \Lambda$ is the intermediate case corresponding to the ‘‘flat’’ Friedmann model.

The Robertson-Walker (R-W) limit follows when $z = r$, $\Phi(t, z) = rR(t)$, $k = k_0r^2$ where $k_0 = \text{const}$ and $B_1 = B_2 = 0$, $C = 4A = 1$. This definition of the R-W limit includes the definition of the limiting radial coordinate (the Szekeres model is covariant with the transformations $z = f(z')$, where $f(z')$ is an arbitrary function).

The Szekeres models are subdivided according to the sign of $g(z)$ into the quasispherical ones (with $g > 0$), quasilplane ($g = 0$), and quasihyperbolic ($g < 0$).¹ Despite suggestions to the contrary made in the literature, the geometry of the latter two classes has, until very recently, not been investigated at all and is not really understood; work on their interpretation has only been begun by Helalby and Krasinski [18]. The sign of $g(z)$ is independent of the sign of $k(z)$, but limitations are imposed on $k(z)$ by the signature of the spacetime: for the signature to be the physical $(+---)$, the function h^2 must be non-negative (possibly zero at isolated points, but not in open subsets), which, via (2.4) means that $g(z) - k(z) \geq 0$ everywhere. Thus, with $g > 0$ (in the quasispherical case) all three possibilities for k are allowed; with $g = 0$ only the two $k \leq 0$ evolutions are admissible, and with $g < 0$, only the $k < 0$ evolution is allowed.

Only the quasispherical model is rather well investigated, and has found useful application in astrophysical cosmology. We recall now its basic properties.

It may be imagined as such a generalization of the Lemaître–Tolman model in which the spheres of constant mass were made nonconcentric. The functions $A(z)$, $B_1(z)$, and $B_2(z)$ determine how the center of a sphere changes its position in a space $t = \text{const}$ when the radius of the sphere is increased or decreased (see a more detailed discussion of this in Ref. [15]). Still, this is a rather simple geometry because all the arbitrary functions depend on just one variable, z . They give us some limited possibility to model the real structures in the Universe (see very elegant examples in Refs. [13,14]), but a fully satisfactory model should involve arbitrary functions of all three spatial variables, to allow modeling of arbitrary structures. Such models are still nonexistent, so the Szekeres models are so far the best devices that exist.

Often, it is more practical to reparametrize the arbitrary functions in the Szekeres metric as follows (this parametrization was invented by C. Hellaby [21]). Even if $A = 0$ initially, a transformation of the (x, y) -coordinates can restore $A \neq 0$, so we may assume $A \neq 0$ with no loss of generality (see Ref. [16]). Then let $g \neq 0$. Writing

$$\begin{aligned} (A, B_1, B_2) &= \frac{\sqrt{|g|}}{S}(1, -P, -Q), \quad \varepsilon \stackrel{\text{def}}{=} g/|g|, \\ k &= -4|g| \times 2E, \quad M = 8|g|^{3/2}\tilde{M} \quad \text{and} \\ \Phi &= R\sqrt{|4g|}, \end{aligned} \quad (2.6)$$

¹We stress once again that *the same* Szekeres model may be quasispherical in one part of the spacetime, and quasihyperbolic elsewhere, with the boundary between these two regions being quasilplane, see an explicit simple example in Ref. [18]. In most of the literature published so far, these models have been considered separately, but this was only for purposes of systematic research.

we can represent the metric (2.1) as

$$e^{-\nu} = \sqrt{|4g|\mathcal{E}}, \quad \mathcal{E} \stackrel{\text{def}}{=} \frac{S}{2} \left[\left(\frac{x-P}{S} \right)^2 + \left(\frac{y-Q}{S} \right)^2 + \varepsilon \right],$$

$$ds^2 = dt^2 - \frac{(R_{,z} - R\mathcal{E}_{,z}/\mathcal{E})^2}{\varepsilon + 2E(z)} dz^2 - \frac{R^2}{\mathcal{E}^2} (dx^2 + dy^2). \quad (2.7)$$

When $g = 0$, the transition from (2.1) to (2.7) is $A = 1/(2S)$, $B_1 = -P/(2S)$, $B_2 = -Q/(2S)$, $k = -2E$, $\tilde{M} = M$, and $\Phi = R$. Then (2.7) applies with $\varepsilon = 0$, and the resulting model is quasiplane.

For further reference, the evolution equation (2.3), in the variables of (2.7), becomes

$$R_{,t}^2 = 2E(z) + \frac{2\tilde{M}(z)}{R} + \frac{1}{3}\Lambda R^2. \quad (2.8)$$

From now on, we will use this representation; the tilde over M will be dropped, but it must be remembered that the M in (2.8) is not the same as the one in (2.3).

The representation (2.7) makes the calculations simpler because the arbitrary functions in it are independent (the condition (2.4) has been incorporated in this form). However, it creates the illusion that the cases $\varepsilon = +1, 0, -1$ characterize the whole spacetime, and thus obscures the fact that all three cases can be parts of the same spacetime.

Rotation and acceleration of the dust source are zero, the expansion is

$$\Theta = 3 \frac{R_{,t}}{R} + \frac{R_{,tz} - R_{,t}R_{,z}/R}{R_{,z} - R\mathcal{E}_{,z}/\mathcal{E}}, \quad (2.9)$$

and the shear tensor is

$$\sigma^\alpha_\beta = \frac{1}{3} \Sigma \text{diag}(0, 2, -1, -1),$$

$$\text{where } \Sigma = \frac{\Phi_{,tz} - \Phi_{,t}\Phi_{,z}/\Phi}{\Phi_{,z} - \Phi\nu_{,z}} \equiv \frac{R_{,tz} - R_{,t}R_{,z}/R}{R_{,z} - R\mathcal{E}_{,z}/\mathcal{E}}. \quad (2.10)$$

Definitions of the Szekeres solutions by invariant properties can be found in Ref. [16].

When $\Lambda \neq 0$, the solutions of (2.8) involve elliptic functions. A general formal integral of (2.8) was presented by Barrow and Stein-Schabes [22]. Any solution of (2.8) will contain one more arbitrary function of z that will be denoted $t_B(z)$, and will enter the solution in the combination $(t - t_B(z))$. The instant $t = t_B(z)$ defines the initial moment of evolution; when $\Lambda = 0$ it is necessarily a singularity corresponding to $\Phi = 0$, and it goes over into the Big Bang singularity in the Friedmann limit. When $t_{B,z} \neq 0$ (that is, in general) the instant of singularity is position-dependent, just as it was in the L-T model.

Just as in the L-T model, another singularity may occur where $(e^\beta)_{,z} = 0$ (if this equation has solutions). This is a shell crossing, but it is qualitatively different from that in the L-T model. As can be seen from (2.2), in the quasi-

spherical case, when a shell crossing exists, its intersection with a $t = \text{const}$ space will be a circle, or, in exceptional cases, a single point, not a sphere. In the quasispherical models shell crossings can be avoided altogether if the arbitrary functions are chosen appropriately; see the complete list and derivation in Ref. [15]. In the quasihyperbolic models, shell crossings can be avoided in one sheet of each hyperboloid, but are unavoidable in the other, see Ref. [18]. In the quasiplane model, if the flat surfaces existing in it are interpreted as infinite planes, shell crossings are unavoidable [18]. However, if these surfaces are interpreted as being compact, shell crossings can be avoided—see Sec. IX in the present paper.

Equation (2.8) is identical with the Friedmann equation, but, just like in the L-T limit, with k and M depending on z , each surface $z = \text{const}$ evolves independently of the others.

The models defined by (2.1)–(2.5) contain eight functions of z , but only five of them correspond to independent physical degrees of freedom. One of the eight functions is determined by (2.4), $g(z)$ was made constant by the reparametrization (2.6), and one can be specified by a choice of z , for example, by defining $z' = M$, or $M = z'^3 \times \{\text{a constant}\}$.

A quasispherical Szekeres region can be matched to the Schwarzschild solution across a $z = \text{const}$ hypersurface [6]. The other two Szekeres regions can be matched to the plane- and hyperbolically symmetric counterparts of the Schwarzschild solution (see Ref. [23] for the solutions and [18] for the matching).

In the following, we will represent the Szekeres solutions with $\beta_{,z} \neq 0$ in the parametrization introduced in (2.7). The formula for density in these variables is

$$\kappa\rho = \frac{2(M_{,z} - 3M\mathcal{E}_{,z}/\mathcal{E})}{R^2(R_{,z} - R\mathcal{E}_{,z}/\mathcal{E})}, \quad (2.11)$$

where, let it be recalled, the M above is the \tilde{M} of (2.6).

III. THE PLANE-SYMMETRIC MODELS

The plane-symmetric dust models (first found by Ellis [24]) result from (2.7) when $\varepsilon = 0$ and (P, Q, S) are independent of z . The constant S can then be scaled to 1 by appropriate redefinitions of R , E , and M . Then, with constant P and Q , the coordinate transformation

$$x = P + \frac{2p}{p^2 + q^2}, \quad y = Q + \frac{2q}{p^2 + q^2} \quad (3.1)$$

changes the metric to

$$ds^2 = dt^2 - \frac{R_{,z}^2}{2E(z)} dz^2 - R^2(dp^2 + dq^2), \quad (3.2)$$

while the energy-density simplifies to

$$\frac{8\pi G}{c^2}\rho = \frac{2M_{,z}}{R^2 R_{,z}}. \quad (3.3)$$

These models are called plane symmetric because their symmetries are the same as those of the Euclidean plane; in the coordinates of (3.2) they are

$$p' = p + A_1, \quad (3.4a)$$

$$q' = q + A_2, \quad (3.4b)$$

$$(p', q') = (p \cos\alpha + q \sin\alpha, -p \sin\alpha + q \cos\alpha), \quad (3.4c)$$

where A_1 , A_2 , and α are arbitrary constants—the group parameters.

Note that Eqs. (2.8) and (3.3) are identical to their counterparts in the spherically symmetric models. In particular, the function $M(z)$ enters in the same way as the active gravitational mass did in spherical models. However, if we wish to interpret $M(z)$ as a mass contained in a volume, we encounter a problem—see below.

Examples of plane-symmetric spaces are the Euclidean plane and the Euclidean space E_3 with the metric $ds_3^2 = dx^2 + dy^2 + dz^2$. However, the space of constant t in (3.2) can never become flat; its curvature tensor is [18]

$${}^3R^z{}_p{}^p{}_z = {}^3R^z{}_q{}^q{}_z = -\frac{E_{,z}}{RR_{,z}}, \quad {}^3R^{pq}{}_{pq} = -\frac{2E}{R^2}. \quad (3.5)$$

Nevertheless, the surfaces P_2 of constant t and z in (3.2) are flat. Thus, there is some mystery in the geometry of the spacetimes (3.2). One component of the mystery is this: In the quasispherical case, and in the associated spherically symmetric model, the surfaces of constant t and z were spheres, and $M(z)$ was a mass inside a sphere of coordinate radius z . In the plane-symmetric case, if the P_2 surfaces are infinite planes, they do not enclose any finite volume, so where does the mass $M(z)$ reside?

