Rotating Bianchi type *V* dust models generalizing the $k=-1$ Friedmann model

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The Einstein equations are investigated for a rotating Bianchi type V dust model in which one of the Killing fields is spanned on velocity and rotation (case 1.2.2.2 in the classification scheme of the earlier papers). A first integral of the field equations is found, and with a special value of this integral coordinate transformations are used to eliminate two components of the metric. The $k=-1$ Friedmann model is shown to be contained among the solutions in the limit of zero rotation. The field equations for the simplified metric are reduced to 3 second-order ordinary differential equations that determine 3 metric components plus a first integral that algebraically determines the fourth component. First derivatives of the metric components are subject to a constraint (a second-degree polynomial with coefficients depending on the functions). It is shown that the set does not follow from a Lagrangian of the Hilbert type. The group of Lie point-symmetries of the set is found, it is two-dimensional noncommutative. Finally, a method of searching for first integrals (for sets of differential equations) that are polynomials of degree 1 or 2 in the first derivatives is applied. No such first integrals exist. The method is used to find a constraint (of degree 1 in first derivatives) that could be imposed on the metric, but it leads to a vacuum solution, and so is of no interest for cosmology.
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I. STATEMENT OF THE PROBLEM AND SUMMARY OF THE PAPER

This paper is a continuation of a series of papers on rotating dust models in relativity.[1–3] The initial motivation for this research was the desire to find a rotating generalization of the Friedmann models. In spite of much effort spent on investigating solutions of Einstein equations with a rotating matter source, no such generalization has been found so far; see literature surveys in Refs. 3 and 4. References 1, 2 and 3 provided a complete classification scheme for hypersurface–homogeneous rotating perfect fluid models with zero acceleration. Unlike previous approaches, the classification includes also timelike and null symmetry orbits, and so it is the farthest-reaching application of the Bianchi classification to rotating and nonaccelerating perfect fluid models in relativity. The models split into 3 general classes: I, in which two of the Killing fields are everywhere spanned on the vector fields of velocity u^α and rotation w^α (Ref. 1); II, in which only one Killing field is spanned on u^α and w^α (Ref. 2); and III, in which all Killing fields are linearly independent of u^α and w^α (Ref. 3). The many particular cases arise because of several possible alignments or misalignments among the 3 Killing fields and u^α and w^α.

By the Bianchi type of the symmetry algebra and by the relation of the velocity field to the symmetry orbits it can be recognized in which cases generalizations of the Friedmann models can be expected. Two such candidate cases were found in class II, and five more in class III. Those of class III were prohibitively complicated, but one of the cases of class II allowed for some progress, and this one is presented in the present paper. It is the Bianchi type V subcase of the case 1.2.2.2, given by Eq. (5.19) in Ref. 2. For the other candidate case found in class II, Eq. (5.10) in Ref. 2,

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the $k=0$ Friedmann limit is calculated in a much more complicated way, this will be a subject of another paper.

In Sec. II, the metric is simplified by a coordinate transformation, and a first integral of the Einstein equations is found. With a zero value of this integral, coordinate transformations can be used to eliminate two components of the metric tensor, and the number of nontrivial Einstein equations is reduced to 7. Although there are only 4 functions+matter density to be determined by these 7 equations, the set later turns out to be self-consistent. In Sec. III, it is shown that the $k=-1$ Friedmann models are contained among the metrics that result in the limit of zero rotation. In Sec. IV, the Einstein equations are reduced to a set S of 3 second-order equations to determine 3 metric components+a quadrature Q to determine the fourth component (g_{33}). Of the Einstein equations derived in Sec. II, one is fulfilled identically in consequence of the set $\{S \cup Q\}$, one turns out to be a constraint imposed on the initial data, and the one that determines the matter-density turns out to provide a first integral. The first integral determines g_{33} algebraically in terms of the other components, and so it is a replacement for the quadrature Q. It is also shown that the set S cannot be obtained as the Euler–Lagrange equations from a variational principle of the Hilbert type. Finally, it is shown in Sec. IV how the set $\{S \cup Q\}$ reproduces the Friedmann equations in the limit of zero rotation and zero shear. In Sec. V, the two-dimensional (non-Abelian) group of Lie point-symmetries of the set is found. In Sec. VI, a method of systematic search for polynomial first-order first integrals of a set of ordinary differential equations is applied to the set S of Sec. IV. It is shown that no first integrals that are polynomials of degree 1 or 2 in the first derivatives exist. The same method is used to reveal the existence of a possible constraint on initial data, which is of degree 1 in first derivatives, that is preserved by the set S. However, the constraint necessarily implies zero matter-density, and so it is not interesting for cosmology.

Equations that are of secondary importance for the main text, but are difficult to reproduce, can be found in the preprint version of this paper.[5] They have been deleted from this text at the request of the referee.

II. THE EINSTEIN EQUATIONS, THEIR FIRST INTEGRAL AND IMPLICATIONS OF THE ZERO VALUE OF THIS INTEGRAL

The subject of the present paper are the Einstein equations for the Bianchi type V subcase of case 1.2.2.2 of Ref. 2. For reference, the initial formulas are recalled in their original notation.