With M being a constant, the metric (3.2) becomes vacuum—the plane-symmetric analogue of the Schwarzschild spacetime.

In the quasispherical Szekeres, and in spherically symmetric, solutions, analogies exist between the relativistic and the Newtonian models. We will thus compare the plane-symmetric model with its possible Newtonian counterparts. For this purpose, let us note the pattern of expansion in (2.8) and (3.2) with $\Lambda = 0$. When $R_{,t} \neq 0$, Eq. (2.8) implies

$$R_{,tt} = -M/R^2. \quad (3.6)$$

Note that $R_{,tt} = 0$ implies $M = 0$, which is a vacuum (in fact, Minkowski in strange coordinates) metric. Now take a pair of dust particles, located at (t, z_1, p_0, q_0) and at (t, z_2, p_0, q_0) , and consider the affine distance between them:

$$\ell_{12}(t) = \int_{z_1}^{z_2} \frac{R_{,z} dz}{\sqrt{2E}} \Rightarrow \frac{d^2\ell_{12}}{dt^2} = \int_{z_1}^{z_2} \frac{R_{,ttz} dz}{\sqrt{2E}}. \quad (3.7)$$

Thus, the two particles will be receding from each other (or approaching each other if collapse is considered) with acceleration that can never be zero.²

Take another pair of dust particles, located at (t, z_0, p_1, q_0) and at (t, z_0, p_2, q_0) . The distance between them, measured within the symmetry orbit, is

$$\begin{aligned} \ell_{34}(t) &= \int_{p_1}^{p_2} R dp \equiv R(p_2 - p_1) \\ &\Rightarrow \frac{d^2\ell_{34}}{dt^2} = R_{,tt}(p_2 - p_1), \end{aligned} \quad (3.8)$$

i.e., the acceleration of the expansion can never vanish in this direction, either, unless $M = 0$. The same result will follow for any direction in the (p, q) surface. Thus, the expansion or collapse in this model proceeds with acceleration in every spatial direction. We will compare this result with the Newtonian situation.

IV. A NEWTONIAN ANALOGUE OF THE PLANE-SYMMETRIC DUST SPACETIME

At first sight, it seems obvious that the Newtonian model analogous to the plane-symmetric dust model in relativity should be dust whose density is constant on parallel (x, y) -planes, and depends only on z . Let us follow this idea.

If the potential is plane-symmetric, then, in the adapted coordinates, it depends only on z . Thus, the Poisson equation simplifies to

$$\frac{d^2V}{dz^2} = 4\pi G\rho(z). \quad (4.1)$$

The general solution of this is

$$V = 4\pi G \int_{z_0}^z dz' \int_{z_0}^{z'} d\tilde{z} \rho(\tilde{z}) + Az + B, \quad (4.2)$$

where A and B are integration constants; z_0 is a reference value of z at which we can specify an initial condition. If we wish to have $V = \text{const}$ (i.e., zero force) when $\rho \equiv 0$, we must take $A = 0$, and then $V(z_0) = B$.³ The equations of motion in this potential are

$$\frac{dv^i}{dt} = -\frac{\partial V}{\partial x^i}, \quad (4.3)$$

where v^i are components of the velocity field of matter, so

$$\frac{dv^x}{dt} = \frac{dv^y}{dt} = 0, \quad \frac{dv^z}{dt} = -\frac{dV}{dz} = -4\pi G \int_{z_0}^z dz' \rho(z'). \quad (4.4)$$

This, however, gives a pattern of expansion different from

²The acceleration would be zero if $R_{,ttz} = 0$, which leads to a contradiction in the Einstein equations.

³An $A \neq 0$ would be qualitatively similar to the cosmological constant in relativity.

that in the relativistic plane-symmetric model. In the above, expansion with acceleration proceeds only in the z -direction, while in the directions orthogonal to z there is no acceleration, or, in a special case, not even any expansion. Consequently, no obvious Newtonian analogue exists for the relativistic plane-symmetric model.⁴

Equation (4.4) shares one property with the relativistic evolution Eq. (2.8). If ρ is bounded in the range of integration, then the force that drives the motion of the fluid is finite, giving the illusion that the potential is generated by some finite mass. However, if we wanted to calculate $V(z)$ by summing up contributions to it from all the volume elements of the fluid, like is done in calculating the gravitational potentials of finite portions of matter, then the result would be an infinite value of V , in consequence of the source having infinite extent in the (x, y) -plane. Thus, if we want to interpret the right-hand side in (4.3) as being generated by a mass, then the mass that drives the evolution is not the total mass in the source, but the mass of a finite portion of the source.

We now provide a solution of the Poisson equation that qualitatively mimics the pattern of expansion of the plane-symmetric relativistic model. The equipotential surfaces of the potential in question will be locally plane symmetric, but their symmetries will not be symmetries of the whole space.

Consider two families of cones given by the equations (see Fig. 1)

$$u = z - \alpha r, \quad v = z + r/\alpha, \quad r \stackrel{\text{def}}{=} \sqrt{x^2 + y^2}, \quad (4.5)$$

where α is a constant and (x, y, z) are Cartesian coordinates. The cones of constant u are orthogonal to the cones of constant v , and the two families are coaxial. We choose u and v as two coordinates in space; the third coordinate will be the angle φ around the axis of symmetry. We begin with the Euclidean metric in the cylindrical coordinates, $ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$, and transform this to the (u, v, φ) coordinates by

$$r = \frac{\alpha(v - u)}{1 + \alpha^2}, \quad z = \frac{u + \alpha^2 v}{1 + \alpha^2}. \quad (4.6)$$

The transformed metric is

$$ds^2 = \frac{du^2 + \alpha^2 dv^2}{1 + \alpha^2} + \frac{\alpha^2(v - u)^2 d\varphi^2}{(1 + \alpha^2)^2}. \quad (4.7)$$

The Laplace operator, which in the cylindrical coordinates is

⁴Incidentally, there will be no Newtonian analogue for the hyperbolic model, since the orbits of hyperbolic symmetry cannot be embedded in a Euclidean space at all. They can be embedded in a flat three-dimensional space, but the space then must have the signature $(- + +)$.

$$\Delta V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}, \quad (4.8)$$

in the (u, v, φ) -coordinates becomes

$$\Delta V = (1 + \alpha^2) \left[-\frac{1}{v - u} \frac{\partial V}{\partial u} + \frac{\partial^2 V}{\partial u^2} + \frac{1}{\alpha^2(v - u)} \frac{\partial V}{\partial v} + \frac{1}{\alpha^2} \frac{\partial^2 V}{\partial v^2} + \frac{1 + \alpha^2}{\alpha^2(v - u)^2} \frac{\partial^2 V}{\partial \varphi^2} \right]. \quad (4.9)$$

Thus, if V depends only on u , then the Poisson equation says:

$$(1 + \alpha^2) \left(-\frac{1}{v - u} \frac{\partial V}{\partial u} + \frac{\partial^2 V}{\partial u^2} \right) = -4\pi G\rho. \quad (4.10)$$

This density has a singularity at $u = v$, i.e., at $r = 0$ in the cylindrical coordinates.

The gradient of $V(u)$ has nonzero components in all directions, and so will create expansion decelerated in all directions. The expansion will be isotropic with respect to the $u = v$ axis, and the anisotropy between the (x, y) and the z -directions is controlled by α .

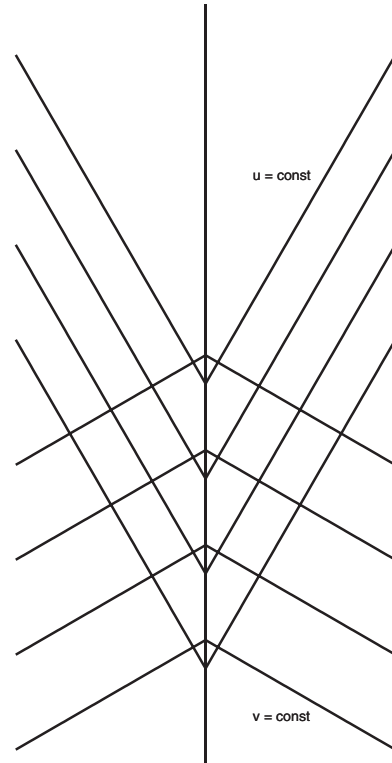


FIG. 1. The cones where u is constant are orthogonal to the cones where v is constant. The coordinates in space are u , v , and the angle around the axis of symmetry. The figure shows an axial cross section through the setup. A gravitational potential which depends only on u in these coordinates gives an expansion pattern that is qualitatively similar to the one in a plane-symmetric dust spacetime.

This potential was introduced here for illustrative purposes, and it is doubtful that it could correspond to any physical configuration. In order to make it credible, one should solve the continuity equation and the Euler equations of motion in it. We do not quote here the appropriate calculations because they lead to an intransparent tangle of differential equations. For dust, that set of equations is overdetermined, so it probably has no solutions.

V. PLANE-SYMMETRIC 3-SPACES INTERPRETED AS TORI

Although known for a long time (see Ref. [24]), the plane-symmetric model has not been investigated for its geometrical and physical properties.

Since a flat spatial geometry is not possible in it (see Eq. (3.5)), we now consider other possible 3-geometries with planar symmetry. The next simplest is a space of constant curvature. From (3.5), the space of constant $t = t_0$ will have constant curvature when

$$2E = \pm C^2 R^2, \quad (5.1)$$

where C is a constant. The curvature is positive when $E < 0$ and negative when $E > 0$. Since the signature of spacetime requires $E \geq 0$, we follow only the $+$ case. Choosing $R(t_0, z) = R$ as the spatial coordinate in this space, we get:

$$ds_3^2 = S^2 \left[\frac{dR^2}{C^2 R^2} + R^2(dp^2 + dq^2) \right]. \quad (5.2)$$

Note that only one hypersurface can have the property (5.1) (since E is independent of t while R depends on t). Thus, the 3-geometry of a space of constant t can evolve away from or toward (5.2), or through (5.2), but cannot preserve this geometry over a finite time.

The surface of constant q in (5.2) has the metric $ds_2^2 = dR^2/(C^2 R^2) + R^2 dp^2$. To visualize it, we embed it now in a three-dimensional Euclidean space with the metric

$$ds_3^2 = dZ^2 + dR^2 + R^2 dp^2. \quad (5.3)$$

Our ds_2^2 is the metric of the surface $Z = Z_0(R)$, where $Z_{0,R}^2 + 1 = 1/(CR)^2$, thus

$$\begin{aligned} Z_0 &= \pm \int \frac{\sqrt{1 - C^2 R^2}}{CR} dR \\ &= \pm \frac{1}{C} \left[\ln \left(\frac{CR}{1 + \sqrt{1 - C^2 R^2}} \right) + \sqrt{1 - C^2 R^2} \right]. \end{aligned} \quad (5.4)$$

This embedding is possible only in the range $R \leq 1/C$. The $R > 1/C$ part of the surface can be embedded in a flat space of the signature $(- + +)$.⁵ The metric is then

$$ds_3^2 = -dZ_1^2 + dR^2 + R^2 dp^2, \quad (5.5)$$

⁵A similar phenomenon is known from the maximally extended Reissner-Nordström spacetime, when the region inside the interior horizon is depicted, see Ref. [16].

and the embedding equation is

$$\begin{aligned} Z_1 &= \pm \int \frac{\sqrt{C^2 R^2 - 1}}{CR} dR \\ &= \pm \frac{1}{C} \left[\sqrt{C^2 R^2 - 1} - 2 \arctan(CR + \sqrt{C^2 R^2 - 1}) \right. \\ &\quad \left. + \pi/2 \right] \end{aligned} \quad (5.6)$$

(the constant of integration was chosen so that $Z_0(1/C) = Z_1(1/C)$). The functions $Z(R)$ and $Z_1(R)$ are shown in Fig. 2. Note that in both embeddings, (5.3) and (5.5), p appears as the polar angle in the plane (R, p) . If p is to be interpreted as actually being a polar angle, with the period 2π , then all points with the coordinates $(t, z, p + 2\pi n, q)$, where n is any integer, should be identical with the point of coordinates (t, z, p, q) . Since $p \rightarrow p + \text{constant}$ are symmetry transformations of the spacetime (3.1), there is no problem with such an identification. Thus we should imagine the (R, p) surface as being created by rotating the curve from Fig. 2 around the Z axis.

However, the same picture would be obtained for an (R, q) surface in (5.2), given by $p = \text{const}$. We would find that in that surface, q is the angular coordinate of the polar coordinates (R, q) , and points of coordinates $(t, z, p, q + 2\pi m)$ can be identified with the point of coordinates (t, z, p, q) . We are thus led to conclude that (p, q) are both angular coordinates with the period 2π , and that the points of coordinates (p, q) have to be identified with the points of coordinates $(p + 2n\pi, q + 2m\pi)$, where n and m are arbitrary integers. The tentative conclusion is that the (p, q) -surface is a flat torus.

The conclusion is tentative in the sense that, while we identify the set $p = p_0$ with the set $p = p_0 + 2\pi$, we are still free to carry out the symmetry transformations within the set $p = p_0$. Thus, the identification can possibly be done with a twist, that will turn a square into a Möbius strip, or with a two-way twist, that will turn it into a projective plane. We will see later that the latter case is in fact forced upon us in the full nonsymmetric case. We will use the term “toroidal topology” that will be meant to include an ordinary torus, and also the identifications with twists.

The conclusions drawn from an embedding can be misleading. As an example, consider the hyperbolically symmetric counterpart of (3.2), which can be written in the form

$$ds^2 = dt^2 - \frac{R_{,z}^2}{2E(z) - 1} dz^2 - R^2(d\vartheta^2 + \sinh^2 \vartheta d\varphi^2). \quad (5.7)$$

The surface of constant t and constant φ has the metric $ds_2^2 = R_{,z}^2 dz^2 / (2E(z) - 1) - R^2 d\vartheta^2$, and embedding it in a Euclidean space we would conclude that ϑ is the polar coordinate with the period 2π . However, in this case

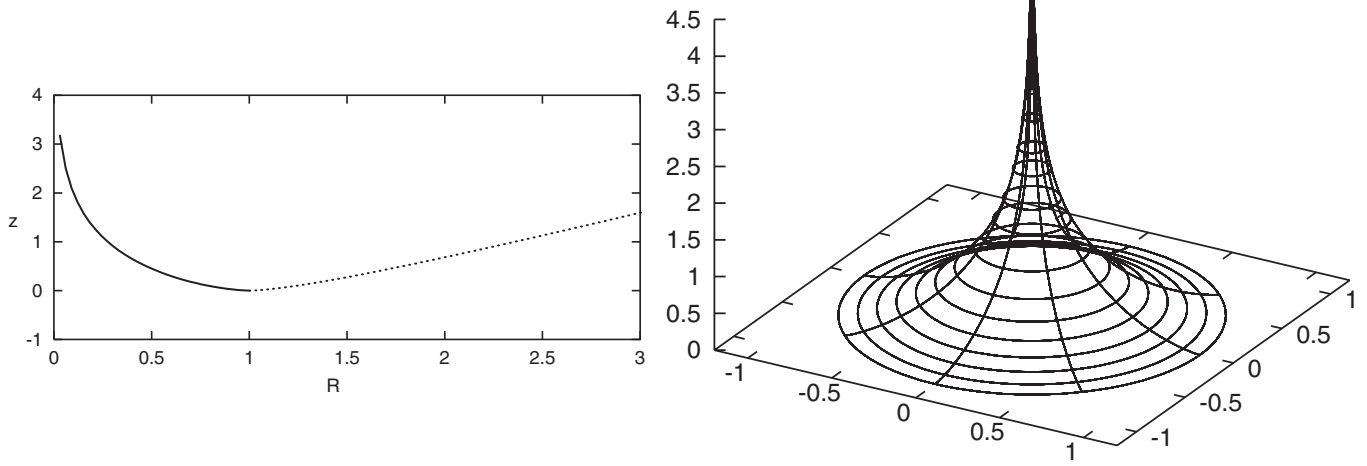


FIG. 2. *Left panel:* The function $Z_0(R)$ (from 0 to 1, solid line) and the function $Z_1(R)$ (from 1 to 3, dotted line), given by Eqs. (5.4) and (5.6), respectively. In the graph we chose the $-$ sign for $Z_0(R)$ and the $+$ sign for $Z_1(R)$. The graph is the cross section of the (R, p) -surface in the spaces (5.3) (left part of the curve) and (5.5) (right part of the curve). *Right panel:* The (R, p) (or (R, q)) surface obtained by rotating the graph from the left panel around the $R = 0$ axis. The lower end of the funnel is where the embedding in the Euclidean space breaks down, i.e., where the solid line meets the dotted line in the left panel. Upwards, the funnel goes infinitely far and becomes infinitely thin.

$\vartheta \rightarrow \vartheta + \text{constant}$ are not symmetry transformations of the spacetime (or of a space of constant t), and so identifications of points with different values of ϑ are not permitted. Thus, the embedding in this case is not a one-to-one mapping. Consequently, the toroidal interpretation of the plane-symmetric case must be treated as one possibility, and not as a definitive conclusion.

Note from (5.2) that the length of any segment of a curve given by $p = \text{const}$ and $q = \text{const}$ that goes into the point $R = 0$ is infinite. The length of such a “radial” line between the values R_1 and R_2 is

$$\ell_{12} = \left| \int_{R_1}^{R_2} \frac{dR}{CR} \right| = \left| \frac{1}{C} \ln\left(\frac{R_2}{R_1}\right) \right|_{R_2 \rightarrow 0} \rightarrow \infty.$$

Thus, Fig. 2 correctly suggests that a surface of constant p or constant q in the space (5.2) has the shape of an infinite funnel, and the point with coordinate $R = 0$ is not accessible (does not in fact belong to this surface). This conclusion is consistent with the observation made in Ref. [18] that in the planar Szekeres metric “there is no real origin, but R, M and E can asymptotically approach zero.”⁶

It can be seen from (3.2) that the (p, q) surface should have a toroidal topology with any form of E . The 3-metric of a $t = t_0$ space is

$$ds_3^2 = \frac{dR_0^2}{2E} + R_0^2(dp^2 + dq^2), \quad (5.8)$$

where $R_0(z) \stackrel{\text{def}}{=} R(t_0, z)$. We can now embed a surface of constant p or a surface of constant q in a three-dimensional flat space by the same method that we used for (5.2), only

⁶In this quote, notation has been adapted to the presently used one.

the equation of embedding will not be explicit, it will be

$$\pm Z_{,R}^2 + 1 = 1/(2E) > 0, \quad (5.9)$$

the upper sign being for embedding in the Euclidean space, the lower sign for the embedding in the pseudo-Euclidean space. In each case the coordinates p and q turn out to be the azimuthal coordinates. As argued in Ref. [18], if a nonsingular origin (where $R = E = 0$) is to exist, then it will be infinitely far from every point of the $t = \text{const}$ space. This implies the infinite funnel geometry of Fig. 2. A sketch of such a torus is shown in Fig. 3.

The toroidal geometry and topology of the (p, q) surfaces neatly explains the pattern of expansion. The circumference of the torus along the p - or q -direction is $2\pi R$ in the coordinates of (5.8). Thus, as R increases with time, the circumference of the torus increases in proportion to R , which causes transversal expansion.

The toroidal topology also solves the problem of where the mass generating the gravitational field resides. As observed in Ref. [18], the regularity conditions at an origin, derived for the Szekeres models in Ref. [15], are independent of ε (it is not guaranteed that they can always be fulfilled; with $\varepsilon < 0$ they cannot). Thus, $E/M^{2/3}$ must tend to a nonzero constant as $z \rightarrow z_c$, where z_c is the origin. Knowing this, let us calculate the amount of rest mass in an arbitrary volume \mathcal{V} , from (3.2) and (3.3). That amount is $\mathcal{M} \stackrel{\text{def}}{=} \int_{\mathcal{V}} \rho \sqrt{g_3} d_3x$, where g_3 is the determinant of the 3-metric of a $t = \text{const}$ subspace of (3.2). Thus

$$\mathcal{M} = \frac{c^2}{4\pi G} \int_{\mathcal{V}} \frac{M_{,z}}{\sqrt{2E}} dpdqdz \equiv \frac{c^2}{4\pi G} \int_{\mathcal{V}} \frac{1}{\sqrt{2E}} dpdq dM. \quad (5.10)$$

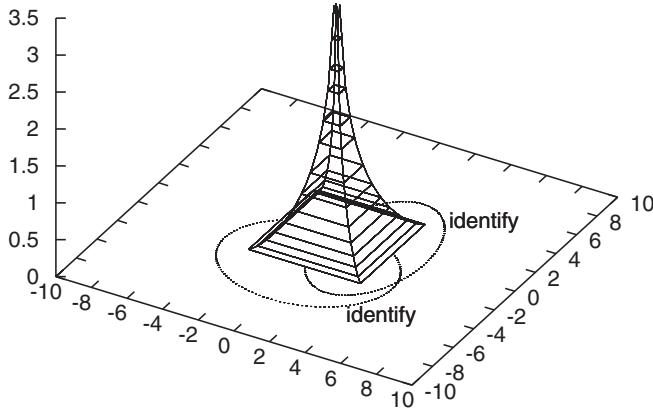


FIG. 3. A sketch of the full 3-space of the plane-symmetric toroidal model (a faithful picture cannot be drawn because the two-dimensional flat torus cannot be embedded in a Euclidean space, and the 3-space of a planar Szekeres model cannot be made flat). Each square section of the funnel represents a two-dimensional flat torus, so its front edge coincides with the back edge, and the left edge coincides with the right edge. Each torus is an orbit of the symmetry group of the model. Each smaller torus is contained within all the larger ones, but the inclusion relation cannot be depicted in the Euclidean space of the picture. Also, in the curved 3-space, the 2-tori surround the asymptotic origin, which is the tip of the funnel, situated infinitely high above the plane drawn in the picture.