The Bianchi type V symmetry results when $c=0$ in Eqs. (5.19) of Ref. 2 and when, in addition, $j=-a$ in Eqs. (5.16). Hence, the metric is

$$\begin{aligned} \mathrm{d}s^2 = {} & \mathrm{d}t^2 + 2y\,\mathrm{d}t\,\mathrm{d}x + y^2 h_{11}\,\mathrm{d}x^2 + 2h_{12}\,\mathrm{d}x\,\mathrm{d}y + 2y^2 h_{13}\,\mathrm{d}x\,\mathrm{d}z \\ & + (h_{22}/y^2)\mathrm{d}y^2 + 2h_{23}\,\mathrm{d}y\,\mathrm{d}z + y^2 h_{33}\,\mathrm{d}z^2, \end{aligned} \tag{2.1}$$

where the coordinates are $\{x^\alpha\} = \{x^0, x^1, x^2, x^3\} = \{t, x, y, z\}$, and $h_{ij}, i, j = 1,2,3$ are unknown functions of the variable

$$v = \mathrm{e}^t y^{C_2/a}, \tag{2.2}$$

a and C_2 being arbitrary constants. The velocity field u^α, the rotation field w^α and the Killing fields ${k_{(i)}}^\alpha$, $i=1,2,3$ are given by

$$\begin{aligned} & u^\alpha = \delta^\alpha{}_0, \quad w^\alpha = (\rho/y)\,\delta^\alpha{}_0, \quad {k_{(1)}}^\alpha = \delta^\alpha{}_1, \quad {k_{(3)}}^\alpha = \delta^\alpha{}_3, \\ & {k_{(2)}}^\alpha = C_2\,\delta^\alpha{}_0 + a(x\,\delta^\alpha{}_1 - y\,\delta^\alpha{}_2 + z\,\delta^\alpha{}_3), \end{aligned} \tag{2.3}$$

where ρ is the matter-density of dust. The rotation tensor $\omega_{\alpha\beta}$ has only one algebraically independent nonzero component:

$$\omega_{12} = \tfrac{1}{2}, \tag{2.4}$$

and therefore the coordinates used here are ill-suited for considering the limit $\omega \to 0$.

As shown in Ref. 1, it follows from the equations of motion and from the equation of conservation of the number of particles that

$$g := \det(g_{\alpha\beta}) = -(y/\rho)^2. \tag{2.5}$$

This is the form in which the metric resulted from the Killing equations in Ref. 2. It is advantageous to transform the coordinates as follows:

$$t = t' - (C_2/a)\ln y', \quad x = x' - C_2/(ay'), \quad (y,z) = (y',z'). \tag{2.6}$$

The result is equivalent to substituting $C_2=0$ and $a=1$ in Eqs. (2.1)–(2.4), i.e., the forms of the metric (2.1), of the vector fields u^α, w^α, ${k_{(1)}}^\alpha$ and ${k_{(3)}}^\alpha$ in (2.3) and of the rotation tensor $\omega_{\alpha\beta}$ in (2.4) do not change [although the new h'_{ij} in (2.1) will be linear combinations of the old h_{ij}], while the new ${k_{(2)}}^\alpha$ basis vector will be

$${k_{(2)}}^\alpha = x\,\delta^\alpha{}_1 - y\,\delta^\alpha{}_2 + z\,\delta^\alpha{}_3, \tag{2.7}$$

and the argument of h_{ij} will now be $v = \mathrm{e}^{t'}$, i.e., the h_{ij} are from now on unknown functions of the time-coordinate t.

It is convenient to parametrize the metric as follows:

$$\mathrm{d}s^2 = (\mathrm{d}t + y\,\mathrm{d}x)^2 - (yK_{11}\,\mathrm{d}x)^2 - (K/y)^2(\mathrm{d}y + y^2h\,\mathrm{d}x)^2 - {K_{33}}^2[yg\,\mathrm{d}x + (f/y)\mathrm{d}y + y\,\mathrm{d}z]^2, \tag{2.8}$$

where K_{11}, K, K_{33}, h, f and g are unknown functions of t. The components of the Einstein tensor referred to below are tetrad components $G_{IJ} = e^\alpha{}_I e^\beta{}_J G_{\alpha\beta}$, i.e., projections of the coordinate components $G_{\alpha\beta}$ onto the orthonormal tetrad $e^I := e^I{}_\alpha\,\mathrm{d}x^\alpha$ implied by (2.8):

$$e^0 = \mathrm{d}t + y\,\mathrm{d}x, \quad e^1 = yK_{11}\,\mathrm{d}x, \quad e^2 = (K/y)(\mathrm{d}y + y^2h\,\mathrm{d}x),$$
$$e^3 = K_{33}[yg\,\mathrm{d}x + (f/y)\mathrm{d}y + y\,\mathrm{d}z], \tag{2.9}$$

where $e^\alpha{}_I$ is the inverse matrix to $e^I{}_\alpha$, i.e., $e^\alpha{}_I e^I{}_\beta = \delta^\alpha{}_\beta$, $e^\alpha{}_J e^I{}_\alpha = \delta^I{}_J$. In the parametrization (2.8), the determinant of the metric is

$$g = -(yK_{11}KK_{33})^2. \tag{2.10}$$

The tetrad components of the Einstein tensor corresponding to the metric (2.8) are given in Appendix A of Ref. 5. As seen from there, two combinations of those equations are of first order, they are $K_{11}G_{03} + G_{13} = 0$, i.e.,

$$\left(\tfrac{3}{2}K_{33}/K_{11}\right)[({K_{11}}^2 - 1)K^{-2}f,_t + h(hf,_t - g,_t)] = 0, \tag{2.11}$$

and $K_{11}G_{02} + G_{12} = 0$, i.e.,

$$(K_{11}K)^{-1}[-\tfrac{3}{2}K^2hh,_t + \tfrac{1}{2}h - K_{11}K_{11},_t + ({K_{11}}^2 - 1)(2K,_t/K - K_{33},_t/K_{33})] = 0. \tag{2.12}$$