With $E \propto M^{2/3}$ in the vicinity of $M = 0$, the integral with respect to M is finite. With a toroidal topology, the ranges of p and q are finite, so the integrals over p and q also give finite values. Thus, the total amount of mass in each space $t = \text{const}$ is finite.

The relation $\mathcal{M}_{,z} = M_{,z}/\sqrt{2E}$ that follows from (5.10) is similar to the relation $\mathcal{M}_{,r} = M_{,r}/\sqrt{1 + 2E}$ that held in the spherically symmetric and quasispherical models. By analogy with that case, we may thus say that in the plane-symmetric spacetime the factor $1/\sqrt{2E}$ measures the relativistic mass defect/excess, i.e., the discrepancy between the active gravitational mass M and the sum of rest masses \mathcal{M} .

VI. NO APPARENT HORIZONS IN THE QUASI-PLANE AND QUASI-HYPERBOLIC MODELS

An apparent horizon is the envelope of the region of trapped surfaces. A (past or future) trapped surface is such a surface on which both the ingoing and outgoing (past- or future-directed, respectively) families of null geodesics converge. A future apparent horizon (AH) always forms in spherically symmetric or quasi-spherical-Szekeres collapse before the Big Crunch singularity is achieved; a past AH always exists after the Big Bang singularity.

It turns out that the AHs do not exist in the quasiplane and quasihyperbolic Szekeres models, and, consequently, neither do they exist in the plane- and hyperbolically

symmetric dust models. Actually, a stronger result holds: these spacetimes remain trapped for all the time. This follows by the method invented by Szekeres [2], which applies here almost unchanged—only the final conclusion is radically different in consequence of the different sign of ε . To avoid getting into complicated details, we begin by using the general form (2.1) of the metric. Suppose a trapped surface exists, and call it Σ .

We assume Σ to be one of the orbits of the quasisymmetry, i.e., to have its equation of the form $\{t = \text{constant}, z = \text{constant}\}$. It will be explained later (see after (6.9)) why it is sufficient to consider such surfaces to prove the conclusion. The traditional definition of a trapped surface requires that it be compact. With the toroidal topology in the planar model, our Σ will be compact indeed. With the infinite topology, and in the quasihyperbolic model, Σ will be infinite. In view of the final result of our consideration, this fact will turn out to be unimportant. We choose these infinite surfaces because of their simple geometry.

Consider any family of null geodesics intersecting Σ orthogonally, and let the tangent vector field of those geodesics be k^μ . Let $(t, z, x, y) = (x^0, x^1, x^2, x^3)$. We have then

$$k_\mu k^\mu = 0, \quad k^\nu k^\mu_{;\nu} = 0 \quad \text{everywhere} \quad (6.1)$$

because k^μ is tangent to null geodesics, and

$$k^2 = k^3 = 0, \quad (k^0)^2 - e^{2\alpha}(k^1)^2 = 0 \quad \text{on } \Sigma \quad (6.2)$$

because k^μ is assumed orthogonal to Σ , so at the points of Σ it must be spanned on the vector fields normal to Σ , which are $(1, 0, 0, 0)$ and $(0, e^{-\alpha}, 0, 0)$. The affine parameter along each null geodesic may be chosen so that

$$k^0 = e^\alpha, \quad k^1 = e = \pm 1 \quad \text{on } \Sigma, \quad (6.3)$$

where we will call the geodesics with $e = -1$ “ingoing,” and those with $e = +1$ “outgoing.”⁷ A surface Σ is trapped when the expansion $k^\mu_{;\mu}$ calculated on Σ is negative for both families. We have on Σ , using (6.2):

$$k^\mu_{;\mu} = k^0_{,t} + k^1_{,z} + e^\alpha(\alpha_{,t} + 2\beta_{,t}) + e(\alpha_{,z} + 2\beta_{,z}). \quad (6.4)$$

In order to simplify this, we now differentiate the first of (6.1) by t , and write out the second of (6.1) for $\mu = 1$, in both cases taking the result on Σ , i.e., making use of the simplifications given in (6.2):

$$\begin{aligned} k^0_{,t} - ee^\alpha k^1_{,t} - e^\alpha \alpha_{,t} &= 0, \\ e^\alpha k^1_{,t} + e(k^1_{,z} + 2e^\alpha \alpha_{,t}) + \alpha_{,z} &= 0. \end{aligned} \quad (6.5)$$

Eliminating $k^1_{,t}$ from (6.5), and using the result to substitute for $k^0_{,t} + k^1_{,z}$ in (6.4) we get

⁷When the surface of constant t and z is infinite, it cannot be closed, therefore the labelling “ingoing” and “outgoing” is only conventional.

$$k^\mu{}_{;\mu} = 2(e^\alpha \beta_{,t} + e \beta_{,z}). \quad (6.6)$$

Using now the expressions for e^α and e^β in the notation of (2.7), i.e.,

$$e^\alpha = \frac{R_{,z} - R \mathcal{E}_{,z} / \mathcal{E}}{\sqrt{\varepsilon + 2E(z)}}, \quad e^\beta = \frac{R}{\mathcal{E}} \quad (6.7)$$

we get in (6.6)

$$k^\mu{}_{;\mu} = 2 \left(\frac{R_{,z}}{R} - \frac{\mathcal{E}_{,z}}{\mathcal{E}} \right) \left(\frac{R_{,t}}{\sqrt{\varepsilon + 2E}} + e \right). \quad (6.8)$$

The first factor changes sign only at shell crossings (see Ref. [15]), so we take it to be positive. Consider collapse, $R_{,t} < 0$. For the ingoing family, $e = -1$, we have $k^\mu{}_{;\mu} < 0$, without further conditions. For the outgoing family, $e = +1$, $k^\mu{}_{;\mu}$ will be negative when $R_{,t} / \sqrt{\varepsilon + 2E} < -1$, which, with negative $R_{,t}$, means that $R_{,t}^2 > \varepsilon + 2E$. Using (2.8) with $\Lambda = 0$ for $R_{,t}^2$, we then obtain

$$2M/R > \varepsilon. \quad (6.9)$$

With $\varepsilon = 0$ and $\varepsilon = -1$, this is always fulfilled,⁸ with the only exception of the ‘‘asymptotic origin’’ in the planar model, where $M/R = \varepsilon = 0$.

A surface given by $\{t = \text{constant}, z = \text{constant}\}$ passes through every point of the spacetime. Since each such surface has now been shown to be trapped at all of its points, this means that all points of the whole spacetime are trapped.

Thus, the quasihyperbolic and quasiplane model, along with their hyperbolically and plane-symmetric limits, are globally future-trapped (when collapsing), and no apparent horizon exists for them. It follows now easily that the corresponding expanding models are globally past-trapped.

This is consistent with the fact that the corresponding vacuum solutions have no event horizons (see Eq. (6.22) in Ref. [18]) and are globally nonstatic.

With no apparent horizons, no black holes may form (more precisely, the whole Universe is one black hole). This excludes the quasiplane and quasihyperbolic models from an important area of application of the quasispherical and spherically symmetric dust models.

VII. THE TOROIDAL PLANE-SYMMETRIC MODEL IN THE SZEKERES COORDINATES

The metric of a torus given by constant t and z in (3.2) is $ds_2^2 = R^2(dp^2 + dq^2)$, where $2\pi R$ is the circumference of the torus in the p -direction and in the q -direction. In the following, we will consider the torus with $R = 1$, and we will call it the ‘‘elementary square’’ or ‘‘elementary torus.’’

⁸Note that M must be positive, or else (3.6) would imply that collapse is retarded and expansion accelerated. This would be gravitational repulsion.

It will be more convenient to assume that, in the coordinates of (3.2), the elementary torus is the square $\{p, q\} \in [-\pi, \pi] \times [-\pi, \pi]$, shown in Fig. 4, rather than $\{p, q\} \in [0, 2\pi] \times [0, 2\pi]$. The \mathbb{R}^2 -space of the (p, q) coordinates can be imagined as filled with infinitely many copies of this square.

As observed in Ref. [18], the function S in the quasiplane model can be absorbed into the other functions by the redefinition

$$(R, E, M) = (\tilde{R}/S, \tilde{E}/S^2, \tilde{M}/S^3), \quad (7.1)$$

so we can assume $S = 1$ with no loss of generality. We do this in the following.