As shown in Appendix B of Ref. 5, the case $h=0$ does not lead to interesting developments, so we shall proceed further under the assumption

$$h \neq 0. \tag{2.13}$$

Then, Eq. (2.11) implies

$$g,_t = [h + (K_{11}{}^2 - 1)/(hK^2)]f,_t. \tag{2.14}$$

With this, the equations $G_{03} = G_{13} = G_{23} = 0$ turn out to be equivalent, and they can be written as follows:

$$-\frac{1}{2}\left(\frac{K_{11}{}^2 - 1}{h} \cdot \frac{K_{33}{}^3 f,_t}{K_{11}K}\right),_t + \frac{K_{33}{}^3 f,_t}{K_{11}K} = 0. \tag{2.15}$$

This invites the introduction of the new variable $u(t)$ by $u,_t = h/(K_{11}{}^2 - 1)$, and then (2.15) becomes

$$\left(\frac{K_{33}{}^3 f,_u}{K_{11}K}\right),_u - 2\frac{K_{33}{}^3 f,_u}{K_{11}K} = 0, \tag{2.16}$$

which has the first integral $K_{33}{}^3 f,_u/(K_{11}K) = C\mathrm{e}^{2u}$, $C = \text{const}$, i.e.,

$$f,_t = C\mathrm{e}^{2u} h K_{11} K/[K_{33}{}^3 (K_{11}{}^2 - 1)]. \tag{2.17}$$

From here on, we shall follow only the special case $C = 0$, which is a solution of the Einstein equations, but not a general one: it is a subcase chosen *ad hoc* for further progress with integration. Then, from (2.17) and (2.14) $f = \text{const}$, $g = \text{const}$, and from (2.9) the coordinate transformation $z' = z + f/y + gx$ leads to

$$f = g = 0, \tag{2.18}$$

without changing any of the other formulas for $g_{\alpha\beta}$, u^α, w^α, $\omega_{\alpha\beta}$ or $k_{(i)}{}^\alpha$.

The Einstein equations $G_{03} = G_{13} = G_{23} = 0$ are now fulfilled identically. We are left with 7 equations that should determine the 4 functions K_{11}, K, K_{33} and h, and the matter density ρ in addition. It will turn out in Sec. IV that the 7 equations are dependent just in the way needed to make the problem self-consistent and determinate.

III. THE FRIEDMANN LIMIT OF THE METRIC

As already stated, the coordinates used in Sec. II are ill-suited for considering the limit $\omega \to 0$. This limit can be calculated after a coordinate transformation and a reparametrization of the metric.

Since $\omega_{12} = -\omega_{21} = \frac{1}{2}$ are the only nonzero components of the rotation tensor, a natural coordinate transformation to consider is

$$y = \omega_0 y', \tag{3.1}$$

where ω_0 is a constant. After the transformation

$$\omega'_{12} = \tfrac{1}{2}\omega_0 = -\omega'_{21} \tag{3.2}$$

(all other $\omega_{\alpha\beta} = 0$), and the limit of zero rotation is $\omega_0 \to 0$. However, before this limit is taken, the metric functions in (2.8) must be reparametrized or else the limit will be singular. The following reparametrizations will do the job:

$$K_{11} = \tilde{K}_{11}/\omega_0, \quad K_{33} = \tilde{K}_{33}/\omega_0, \quad f = \tilde{f}\omega_0. \tag{3.3}$$

The transformation (3.1) and the reparatmetrization (3.3) result in a metric whose limit $\omega_0 \to 0$ (with primes and tildes omitted) is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - (yK_{11}\,\mathrm{d}x)^2 - (K/y)^2\,\mathrm{d}y^2 - K_{33}{}^2[yg\,\mathrm{d}x + (f/y)\mathrm{d}y + y\,\mathrm{d}z]^2. \tag{3.4}$$

The $k=-1$ Friedmann model results when $g=f=0$ and $K_{11}=K=K_{33}:=R(t)$, where $R(t)$ is the Friedmann scale factor. The resulting coordinates are none of the standard ones, but are related by $y=e^u$ to one of the sets used in the literature [see Eq. (1.3.15) in Ref. 6].

The fact that (3.4), the limit $\omega_0\to 0$ of (2.8), is still more general than the Friedmann metric means that shear survives the transition $\omega\to 0$.

It will be shown at the end of Sec. IV that the explicitly written out Einstein equations do allow a continuous limiting transition $\omega\to 0$, $\sigma\to 0$, and in the limit they reproduce exactly the Friedmann equations.

IV. THE INDEPENDENT EINSTEIN EQUATIONS

We shall now proceed with the subcase (2.18). Equation (2.11) is then fulfilled identically. Equation (2.12) does not change, and it can be more conveniently rewritten if K_{11} is parametrized as follows:

$$K_{11}=\cosh(F). \tag{4.1}$$

Then, from (2.12),

$$K_{33},_t=K_{33}[-\tfrac{3}{2}K^2hh,_t/\sinh^2(F)+\tfrac{1}{2}h/\sinh^2(F)+2K,_t/K-\cosh(F)F,_t/\sinh(F)]. \tag{4.2}$$

When this is substituted into the remaining Einstein equations [(A.1)–(A.10) in Ref. 5], the function K_{33} disappears from the set completely, i.e., we are left with 6 equations to determine h, K, F and the matter-density plus the quadrature implied by (4.2) that allows one to calculate K_{33} once $h(t)$, $K(t)$ and $F(t)$ are known.