The coordinates of points to be identified are related in a more complicated way in the Szekeres coordinates of (2.1), in which the plane-symmetric model is given by

$$ds^2 = dt^2 - \frac{R_{,z}^2}{2E(z)} dz^2 - R^2 \frac{4(dx^2 + dy^2)}{[(x - P)^2 + (y - Q)^2]^2}, \quad (7.2)$$

with P and Q being arbitrary constants. A line $q = q_0$ corresponds, in the (x, y) -coordinates, to

$$(x - P)^2 + (y - Q - 1/q_0)^2 = 1/q_0^2, \quad (7.3)$$

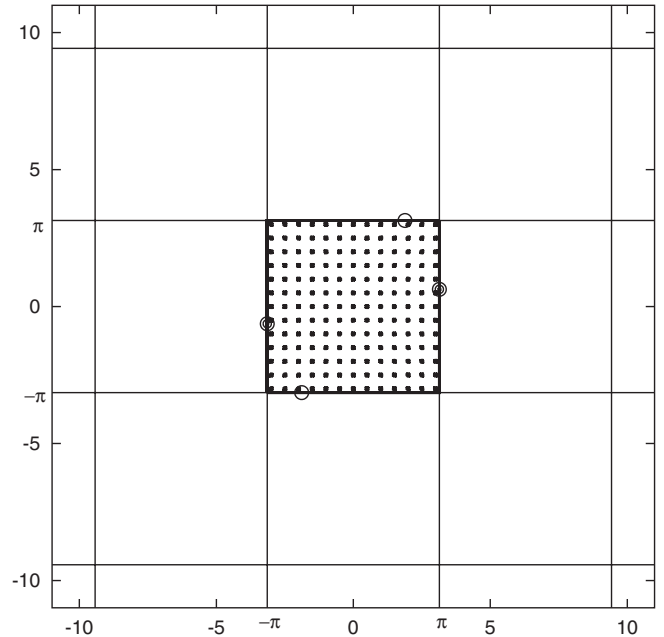


FIG. 4. A map of the elementary torus in the (p, q) coordinates (the central dotted square). Its left edge coincides in space with the right edge, the lower edge coincides with the upper edge. Sometimes it is convenient to consider the torus as a subset of the \mathbb{R}^2 plane, in which case the plane should be imagined as covered with an infinite number of copies of the elementary square. It will later turn out that the identifications have to be done with a twist, i.e., with reflections in $p = 0$ and $q = 0$. The open circles and the full circles show the pairs of points to be identified.

which is in general a circle of radius $1/q_0$ and the center at $(x, y) = (P, Q + 1/q_0)$. In the special case $q_0 = 0$ the image becomes the straight line $y = Q$. Consequently, the lines $q = \pm\pi$ in the (p, q) -coordinates go over into the circles

$$(x - P)^2 + (y - Q \mp 1/\pi)^2 = 1/\pi^2, \quad (7.4)$$

while the lines $p = \pm\pi$ go over into the circles

$$(x - P \mp 1/\pi)^2 + (y - Q)^2 = 1/\pi^2. \quad (7.5)$$

The image of the central point $(p, q) = (0, 0)$ is the infinity of the (x, y) -plane. Conversely, the point $(x, y) = (P, Q)$ is the image of the infinity of the (p, q) -coordinates.

Moreover, from (7.3) follows that the image of the area $\{q^2 < q_0^2\}$ (an infinite strip of the (p, q) plane contained between $q = -q_0$ and $q = q_0 > 0$) is the area *outside* the circles $(x - P)^2 + (y - Q \mp 1/q_0)^2 = 1/q_0^2$. Similarly, the image of the area $\{p^2 < p_0^2\}$ is the area outside the circles $(x - P \mp 1/p_0)^2 + (y - Q)^2 = 1/p_0^2$. Consequently, the image of the elementary torus in the (x, y) coordinates will be the infinite subset of the \mathbb{R}^2 plane lying outside all four circles, see Fig. 5.

This is the explanation to Fig. 5. The image of the line $p = 0$ in the (x, y) -coordinates is the vertical line $x = P$ in the figure, with $(x, y) = (P, Q)$ being the image of infinity of the (p, q) coordinates. Similarly, the image of $q = 0$ is the horizontal line $y = Q$. In the (p, q) -coordinates of (3.2), the torus is the area encircled by the straight lines

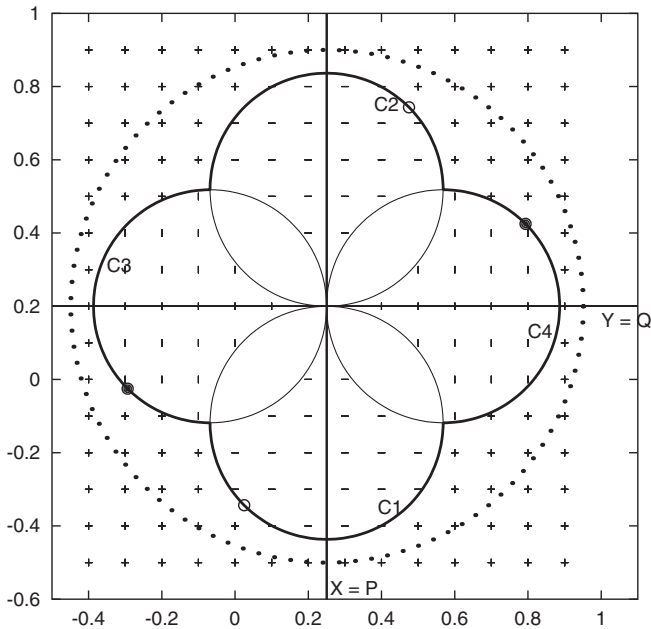


FIG. 5. The image of the torus $\{p, q\} \in [-\pi, \pi] \times [-\pi, \pi]$ in the Szekeres coordinates. The values of P and Q were chosen arbitrarily, but other elements of the figure are drawn to scale. The small empty circles and the small solid circles mark pairs of points to be identified in the general, nonsymmetric case (see Sec. VIII). More explanation in the text.

$p = -\pi, p = \pi, q = -\pi,$ and $q = \pi$. The image of the line $q = -\pi$ is the circle C1, of radius $1/\pi$. The image of the line $q = \pi$ is the circle C2, of the same radius. The image of the torus must be contained outside these two circles—in the area covered with vertical strokes. Then, the image of $p = -\pi$ is the circle C3, and the image of $p = \pi$ is the circle C4, both of the same radius $1/\pi$. Consequently, the image of the torus must be contained outside these circles—in the area covered by horizontal strokes. Thus, the image of the whole torus is the common subset of these two areas—the area in the figure outside the thick line and covered with crosses. The area inside the thick closed curve contains an infinite number of images of copies of the elementary torus.

Each circle in the figure with the center at $(x, y) = (P, Q)$ and with radius $a > 2/\pi$ is an image of a circle of radius $2/a < \pi$ centered at $(p, q) = (0, 0)$ that lies all within a single copy of the elementary torus (one such circle is shown in the figure with a dotted line). In particular, this applies to a circle of radius 1.

VIII. A NONSYMMETRIC TOROIDAL PLANAR MODEL

We will verify now whether the toroidal interpretation can be applied to the full nonsymmetric planar Szekeres model.

In the plane-symmetric subcase, with the metric (3.2), it would in fact be possible to identify points for which $p = p_0$ with points for which $p = -p_0$ at any fixed p_0 because the metric is invariant under the reflections $p \rightarrow -p$; and the same is true for q . The discrete symmetries in the general planar case, (2.7) with $\varepsilon = 0$, are more restricted and they will point out more precisely which points can be identified with which.

In the general planar Szekeres model, as seen from (7.5), identifying the line $p = \pi$ with the line $p = -\pi$ means identifying the circle C_4 in Fig. 5, on which

$$(x - P)^2 + (y - Q)^2 = \frac{2}{\pi}(x - P), \quad (8.1)$$

with the circle C_3 , on which

$$(x - P)^2 + (y - Q)^2 = -\frac{2}{\pi}(x - P). \quad (8.2)$$

To see what effect this has on the metric, we write out Eq. (2.7) with $\varepsilon = 0$ and $S = 1$:

$$ds^2 = dt^2 - \frac{1}{2E(z)} \left[R_{,z} + 2R \frac{(x - P)P_{,z} + (y - Q)Q_{,z}}{(x - P)^2 + (y - Q)^2} \right]^2 dz^2 - \frac{R^2(dx^2 + dy^2)}{[(x - P)^2 + (y - Q)^2]^2}. \quad (8.3)$$

Identifying (8.1) with (8.2) is clearly possible within the (x, y) -surface, where this is an isometry. To make it an isometry also in the spacetime, we have to do this identi-

fication simultaneously with the substitutions

$$x - P \rightarrow P - x, \quad y - Q \rightarrow Q - y. \quad (8.4)$$

The first is a reflection in the $x = P$ axis in Fig. 5, which clearly must accompany the identification of (8.1) with (8.2). The second of (8.4) is a reflection in the $y = Q$ axis, which is required anyway if we identify the circles C_1 and C_2 .

The conclusion is thus: the toroidal interpretation is admissible also in the general case, provided the identifications within each (x, y) surface occur in the way marked by the large points in Figs. 4 and 5, i.e., with reflections in the $x = P$ and $y = Q$ axes. A square with sides identified in this way is in fact a projective plane (for an elementary introduction to projective geometry see Ref. [25]). Such a topology of the (x, y) surfaces solves some problems, but at the same time creates some others. We will come back to this later.

IX. SHELL CROSSINGS IN THE TOROIDAL MODEL

We showed in Ref. [18] that the planar Szekeres metric, if its (x, y) surfaces are interpreted as infinite planes, always develops shell crossings unless $\mathcal{E}_{,z}/\mathcal{E}$ is constant, in which case the spacetime becomes plane symmetric. The toroidal topology can solve this problem.

Let us recall, after Ref. [18], that the surfaces of constant t and constant z in the planar model, (2.7) with $\varepsilon = 0$, can be parametrized by the variables (θ, ϕ) related to (x, y) by

$$x - P = \frac{2}{\theta} \cos\phi, \quad y - Q = \frac{2}{\theta} \sin\phi \quad (9.1)$$

(we assumed that S was absorbed into the other functions as in (7.1)). The value of $1/\theta$ is the distance from the point $(x, y) = (P, Q)$ in Fig. 5. A shell crossing occurs at such (x, y, z) where $R_{,z} - R\mathcal{E}_{,z}/\mathcal{E} = 0$ in (2.7), while $E \neq 0$. In the (θ, ϕ) variables we have

$$\frac{\mathcal{E}_{,z}}{\mathcal{E}} = -\theta(P_{,z} \cos\phi + Q_{,z} \sin\phi), \quad (9.2)$$

and so the locus of a shell crossing is at

$$\theta = -\frac{R_{,z}}{R(P_{,z} \cos\phi + Q_{,z} \sin\phi)}. \quad (9.3)$$

As long as $(P_{,z} \cos\phi + Q_{,z} \sin\phi)$ is not identically zero, i.e., as long as $P_{,z}$ and $Q_{,z}$ do not vanish simultaneously, this equation has a solution for θ .⁹ However, if the θ implied by (9.3) is large enough, then the corresponding $1/\theta$ will be small, and the point with such a value of θ will lie inside the closed curve in Fig. 5, i.e., outside the elementary torus. For this to be the case, a sufficient

⁹By definition, $\theta \geq 0$. But the correct sign of θ can always be achieved by changing $\phi \rightarrow \phi + \pi$.

condition is that the $1/\theta$ of (9.3) is smaller than $\sqrt{2}/\pi$ —the smallest distance of the contour in Fig. 5 from the origin at $(x, y) = (P, Q)$:

$$\left| \frac{R(P_{,z} \cos\phi + Q_{,z} \sin\phi)}{R_{,z}} \right| < \frac{\sqrt{2}}{\pi}. \quad (9.4)$$

Since $|P_{,z} \cos\phi + Q_{,z} \sin\phi| \leq \sqrt{P_{,z}^2 + Q_{,z}^2}$, a sufficient condition for (9.4) to hold at all values of ϕ is

$$P_{,z}^2 + Q_{,z}^2 \leq \frac{2R_{,z}^2}{\pi^2 R^2} \quad (9.5)$$

at all t and all z . Now we have to verify whether this condition can hold during all of the evolution with $P_{,z}^2 + Q_{,z}^2 > 0$. We know (see, for example, Refs. [15,26] or [16]) that $R_{,z} \neq 0$ can be achieved at all times and all z —this is identical to one of the conditions that guarantee no shell crossings in the Lemaître-Tolman model. However, we have to verify that (9.5) can still be fulfilled with $P_{,z}^2 + Q_{,z}^2 > 0$ when $R \rightarrow \infty$.

For this purpose, we follow Hellaby and Lake [26] and use the formulae of Ref. [16], Sec. 18.10. We do not have to consider the case when $E = 0$ in (2.8) because, as stated earlier, in the planar model E can vanish at isolated values of z , but not on open intervals (in fact, $E \rightarrow 0$ only at the asymptotic origin). Thus, the behavior of the model at $E \rightarrow 0$ can be calculated from the formulae given below in the appropriate limit. With $E > 0$ and $\Lambda = 0$, the solution of (2.8) can be represented in a parametric form as

$$\begin{aligned} R(t, z) &= \frac{M}{2E} (\cosh\eta - 1), \\ \sinh\eta - \eta &= \frac{(2E)^{3/2}}{M} [t - t_B(z)], \end{aligned} \quad (9.6)$$

where $t_B(z)$ is the (arbitrary) bang time function. From here we then find

$$\begin{aligned} \frac{R_{,z}}{R} &= \frac{M_{,z}}{M} (1 - \Phi_3) + \frac{E_{,z}}{E} \left(\frac{3}{2} \Phi_3 - 1 \right) \\ &\quad - \frac{(2E)^{3/2}}{M} t_{B,z} \Phi_4(\eta) \end{aligned} \quad (9.7)$$

where

$$\Phi_3(\eta) \stackrel{\text{def}}{=} \frac{\sinh\eta(\sinh\eta - \eta)}{(\cosh\eta - 1)^2}, \quad \Phi_4(\eta) \stackrel{\text{def}}{=} \frac{\sinh\eta}{(\cosh\eta - 1)^2}. \quad (9.8)$$

The functions $\Phi_3(\eta)$ and $\Phi_4(\eta)$ have the following properties:

$$\begin{aligned}
\lim_{\eta \rightarrow 0} \Phi_3(\eta) &= 2/3, & \lim_{\eta \rightarrow \infty} \Phi_3(\eta) &= 1, \\
\lim_{\eta \rightarrow 0} \Phi_4(\eta) &= \infty, & \lim_{\eta \rightarrow \infty} \Phi_4(\eta) &= 0, \\
\frac{d\Phi_3}{d\eta} &> 0 \quad \text{for } \eta > 0, & \frac{d\Phi_4}{d\eta} &< 0 \quad \text{for } \eta > 0.
\end{aligned} \tag{9.9}$$

Then, as shown for the Lemaître-Tolman model, $R_{,z} > 0$ for all (t, z) if and only if

$$M_{,z} > 0, \quad E_{,z} > 0, \quad t_{B,z} < 0. \tag{9.10}$$

These can be fulfilled by the appropriate choice of the arbitrary functions. Consequently, the properties of the functions $\Phi_3(\eta)$ and $\Phi_4(\eta)$ guarantee that $R_{,z}/R > 0$ for all t and z , and so, with sufficiently small nonzero $|P_{,z}|$ and $|Q_{,z}|$, Eq. (9.5) can always be fulfilled, i.e., shell crossings can be avoided throughout the evolution.

This conclusion remains true when we require that $1/\theta < \ell\sqrt{2}/\pi$ in (9.3) and (9.4), where ℓ is a natural number, i.e., that the region free of shell crossings is a multiple of the elementary square. This observation will prove useful below.

X. FORMATION OF STRUCTURES IN THE PLANAR MODEL

As shown for the Lemaître-Tolman models (see Refs. [27] and [16], Sec. 18.19), in the ever-expanding case $E > 0$ an increasing density perturbation, $\rho_{,z}/\rho$, freezes asymptotically into the background—i.e., it tends to a finite value determined uniquely by the initial conditions. Consequently, it is impossible in these models to describe the formation of condensations that collapse to a very high density, such as a galaxy with a central black hole. Since the evolution of the quasilane and quasihyperbolic models is described by the same equations, they will suffer from the same problem. (And we have already found in Sec. VI that these models cannot describe black holes.)

Thus, these models can be used for considering the formation of moderate-amplitude condensations and voids.

XI. INTERPRETATION OF THE MASS FUNCTION $M(r)$ IN THE TOROIDAL QUASIPANE MODEL

Let us recall some basic facts about the mass function $M(r)$ in the quasispherical model.

The amount of rest mass within a sphere of coordinate radius z at coordinate time t , when $z = z_c$ is the center of the sphere, equals $\mathcal{M} = \int_{\mathcal{V}} \rho \sqrt{|g_3|} d_3x$, where \mathcal{V} is the volume of the sphere, ρ is the mass-density given by (2.11), and g_3 is the determinant of the 3-metric $t = \text{const}$ in (2.7). Substituting for ρ and for g_3 we get

$$\begin{aligned}
\mathcal{M} &= \frac{1}{4\pi} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{z_c}^z du \left[\frac{M_{,u}(u)}{\sqrt{1+2E\mathcal{E}^2}} \right. \\
&\quad \left. - \frac{3M\mathcal{E}_{,u}}{\sqrt{1+2E\mathcal{E}^3}} \right], \tag{11.1}
\end{aligned}$$

where u is the running value of z under the integral. Note that \mathcal{E} is the only quantity that depends on x and y ; it is an explicitly given function, and so the integration over x and y can be carried out. As is easy to verify, we have¹⁰

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{\mathcal{E}^2} = 4\pi, \quad \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dq \frac{\mathcal{E}_{,z}}{\mathcal{E}^3} = 0. \tag{11.2}$$

(The first of these just confirms that this is the surface area of a unit sphere.) Using this in (11.1) we get

$$\mathcal{M} = \int_{z_c}^z \frac{M_{,u}}{\sqrt{1+2E}}(u) du, \tag{11.3}$$

which is the same relation as in the L-T model, and shows that $1/\sqrt{1+2E}$ is the relativistic energy defect/excess function (when $2E > 0$ and $2E < 0$ respectively).

If we interpret the quasilane model as having the toroidal topology, then the calculation given above can be done in a very similar way. The integrals over x and y in the analogue of (11.1) should now extend over the area Σ outside the cloverlike contour in Fig. 5. Just as in the quasispherical case, since only \mathcal{E} depends on x and y , we can integrate over x and y first. For this purpose, it will be convenient to transform (x, y) by (3.1), since this will greatly simplify the definition of the area of integration: in the (p, q) variables, this is just the square $[-\pi, \pi] \times [-\pi, \pi]$. We have

$$d_2xy = -\frac{4}{(p^2 + q^2)^2} d_2pq, \tag{11.4}$$

$$\frac{1}{\mathcal{E}^2} = \frac{1}{4}(p^2 + q^2)^2. \tag{11.5}$$

The volume-element in calculating the mass is non-oriented, so in transforming the integral we take the absolute value of the surface-element. Thus:

$$\int_{\Sigma} d_2xy \frac{1}{\mathcal{E}^2} = \int_{-\pi}^{\pi} dp \int_{-\pi}^{\pi} dq = 4\pi^2, \tag{11.6}$$

$$\int_{\Sigma} d_2xy \frac{\mathcal{E}_{,z}}{\mathcal{E}^3} = \int_{-\pi}^{\pi} dp \int_{-\pi}^{\pi} dq (pP_{,z} + qQ_{,z}) = 0. \tag{11.7}$$

However, here we encounter a problem with the geometrical interpretation. A projective plane P_2 is one-sided, and so nonorientable [25]. Thus, it is impossible to divide a 3-space in which P_2 is immersed into the interior of P_2 and

¹⁰This fact was first proven by Szekeres [2].

its exterior. We will clearly need such a distinction if we want to speak about a mass contained inside P_2 . (The surface element (11.4) gives no warning about the change of orientation after going all around the plane, but this must be an artifact of the coordinate map.) This problem may be solved, for example, as follows. In Fig. 4, we identify the line $p = \pi$ with the line $p = -\pi$ not in the same elementary square \square_1 , but in an identical duplicate copy \square_2 of the elementary square. After proceeding from $p = -\pi$ to $p = \pi$ in \square_2 , we identify the set $p = \pi$ of \square_2 with $p = -\pi$ of \square_1 . Then we repeat the trick in the q -direction: we identify the line $q = \pi$ in \square_1 with $q = -\pi$ in \square_3 , still another duplicate copy of \square_1 . In \square_3 , we identify the set $q = \pi$ with $q = -\pi$ of \square_1 , and the set $p = \pi$ with $p = -\pi$ in \square_4 , a fourth copy of \square_1 . The setup is illustrated in Fig. 6. In this way, while going across the elementary square, we can arrive back to the starting point after crossing at least twice its length. The surface obtained in this way is two-sided and orientable. Its total surface area is then $16\pi^2$ instead of the $4\pi^2$ calculated in (11.6), but still finite, and the integration in (11.6) effectively covers the region Σ four times, so we denote the area of integration by 4Σ .

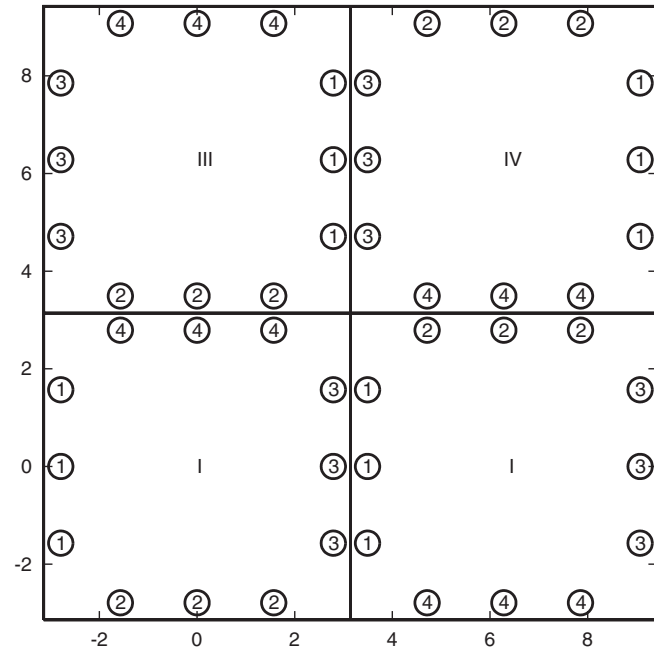


FIG. 6. Four identical copies of a projective plane patched together to form a two-sided surface (the 4Σ of Eq. (11.8)). Numbers in circles identify the edges of the elementary square; roman numbers label the copies of the elementary square. Copy II is attached to copy I so that it mimics the twist in the projective plane. Similarly, copy III is attached to copy I in the same way. Copy IV is attached to copies II and III to close the picture. The right edge of the large square is then identified with the left edge without a twist; its upper edge is identified with the lower edge also without a twist. The surface thus obtained is two-sided, so orientable, and it makes sense to speak about the volume inside it.