Since (2.12) is now satisfied, the equations $G_{02}=0$ and $G_{12}=0$ are equivalent, and they can be written as

$$\begin{aligned} h,_{tt}=&\tfrac{3}{2}K^2hh,_t^2/\sinh^2(F)-5K,_th,_t/K+(2\cosh^2(F)-1)F,_th,_t/\sinh(F)\cosh(F)\\ &+hh,_t/\sinh^2(F)+K,_t/K^3+F,_t/K^2\cosh(F)\sinh(F)-\tfrac{1}{2}h/(K\sinh(F))^2. \end{aligned} \tag{4.3}$$

This is used to eliminate $h,_{tt}$ from the other Einstein equations. The equation $G_{01}=0$ can then be solved for $F,_{tt}$ (see Appendix C in Ref. 5) and this is used to eliminate $F,_{tt}$ from the diagonal components of the Einstein tensor (all the nondiagonal Einstein equations have been used up at this point). After such a substitution, the following identity is fulfilled:

$$G_{11}+G_{33}-2G_{22}\equiv 0, \tag{4.4}$$

i.e., one of the three equations $G_{11}=G_{22}=G_{33}=\Lambda$ can be discarded because it is a consequence of the remaining two. We choose to discard $G_{33}=\Lambda$.

Then, $K,_{tt}$ can be calculated from $G_{22}-G_{11}=0$. The result is

$$\begin{aligned} K,_{tt}=&\tfrac{1}{4}K^3\sinh^{-2}(F)h,_t^2-\tfrac{3}{2}K^3h\cosh(F)\sinh^{-3}(F)F,_th,_t\\ &-\cosh^{-1}(F)\sinh^{-1}(F)K,_tF,_t+2\cosh(F)\sinh^{-1}(F)K,_tF,_t-K\cosh^2(F)\sinh^{-2}(F)F,_t^2\\ &-\tfrac{3}{4}Kh,_t+\tfrac{3}{2}K^3h^2\sinh^{-4}(F)h,_t+\tfrac{3}{4}K^3h^2\sinh^{-2}(F)h,_t\\ &-\tfrac{3}{2}hK,_t-h\sinh^{-2}(F)K,_t-Kh\cosh^{-1}(F)\sinh^{-1}(F)F,_t+\tfrac{3}{2}Kh\cosh^3(F)\sinh^{-3}(F)F,_t\\ &-\tfrac{1}{4}K^{-1}\cosh^2(F)\sinh^{-2}(F)-\tfrac{1}{2}Kh^2\sinh^{-4}(F)-\tfrac{1}{4}Kh^2\sinh^{-2}(F). \end{aligned} \tag{4.5}$$

This is used to eliminate $K,_{tt}$ from the right-hand side of the equation determining $F,_{tt}$ (see Appendix C in Ref. 5), and the result is

$$
\begin{aligned}
F,_{tt}= & -\tfrac{3}{4}K^2\cosh^{-1}(F)\sinh^{-1}(F)h,_t^2-\tfrac{3}{2}Kh\cosh^{-1}(F)\sinh^{-1}(F)K,_th,_t\\
& +2K^{-2}\cosh^{-1}(F)\sinh(F)K,_t^2-K^{-1}K,_tF,_t-\cosh(F)\sinh^{-1}(F)F,_t^2\\
& +\tfrac{3}{4}K^2h^2\cosh^{-1}(F)\sinh^{-1}(F)h,_t+\tfrac{3}{2}K^2h^2\cosh^{-1}(F)\sinh^{-3}(F)h,_t\\
& +\tfrac{1}{2}\cosh^{-1}(F)\sinh^{-1}(F)h,_t-\tfrac{3}{4}\cosh^{-1}(F)\sinh(F)h,_t-\tfrac{3}{2}K^{-1}h\cosh^{-1}(F)\sinh^{-1}(F)K,_t\\
& -\tfrac{3}{2}K^{-1}h\cosh^{-1}(F)\sinh(F)K,_t+h\sinh^{-2}(F)F,_t+\tfrac{5}{2}hF,_t-\tfrac{1}{4}K^{-2}\cosh^{-1}(F)\sinh(F)\\
& -\tfrac{3}{4}K^{-2}\cosh^{-1}(F)\sinh^{-1}(F)-\tfrac{1}{2}h^2\cosh^{-1}(F)\sinh^{-3}(F)-\tfrac{1}{4}h^2\cosh^{-1}(F)\sinh^{-1}(F).
\end{aligned}
\tag{4.6}
$$

With (4.2), (4.3), (4.5) and (4.6) all substituted, the equation $G_{11}=\Lambda$ reduces to the following form:

$$
\begin{aligned}
G_{11}= & \tfrac{1}{4}K^2\cosh^{-2}(F)h,_t^2+\tfrac{3}{2}Kh\cosh^{-2}(F)K,_th,_t+\tfrac{3}{2}K^2h\cosh^{-1}(F)\sinh^{-1}(F)F,_th,_t\\
& -2K^{-2}\cosh^{-2}(F)\sinh^2(F)K,_t^2-2K^{-1}\cosh^{-1}(F)\sinh(F)F,_tK,_t+F,_t^2\\
& +\tfrac{3}{2}K^2h^2\cosh^{-2}(F)h,_t-3K^2h^2\sinh^{-2}(F)h,_t-\tfrac{1}{2}\cosh^{-2}(F)h,_t+\tfrac{3}{2}h,_t\\
& +\tfrac{5}{2}K^{-1}h\cosh^{-2}(F)K,_t+3K^{-1}hK,_t-\tfrac{5}{2}h\cosh^{-1}(F)\sinh^{-1}(F)F,_t\\
& -3h\cosh^{-1}(F)\sinh(F)F,_t+\tfrac{1}{4}K^{-2}\cosh^{-2}(F)+\tfrac{3}{2}K^{-2}+\tfrac{1}{2}h^2\cosh^{-2}(F)+h^2\sinh^{-2}(F)\\
= & \Lambda.
\end{aligned}
\tag{4.7}
$$

Now it may be verified that $G_{11}=\text{const}$ is preserved by Eqs. (4.3), (4.5) and (4.6). This is done as follows. The derivative $(\mathrm{d}/\mathrm{d}t)\,G_{11}$ is calculated, and $h,_{tt}$, $K,_{tt}$ and F_{tt} that reappear are eliminated using (4.3), (4.5) and (4.6). Then, $K,_t^2$ is found from (4.7) and used to eliminate $K,_t^3$ and $K,_t^2$ from $(\mathrm{d}/\mathrm{d}t)\,G_{11}$. The result is the identity $(\mathrm{d}/\mathrm{d}t)\,G_{11}\equiv 0$. This means that, in virtue of the other field equations, if $G_{11}=\Lambda$ holds at any given time, it will remain constant at all other times. Hence, $G_{11}=\Lambda$ is a limitation imposed by the Einstein equations on the initial data for Eqs. (4.3), (4.5), (4.6), and it defines the cosmological constant in terms of the other constants that will appear after (4.3), (4.5) and (4.6) are solved. If $\Lambda=0$, then $G_{11}=0$ reduces the number of arbitrary constants by 1.

Hence, with (4.4), we are left with only four equations: (4.2), (4.7) and any two equations from the set $S=\{(4.3),(4.5),(4.6)\}$, to determine the four functions K_{33}, h, K and F. The third equation in S is implied by the remaining two together with (4.7). The only field equation that has not yet been used up is

$$G_{00}=(8\pi G/c^4)\rho-\Lambda. \tag{4.8}$$

This may be expected to simply define the matter-density in terms of the metric functions. However, in the formulation used in this paper, matter-density enters the equations in two ways: as a source term in G_{00} above, and also through (2.5). From (2.5) and (2.10) it follows that ρ must be related to the other functions by

$$\rho=(K_{11}KK_{33})^{-1}. \tag{4.9}$$

Together with (4.8) and (4.1) this implies that the following must hold:

$$[(G_{00}+\Lambda)\cosh(F)KK_{33}],_t\equiv 0. \tag{4.10}$$

Indeed, this is an identity. This is verified as follows. First, (4.1), (4.2), (4.3), (4.5) and (4.6) are substituted into G_{00} (with $f=g=0$) to eliminate all second derivatives. Then, (4.10) is calculated, and (4.2), (4.3), (4.5) and (4.6) are used to eliminate $K_{33},_t$ and all second derivatives again. Finally, (4.7) is used to eliminate $K,_t^3$ and $K,_t^2$ from the left-hand side of (4.10). In the end, the identity (4.10) results. Hence, (2.5) and (4.8) are consistent in virtue of the other field equations, and moreover $(G_{00}+\Lambda)\cosh(F)KK_{33}=C=\text{const}$ [with second derivatives of h, K and F eliminated by (4.3), (4.5) and (4.6) and with $K,_t^2$ eliminated by (4.7)] is the following first integral of the Einstein equations:

$$K_{33}\Big[-3Kh\sinh(F)F,_t-\frac{3}{2}Kh^2\cosh^{-1}(F)+\frac{3}{2}K\cosh^{-1}(F)\sinh^2(F)h,_t$$
$$+3h\cosh^{-1}(F)\sinh^2(F)K,_t-\frac{3}{2}K^{-1}\sinh^2(F)\cosh^{-1}(F)-\frac{3}{2}K^3h^2\cosh^{-1}(F)h,_t\Big]=C. \tag{4.11}$$

Note that, from (4.8) and (4.9), $C=8\pi G/c^4\neq 0$, and so (4.11) determines K_{33} algebraically. Hence, (4.11) can replace (4.2) as the definition of K_{33}. Thereby, the problem of this paper was reduced to the following procedure.

(1) Find the most general solution of the set $\{(4.3),(4.5),(4.6)\}$. It will contain 6 arbitrary constants $\{C_1,\ldots,C_6\}$.

(2) Impose (4.7) on the $\{h,K,F\}$ found in the previous step. This will be just a definition of Λ in terms of $\{C_1,\ldots,C_6\}$ or, when $\Lambda=0$, an additional constraint imposed on $\{C_1,\ldots,C_6\}$.

(3) Calculate K_{33} from (4.11), with $C=8\pi G/c^4$.

(4) Calculate the matter-density from (4.9).