Consequently, the analogue of (11.1) for $\varepsilon = 0$ is

$$\begin{aligned} \mathcal{M} &= \frac{1}{4\pi} \int_{z_c}^z du \int_{4\Sigma} d_2xy \left[\frac{M_{,u}(u)}{\sqrt{2E}\mathcal{E}^2} - \frac{3M\mathcal{E}_{,u}}{\sqrt{2E}\mathcal{E}^3} \right] \\ &= 16\pi \int_{z_c}^z du \frac{M_{,u}(u)}{\sqrt{2E}}. \end{aligned} \quad (11.8)$$

As already stated (see the comment around Eq. (5.10)), if the regularity conditions are fulfilled, the integral above is finite, and $\mathcal{M}(z_c) = 0$. Consequently, we can interpret $M(z)$ as the active gravitational mass within a volume extending from the origin at $z = z_c$ to the running value of z , and $1/\sqrt{2E}$ is the value of the mass defect/excess factor.

With the toroidal interpretation, the proof that $M(z)$ is a measure of the active gravitational mass was rather simple. The same may be shown also when the quasiplane model is interpreted as infinite in extent, but in a more complicated way. We do show it in the next section—however, this is only for mathematical completeness. As demonstrated in Ref. [18], with the infinite spaces the quasiplane Szekeres solutions have irremovable shell crossings, and so are in fact not acceptable as cosmological models.

XII. INTERPRETATION OF $M(z)$ FOR THE QUASIPANE MODEL WITH INFINITE SPACES

The consideration below was devised for the model with infinite spaces of constant t , but in fact it applies just as well with the toroidal spaces.

Recall that with $\varepsilon = 0$ we are free to rename the functions R , E , and M as in (7.1), and the result will be the same as if $S \equiv 1$. Thus, we assume $S \equiv 1$ throughout this section.

For the beginning let us consider the plane-symmetric subcase of the $\varepsilon = 0$ model, which has $P_{,z} = Q_{,z} = 0$. With $S \equiv 1$, we can write an analogue of the integrals in (11.2). Let us choose a circle of radius 1 centered at $(x, y) = (P, Q)$ (both P and Q being now constant) in every surface of constant t and z . Let d_2xy be the surface element in the (x, y) plane, and let U be the outside of the unit circle. This region has a finite surface area. (From the considerations of Sec. VII we know that if we interpret the (x, y) surfaces as tori then the region U will all be contained within the elementary torus.) Then, introducing (u, φ) by

$$x = P + u \cos\varphi, \quad y = Q + u \sin\varphi, \quad (12.1)$$

we get

$$\int_U d_2xy \frac{1}{\mathcal{E}^2} = \int_0^{2\pi} d\varphi \int_1^\infty \frac{4}{u^3} du = 4\pi, \quad (12.2)$$

in every ($t = \text{const}, z = \text{const}$) surface. Now let \mathcal{V} be a three-dimensional set in a $t = \text{const}$ space, extending from $z = z_c$ to a running value of z , whose every section of

constant z is U —the outside of the unit circle $(x - P)^2 + (y - Q)^2 = 1$. Calculating, as in (11.1), $\mathcal{M} = \int_{\mathcal{V}} \rho \sqrt{|g_3|} d_3x$ with $\varepsilon = 0$, $S = 1$, and P, Q constant we get $\mathcal{E}_{,z} = 0$ and

$$\mathcal{M} = \frac{1}{4\pi} \int_U d_2xy \int_{z_c}^z du \frac{M_{,u}}{\sqrt{2E}\mathcal{E}^2} = \int_{z_c}^z \frac{M_{,u}}{\sqrt{2E}}(x) du. \quad (12.3)$$

Thus, in this case, M behaves as the active gravitational mass contained outside a tube of coordinate height $(z - z_c)$ which has radius equal to 1 at every z value, while $1/\sqrt{2E}$ plays the role of the mass defect/excess factor. We recall that the (x, y) coordinates of (2.7), in the plane-symmetric case $\varepsilon = 0 = \mathcal{E}_{,z}$, $S = 1$ are related by the inversion (3.1) to the Cartesian coordinates (p, q) in a plane, so the outside of the tube in the (x, y) coordinates is in reality the inside of the same tube in the Cartesian coordinates. Thus, physically, M is the active mass within a tube \mathcal{V}_2 .

We calculated the integrals in (12.2) and (12.3) around the central point $(x, y) = (P, Q)$. However, with plane symmetry, the origin of the Cartesian coordinates can be transferred to any other point by a symmetry transformation.¹¹

Let us consider the transformation (3.1), after which the metric becomes (3.2), which is formally (2.7) with $\mathcal{E} = 1$ and (x, y) renamed to (p, q) . In this form, the transformation

$$p = p' + A_p, \quad q = q' + A_q \quad (12.4)$$

(with A_p and A_q being arbitrary constants) is an isometry of (3.2). Thus, the transformation (12.4) does not change either the metric (3.2) or the value of the integral (12.2), which, in the variables (p, q) , becomes simply $4 \int_{S_1} dpdq = 4\pi$, independently of where the center of the circle S_1 is located.

Now let us consider the planar metric $\varepsilon = 0$ that is not plane symmetric, i.e., with $P_{,z}$ and $Q_{,z}$ not vanishing simultaneously. Let $U(z)$ be the outside of a unit circle in an $z = \text{const}$ surface, with the center at $(x, y) = (P(z), Q(z))$. Within each single such surface, applying the transformation of variables (12.1), we get

$$\begin{aligned} \int_U d_2xy \frac{1}{\mathcal{E}^2} &= \int_0^{2\pi} d\varphi \int_1^\infty \frac{4}{u^3} du = 4\pi, \\ \int_U d_2xy \frac{\mathcal{E}_{,z}}{\mathcal{E}^3} &= \int_0^{2\pi} d\varphi \int_1^\infty \frac{-4u \cos\varphi P_{,z} - 4u \sin\varphi Q_{,z}}{u^5} du \\ &= 0. \end{aligned} \quad (12.5)$$

¹¹The circle of unit radius in the Cartesian coordinates, when moved to another point of the (p, q) plane, will not have a unit coordinate radius in the (x, y) coordinates, and the image of the center of the circle will not be the center of the image circle. However, the surface area of the circle and the invariant distances between points are not changed.

These integrals do not depend on $P_{,z}$ or $Q_{,z}$, but the centers of the circles no longer have the same (x, y) coordinates at each z . Thus, to use (12.5) in an analogue of (12.3), we have to take a volume \mathcal{V} which is a wiggly tube: its every cross section with a constant z surface is a unit circle, but the centers of the circles do not lie on a line orthogonal to the $z = \text{const}$ surfaces. Instead, they lie on the curve in the $t = \text{const}$ space given by the parametric equations $x = P(z)$, $y = Q(z)$. Because of the second of (12.3) and (12.5) still follows for this single tube.

The whole 3-space $t = \text{const}$ is now no longer homogeneous with respect to the group of plane symmetries. However, each single $z = \text{const}$ surface in that space is homogeneous. In particular, the surface containing the base of the tube, $z = z_0$, is homogeneous. Thus, we can apply the inversion (3.1) with $P = P(z_0)$, $Q = Q(z_0)$. The inside and outside of the unit circle in the $z = z_0$ will thereby simply interchange, but the resulting transformations in other $z = \text{const}$ surfaces will be more complicated, and the wiggly tube will deform substantially. Still, in the inverted coordinates we are now free to move the center of the base circle (within the $z = z_0$ surface) to any other point.

We now carry out this plan. Let us denote:

$$\begin{aligned} P(z_0) &\stackrel{\text{def}}{=} P_0, & Q(z_0) &\stackrel{\text{def}}{=} Q_0, \\ V &\stackrel{\text{def}}{=} (P_0 - P)^2 + (Q_0 - Q)^2. \end{aligned} \quad (12.6)$$

To the metric (2.7) with $\varepsilon = 0$ and $S = 1$ we apply the inversion adapted to the surface $\{t = \text{const}, z = z_0\}$:

$$x = P_0 + \frac{P}{p^2 + q^2}, \quad y = Q_0 + \frac{q}{p^2 + q^2}. \quad (12.7)$$

After this, the 2-metric $R^2(dx^2 + dy^2)/\mathcal{E}^2$ becomes

$$\begin{aligned} ds_2^2 &= \frac{1}{\tilde{\mathcal{E}}^2} (dx^2 + dy^2), \\ 2\tilde{\mathcal{E}} &= V(p^2 + q^2) + 2(P_0 - P)p + 2(Q_0 - Q)q + 1. \end{aligned} \quad (12.8)$$

In these coordinates, the surface $\{t = \text{const}, z = z_0\}$ is explicitly homogeneous, so we are now free to shift the origin of coordinates to any other point by

$$p = p' + A_1, \quad q = q' + A_2, \quad (12.9)$$

with A_1 and A_2 being constants. After the shift, the metric is still Szekeres with $\varepsilon = 0$, but with complicated expressions for the new P, Q , and S .

After the two transformations, (12.7) and (12.9), the region U of integration in (12.5) (which was the outside of a tube extending out to infinity) goes over into a finite region—the inside of a certain tube whose edge is the

image of the family of unit circles in (x, y) . In the integrals (12.5), the two transformations are just changes of integration variables, so the values of the integrals do not change, and thus (12.3) still applies. This shows that over each point of the surface of constant t and of $z = z_0$ in the Szekeres $\varepsilon = 0$ metric we can find a region of finite volume (a wiggly tube) such that the function M can be interpreted as the active gravitational mass within that tube.

As an illustration of the considerations about wiggly columns and their transport over the manifold, we now consider a special case of the transformation (12.9) with $A_1 \stackrel{\text{def}}{=} \lambda$ and $A_2 = 0$. But first we give the complete transformation that will take us back to the coordinates of (2.7) in the surface $z = z_0$.

To the variables (p', q') of (12.9) we apply the inversion in a circle of radius 1 centered at $(p', q') = (0, 0)$, and the shift by (P_0, Q_0) to the resulting (x', y') coordinates. Calling the final coordinates (x_2, y_2) , we calculate the effect of (12.9) on the variables (x, y) . The complete transformation from (x, y) to (x_2, y_2) is

$$\begin{aligned} x_2 &= P_0 + \frac{1}{W}(x - P_0 + A_1 \mathcal{U}), \\ y_2 &= Q_0 + \frac{1}{W}(y - Q_0 + A_2 \mathcal{U}), \end{aligned} \tag{12.10}$$

where

$$\begin{aligned} \mathcal{U} &\stackrel{\text{def}}{=} (x - P_0)^2 + (y - Q_0)^2, \\ W &\stackrel{\text{def}}{=} 1 + 2[A_1(x - P_0) + A_2(y - Q_0)] + (A_1^2 + A_2^2)\mathcal{U}. \end{aligned} \tag{12.11}$$

This whole set of transformations does not change the metric (2.7) in the hypersurface $z = z_0$, but after the transformations the unit circle $\mathcal{U} = 1$ goes over into the circle

$$(x_2 - P_0 - A_1/\gamma)^2 + (y_2 - Q_0 - A_2/\gamma)^2 = 1/\gamma^2, \tag{12.12}$$

where $\gamma \stackrel{\text{def}}{=} A_1^2 + A_2^2 - 1$.

Now we specialize this to the 1-parameter subgroup $A_1 = \lambda, A_2 = 0$, i.e., to the shift along the p -direction in (12.9). The transformation (12.10)–(12.11) becomes:

$$\begin{aligned} x_2 &= P_0 + \frac{x - P_0 + \lambda\{(x - P_0)^2 + (y - Q_0)^2\}}{W_0}, \\ y_2 &= Q_0 + \frac{y - Q_0}{W_0}, \end{aligned} \tag{12.13}$$

$$W_0 \stackrel{\text{def}}{=} \lambda^2\{(x - P_0)^2 + (y - Q_0)^2\} + 2\lambda(x - P_0) + 1,$$

and its inverse is obtained by replacing λ with $(-\lambda)$, i.e.,

$$\begin{aligned} x &= P_0 + \frac{x_2 - P_0 - \lambda\{(x_2 - P_0)^2 + (y_2 - Q_0)^2\}}{\tilde{W}_0}, \\ y &= Q_0 + \frac{y_2 - Q_0}{\tilde{W}_0}, \end{aligned} \tag{12.14}$$

$$\tilde{W}_0 \stackrel{\text{def}}{=} \lambda^2\{(x_2 - P_0)^2 + (y_2 - Q_0)^2\} - 2\lambda(x_2 - P_0) + 1.$$

The Jacobians of the two transformations are, respectively,

$$\tilde{J} = \left| \frac{\partial(x_2, y_2)}{\partial(x, y)} \right| = \frac{1}{W_0^2}, \quad J = \left| \frac{\partial(x, y)}{\partial(x_2, y_2)} \right| = \frac{1}{\tilde{W}_0^2}. \tag{12.15}$$

The following identities are useful in calculations:

$$\begin{aligned} (x_2 - P_0)^2 + (y_2 - Q_0)^2 &\equiv \frac{(x - P_0)^2 + (y - Q_0)^2}{W_0}, \\ (x - P_0)^2 + (y - Q_0)^2 &\equiv \frac{(x_2 - P_0)^2 + (y_2 - Q_0)^2}{\tilde{W}_0}. \end{aligned} \tag{12.16}$$

The transformation (12.13) takes the circle $(x - P)^2 + (y - Q)^2 = u^2$ to a shifted circle with a different radius, namely

$$(x_2 - A)^2 + (y_2 - B)^2 = (u/p_0)^2, \tag{12.17}$$

where

$$\begin{aligned} A &\stackrel{\text{def}}{=} \frac{\lambda^2 P_0 [(P_0 - P)^2 + (Q_0 - Q)^2 - u^2] - \lambda [u^2 + P_0^2 - P^2 - (Q_0 - Q)^2] + P}{P_0}, \\ B &\stackrel{\text{def}}{=} \frac{\lambda^2 Q_0 [(P_0 - P)^2 + (Q_0 - Q)^2 - u^2] - 2\lambda Q_0 (P_0 - P) + Q}{P_0}, \\ p_0 &\stackrel{\text{def}}{=} \lambda^2 [(P_0 - P)^2 + (Q_0 - Q)^2 - u^2] - 2\lambda (P_0 - P) + 1 \end{aligned} \tag{12.18}$$

(in fact, these will be applied with $u = 1$).

Now using (12.14)–(12.16) we find

$$(x - P)^2 + (y - Q)^2 = \frac{S_0}{\tilde{W}_0} [(x_2 + \alpha)^2 + (y_2 + \beta)^2], \tag{12.19}$$

where

$$\begin{aligned} S_0 &\stackrel{\text{def}}{=} 1 - 2\lambda(P_0 - P) + \lambda^2 [(P_0 - P)^2 + (Q_0 - Q)^2], \\ \alpha &\stackrel{\text{def}}{=} -P_0 + \frac{P_0 - P - \lambda [(P_0 - P)^2 + (Q_0 - Q)^2]}{S_0}, \\ \beta &\stackrel{\text{def}}{=} -Q_0 + \frac{Q_0 - Q}{S_0}. \end{aligned} \tag{12.20}$$

Note that S_0 , α , and β are functions only of z ; they do not depend on x_2 and y_2 . We can see that the \tilde{W}_0^2 that will appear in the transformed integral in (12.5), $\int_U d_2xy/\mathcal{E}^2$, will be canceled by the \tilde{W}_0^2 from the Jacobian of the transformation, and what remains will be

$$\int \frac{1}{\mathcal{E}^2} dx dy = \frac{4}{S_0(z)^2} \int \frac{1}{[(x_2 + \alpha)^2 + (y_2 + \beta)^2]^2} dx_2 dy_2, \quad (12.21)$$

an integral of exactly the form (12.5), except for the additional factor $1/S_0^2$.

The transformation (12.13) affects also the metric. Under (12.13), \mathcal{E} changes as follows:

$$\begin{aligned} \mathcal{E} &= (x - P)^2 + (y - Q)^2 \\ &= \frac{S_0}{\tilde{W}_0} [(x_2 + \alpha)^2 + (y_2 + \beta)^2]. \end{aligned} \quad (12.22)$$

The expression $dx^2 + dy^2$, after the transformation (12.13) goes over into

$$(dx_2^2 + dy_2^2)/\tilde{W}_0^2, \quad (12.23)$$

and so the two equations above imply that after the transformation

$$\frac{dx^2 + dy^2}{\mathcal{E}^2} = \frac{dx_2^2 + dy_2^2}{S_0^2 [(x_2 + \alpha)^2 + (y_2 + \beta)^2]^2}. \quad (12.24)$$

We are in the same Szekeres model as at the beginning, but at a different location. The alien form of the Szekeres metric in these coordinates results from the fact that the transformation (12.13) is not a symmetry, apart from the single surface $z = z_0$. Since the quantities in this equation (α , β , and S_0) depend on the continuous parameter λ , the equation above describes the result of a shift to any location (with $\lambda = 0$ corresponding to identity). Eqs. (12.24) and (12.21) show that the transformation (12.13) is area-preserving.

The second integral in (12.5) does not change its zero value under the transformation (12.13)—since that transformation is simply a change of variables under a definite integral, applied both to the integrand and to the area of integration. Thus, the second integral in (12.5) does not contribute to calculating \mathcal{M} in (12.3).

XIII. SUMMARY

Continuing the research begun in Ref. [18], geometrical properties of the quasiplane Szekeres model were investigated here, along with the corresponding properties of the plane-symmetric model. The following results were achieved:

- (1) The pattern of decelerated expansion in the plane-symmetric model was analyzed and shown to be in complete disagreement with the Newtonian analogues and intuitions (Secs. III and IV). An example

of a Newtonian potential that gives a similar pattern of expansion has coaxial parallel cones as its equipotential surfaces; it has not been investigated whether such a potential can be generated by any realistic matter distribution.

- (2) Embeddings of the constant t and constant z surfaces in the Euclidean space suggest that the flat surfaces contained in the plane-symmetric model can be interpreted as flat tori whose circumferences are proportional to the function $R(t, z)$, and thus vary with time (Sec. V). Such a topology immediately explains the pattern of expansion and implies that the total mass contained within a $z = \text{const}$ surface is finite.
- (3) The quasiplane and quasihyperbolic models are permanently trapped (Sec. VI), so no apparent horizons exist in them. Consequently, these models cannot be used to describe the formation of black holes (the whole Universe is one black hole all the time).
- (4) The toroidal interpretation can be applied to the full quasiplane model (Sec. VIII). However, the flat surfaces defined by the geometry must then be interpreted as projective planes rather than tori.
- (5) The toroidal quasiplane model can be free of shell crossings (Sec. IX), unlike the model with infinite spaces.
- (6) For both interpretations (toroidal and infinite), the quasiplane model cannot describe the formation of structures that collapse to very high densities (Sec. X), since the density perturbations tend to finite values in the asymptotic future.
- (7) To calculate the mass within a volume, the geometrical interpretation of the flat surfaces has to be modified yet again, because the projective plane is one-sided and so has no interior. Thus, instead of identifying the opposite edges of the elementary square, we introduce three extra identical duplicate copies of the elementary square and patch them together into a two-sided surface. With the toroidal interpretation thus modified, the mass function $M(z)$ is proportional to the active gravitational mass contained within a “radial” coordinate z (Sec. XI). With infinite spaces, this function is proportional to the active gravitational mass within a “wiggly tube” of finite radius (Sec. XII).

Whether the toroidal interpretation is a necessity is still unknown. However, this paper demonstrated that with the toroidal topology the quasiplane model becomes in several respects simpler, and, however paradoxical this may sound, more realistic.

The quasiplane model with toroidal spaces may be a testing ground for the idea of a “small Universe,” proposed by Ellis [28]. A small Universe is one with compact spatial sections, in which thus a present observer has already seen several times around the space. Several papers were de-

voted to checking this idea against the observational data (see, for example, Refs. [29–34]; a conclusive proof of any nontrivial topology is, unfortunately, still lacking). However, the background geometry has always been a homogeneous isotropic Robertson-Walker metric with identifications in the underlying manifold. The quasiplane toroidal Szekeres model has a less general topology (identifications in it occur only in two-dimensional surfaces, in the z -direction the space is infinite), but is inhomogeneous and has no symmetry, so might be useful for considering light propagation and comparing the mass distribution in the model with the observed images.

ACKNOWLEDGMENTS

The research for this paper was inspired by a collaboration with Charles Hellaby, initiated in 2006 at the Department of Mathematics and Applied Mathematics in Cape Town. It was supported by the Polish Ministry of Science and Education Grant No. 1 P03B 075 29. Discussions with Ch. Hellaby helped in clarifying several points. I am grateful to Dr. Stanisław Bajtlik for directing me to the relevant references on the observational aspects of the topology of space. Members of the relativity seminar at the Institute of Theoretical Physics, Warsaw University, are gratefully acknowledged for their valuable comments on the geometry of projective planes that led to an improvement in this text.

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