As shown in Ref. 7, an efficient method to find first integrals of a set of equations exists if the set can be obtained from a Lagrangian. Unfortunately, the problem of determining whether a given set of equations is derivable from a Lagrangian is rather complicated and unsolved in general.[8] It is known that the Einstein equations for class B Bianchi metrics may not admit a Lagrangian, even though the general Einstein equations do (see Ref. 9 for an explanation). It is shown in Appendix A that Eqs. (4.3), (4.5), (4.6) do not follow from the most natural Lagrangian conceivable in this case: a second-degree polynomial in the first derivatives of h, K and F, with coefficients being functions of h, K and F.

For further reference, let us consider the limit of zero rotation in (4.2)–(4.3) and (4.5)–(4.7). After the reparametrization (3.3) we have

$$\cosh(F)=\tilde{K}_{11}/\omega_0,\quad \sinh(F)=\sqrt{\tilde{K}_{11}{}^2/\omega_0^2-1},$$
$$F,_t=\tilde{K}_{11,t}/\sqrt{\tilde{K}_{11}{}^2-\omega_0^2}, \tag{4.12}$$

and then (4.2) in the limit $\omega_0\rightarrow 0$ becomes

$$\tilde{K}_{33,t}=\tilde{K}_{33}(2K,_t/K-\tilde{K}_{11,t}/\tilde{K}_{11}), \tag{4.13}$$

which is an identity in the Friedmann limit $\tilde{K}_{11}=K=\tilde{K}_{33}=R(t)$.

The limiting form of (4.3) is

$$h,_{tt}=-5h,_tK,_t/K+2\tilde{K}_{11,t}h,_t/\tilde{K}_{11}+K,_t/K^3. \tag{4.14}$$

The limit $\omega_0\rightarrow 0$ of (4.5) is

$$K,_{tt} = -K\,\widetilde{K}_{11,t}{}^2/\widetilde{K}_{11}{}^2 - \tfrac{3}{4}Kh,_t - \tfrac{3}{2}hK,_t + \tfrac{3}{2}Kh\widetilde{K}_{11,t}/\widetilde{K}_{11} - 1/(4K) + 2K,_t\widetilde{K}_{11,t}/\widetilde{K}_{11}. \tag{4.15}$$

The same limit of (4.6) is

$$\widetilde{K}_{11,tt}/\widetilde{K}_{11} = 2K,_t^2/K^2 - K,_t\widetilde{K}_{11,t}/(K\widetilde{K}_{11}) + \tfrac{3}{2}h\widetilde{K}_{11,t}/\widetilde{K}_{11} - \tfrac{3}{2}h,_t - \tfrac{3}{2}hK,_t/K - 1/(4K^2). \tag{4.16}$$

In the Friedmann limit $\widetilde{K}_{11} = K = R(t)$, Eqs. (4.15) and (4.16) become identical:

$$R,_{tt}/R = R,_t^2/R^2 - \tfrac{3}{4}h,_t - 1/(4R^2). \tag{4.17}$$

Finally, the limit $\omega_0 \to 0$ of (4.7) is

$$-2K,_t^2/K^2 - 2K,_t\widetilde{K}_{11,t}/(K\widetilde{K}_{11}) + \widetilde{K}_{11,t}{}^2/\widetilde{K}_{11}{}^2 + \tfrac{3}{2}h,_t + 3hK,_t/K - 3h\widetilde{K}_{11,t}/\widetilde{K}_{11} + 3/(2K^2) = \Lambda. \tag{4.18}$$

The Friedmann limit of this is

$$-3R,_t^2/R^2 + \tfrac{3}{2}h,_t + 3/(2R^2) = \Lambda. \tag{4.19}$$

Finding $h,_t$ from (4.19) and substituting it in (4.17) we obtain

$$R,_{tt}/R = -R,_t^2/(2R^2) + 1/(2R^2) - \Lambda/2, \tag{4.20}$$

which is exactly one of the Friedmann equations. Incidentally, the $h,_t$ found from (4.19), if substituted in (4.14), leads to (4.20) again. Hence, in the Friedmann limit, (4.14) follows from (4.19) and (4.17). The function h, as seen from (3.4), does not enter the limiting metric $\omega \to 0$ at all because, after the reparametrization (3.3), it is multiplied by ω_0.

Note that also (4.11) has a meaningful Friedmann limit. In order to make this limit finite, it must be assumed that

$$C = \widetilde{C}/\omega_0^2, \tag{4.21}$$

and then the limit $\omega_0 \to 0$ of (4.11) is

$$K_{33}[\Lambda K\widetilde{K}_{11} + 2K,_t^2\widetilde{K}_{11}/K + 2K,_t\widetilde{K}_{11,t} - K\widetilde{K}_{11,t}{}^2/\widetilde{K}_{11} - 3\widetilde{K}_{11}/K] = \widetilde{C}. \tag{4.22}$$

In the Friedmann limit this becomes

$$R(\Lambda R^2 + 3R,_t^2 - 3) = \widetilde{C}. \tag{4.23}$$

Recalling the Friedmann formula for the mass-density, with $k = -1$:

$$3R,_t^2/R^2 - 3/R^2 + \Lambda = (8\pi G/c^2)\rho, \tag{4.24}$$

we recognize in (4.23) the familiar mass-conservation formula of the Friedmann model, $\rho R^3 = c^2\widetilde{C}/(8\pi G) = \text{const}$.

V. THE LIE POINT-SYMMETRIES OF THE EQUATIONS (4.3), (4.5) AND (4.6)

The basic definitions and theorems concerning point-symmetries are presented in detail in Refs. 7 and 8.

Equations (4.3), (4.5) and (4.6) are of the following form:

$$\frac{\mathrm{d}^2 z^i}{\mathrm{d}t^2} = W^i{}_{jk}\frac{\mathrm{d}z^j}{\mathrm{d}t}\frac{\mathrm{d}z^k}{\mathrm{d}t} + V^i{}_j\frac{\mathrm{d}z^j}{\mathrm{d}t} + U^i, \tag{5.1}$$

where $i=0,1,2$; $(z^0,z^1,z^2)=(h,K,F)$ and $W^i{}_{jk}$, $V^i{}_j$ and U^i are functions of the z^i, but not of t. (Incidentally, the independence of t of all these coefficients immediately implies one group of symmetries, $t\rightarrow t'=t+s$, where s is the group parameter.) Let the following be a one-dimensional group of point transformations:

$$t'=t'(t,\{z^j\},\tau), \quad z'^i=z'^i(t,\{z^j\},\tau), \tag{5.2}$$

where τ is the group parameter and $\tau=\tau_0$ corresponds to the identity [so that $t'(t,\{z^j\},\tau_0)\equiv t$, etc.]. The generators of this group [the field of vectors tangent to the orbits of the group (5.2)] are then

$$X=\xi\frac{\partial}{\partial t}+\eta^j\frac{\partial}{\partial z^j}, \tag{5.3}$$

where

$$\begin{bmatrix}\xi\\ \eta^j\end{bmatrix}=\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix}t'\\ z'^j\end{bmatrix}_{\tau=\tau_0}. \tag{5.4}$$

The generator X is extended to arbitrary derivatives $\mathrm{d}^k z/\mathrm{d}t^k := \overset{(k)}{z}$ by the recursive formulas

$$\overset{(0)j}{\eta}=\eta^j, \quad \overset{(k)j}{\eta}=\frac{\mathrm{d}\overset{(k-1)j}{\eta}}{\mathrm{d}t}-\frac{\mathrm{d}^k z^j}{\mathrm{d}t^k}\frac{\mathrm{d}\xi}{\mathrm{d}t}, \tag{5.5}$$

$$\overset{(k)}{X}=\xi\frac{\partial}{\partial t}+\eta^j\frac{\partial}{\partial z^j}+\overset{(1)j}{\eta}\frac{\partial}{\partial \overset{(1)j}{z}}+\cdots+\overset{(k)j}{\eta}\frac{\partial}{\partial \overset{(k)j}{z}}. \tag{5.6}$$

The derivatives $\mathrm{d}/\mathrm{d}t$ in (5.5) are total derivatives, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t,\{z^i\},\{\overset{(1)i}{z}\},\ldots,\{\overset{(k)i}{z}\})=\frac{\partial f}{\partial t}+\frac{\mathrm{d}z^j}{\mathrm{d}t}\frac{\partial f}{\partial z^j}+\sum_{p=1}^{k}\overset{(p+1)j}{z}\frac{\partial}{\partial \overset{(p)j}{z}},$$

and the order n to which the generator X has to be extended is equal to the highest order of derivatives in the set (5.1) ($n=2$ in our case). A generator of a point-symmetry obeys then

$$\overset{(n-1)}{X}\Omega^i=\frac{\mathrm{d}\overset{(n-1)i}{\eta}}{\mathrm{d}t}-\Omega^i\frac{\mathrm{d}\xi}{\mathrm{d}t}, \tag{5.7}$$

where Ω^i is the right-hand side of (5.1). Equations (5.7) must be identities in all the derivatives $\overset{(1)i}{z},\ldots,\overset{(n-1)i}{z}$, and so they imply several separate equations to be obeyed by the ξ and η^i.

For our equations (5.1), Eqs. (5.7) imply the following four relations:

$$\xi,_{kl}+W^j{}_{kl}\xi,_j=0, \tag{5.8}$$

$$\eta^i,_{kl}=W^i{}_{kl,s}\eta^s+2W^i{}_{s(l}\eta^s,_{k)}-W^s{}_{kl}\eta^i,_s+\delta^i{}_{(l}V^s{}_{k)}\xi,_s+V^i,_{(l}\xi,_{k)}+2\frac{\partial^2\xi}{\partial t\,\partial z^{(k}}\delta^i{}_{l)}, \tag{5.9}$$

where parentheses on indices denote symmetrization,

$$\frac{\partial^2 \eta^i}{\partial t\,\partial z^k} = W^i{}_{ks}\frac{\partial \eta^s}{\partial t} + \frac{1}{2}V^i{}_{k,s}\eta^s + \frac{1}{2}V^i{}_s\eta^s{}_{,k} - \frac{1}{2}V^s{}_k\eta^i{}_{,s} + \frac{1}{2}V^i{}_k\frac{\partial \xi}{\partial t} + U^i\xi_{,k} + \frac{1}{2}\delta^i{}_k U^s\xi_{,s} + \frac{1}{2}\delta^i{}_k\frac{\partial^2 \xi}{\partial t^2}, \tag{5.10}$$

$$\frac{\partial^2 \eta^i}{\partial t^2} = V^i{}_s\frac{\partial \eta^s}{\partial t} + U^i{}_{,s}\eta^s - U^s\eta^i{}_{,s} + 2U^i\frac{\partial \xi}{\partial t}. \tag{5.11}$$

The general solution of these equations [with $W^i{}_{kl}$, $V^i{}_k$ and U^i read off from (4.3), (4.5) and (4.6)] is

$$X = A\frac{\partial}{\partial t} + B\left(t\frac{\partial}{\partial t} - h\frac{\partial}{\partial h} + K\frac{\partial}{\partial K}\right), \tag{5.12}$$

where A and B are arbitrary constants. The proof that this is the most general solution is laborious but straightforward, it is given in Appendix E of Ref. 5. Hence, our set of equations has a two-dimensional symmetry group whose generators are

$$X_{(1)} = \frac{\partial}{\partial t}, \quad X_{(2)} = t\frac{\partial}{\partial t} - h\frac{\partial}{\partial h} + K\frac{\partial}{\partial K}, \tag{5.13}$$

and the corresponding finite symmetry transformations are

$$t' = t + \tau_1, \quad (h', K', F') = (h, K, F);$$
$$t' = e^{\tau_2}t, \quad h' = e^{-\tau_2}h, \quad K' = e^{\tau_2}K, \quad F' = F, \tag{5.14}$$

where τ_1 and τ_2 are the group parameters.

Unfortunately, these symmetries do not lead to any discernible simplification of the set $S = \{(4.3),(4.5),(4.6)\}$. In variables adapted to the generator $X_{(1)}$, the independent variable is K, and $t(K)$ is one of the functions. The set (5.1) thus transformed is of first order in $\phi(K) := dt/dK$, but the first-order equation is still a member of a complicated set and none of the equations separate out. Moreover, after the transformed set is algebraically solved for $t_{,KK}$, $h_{,KK}$ and $F_{,KK}$, the right-hand sides become polynomials of *third* degree in $t_{,K}$, $h_{,K}$ and $F_{,K}$.

The variables adapted to the generator $X_{(2)}$ are (t', h', K'), where

$$t = e^{K'}t', \quad K = e^{K'}, \quad h = e^{-K'}h'. \tag{5.15}$$

In these variables, the set (5.1) becomes of first order in $\psi(t') = K'_{,t'}$. However, after it is solved for $h'_{,t't'}$, $K'_{,t't'}$ and $F_{,t't'}$, the right-hand sides of $h'_{,t't'}$ and $F_{,t't'}$ contain rational functions of the form $W/(1 + t'K'_{,t'})$, where W is a monomial of second degree in some of the $h'_{,t'}$, $K'_{,t'}$ and $F_{,t'}$. Neither equation separates out. It is not possible to adapt the variables to both the generators simultaneously because the group is nonabelian. This author was not able to make any use of the new variables.

VI. FIRST INTEGRALS THAT ARE POLYNOMIALS IN $(h_{,t}, K_{,t}, F_{,t})$

Suppose that the set $\tilde{S} = \{(4.3),(4.5),(4.6),(4.7)\}$ has a first integral of the form

$$I := Q_{ij}\dot{z}^i\dot{z}^j + L_i\dot{z}^i + E = C = \text{const}, \tag{6.1}$$

where C is an arbitrary constant, $Q_{ij} = Q_{ji}$, L_i and E are unknown functions of (h, K, F), $i,j = 1,2,3$, $z^1 = h, z^2 = K, z^3 = F$. Then $dI/dt \equiv 0$ in virtue of $\tilde{S}$, i.e., using (5.1) to eliminate $\ddot{z}^i$:

$$(2Q_{ij}\dot{z}^j+L_i)(W^i{}_{kl}\dot{z}^k\dot{z}^l+V^i{}_k\dot{z}^k+U^i)+Q_{ij,k}\dot{z}^i\dot{z}^j\dot{z}^k+L_{i,j}\dot{z}^i\dot{z}^j+E_{,i}\dot{z}^i\equiv 0. \tag{6.2}$$

It can be verified that first integrals of the form (6.1) do not exist for our set $\tilde{S}$. The calculations are conceptually straightforward, but lead through horrible intermediate expressions, so they are not reported here. The hypothesis that (6.1) is a first integral uniquely leads to an equation that is equivalent to (4.7).

The same method may be used to test whether our set of equations admits a constraint that would be a polynomial of degree 1 or 2 in the first derivatives. The only difference with respect to the procedure of looking for a first integral is that in verifying whether (6.2) is zero, Eq. (6.1) is used, too. If a nontrivial solution of (6.2) with this additional simplification is found, then it means that the derivative of (6.1) by t is zero if (6.1) holds for any fixed t. Then, such (6.1) is a constraint preserved by the set $\tilde{S}$. However, even this attempt has not led to useful results. Constraints of degree 2, i.e., those with $Q_{ij}\neq 0$, lead to prohibitively complicated equations and could not be investigated. One constraint of the form (6.1) with $Q_{ij}=0$ was found, but it is equivalent to the square bracket in (4.11) being zero, and so implies zero matter density. Again, the details are not reported because they contain complicated equations, but no ingenious ideas. This result proves the usefulness of the method—a sensible constraint was revealed—but the solution with zero density is not interesting for cosmology, and thus not necessarily worth investigating.

The zero-density constraint was found without using Eq. (4.7). Equation (4.7) would reduce the number of unknown functions by one, but the resulting set of equations is prohibitively complicated and no progress was achieved.

VII. SUMMARY OF RESULTS

It was shown that the Einstein equations for the metric (2.8) with $f=g=0$ are self-consistent and solvable. They reduce to the set $S=\{(4.3),(4.5),(4.6)\}$ to determine h, K and $K_{11}=\cosh(F)$, and (4.11) to determine K_{33} (where $C=8\pi G/c^4$). The matter density is found from (4.9). The first derivatives of the functions obeying the set S must obey (4.7).

The Friedmann solution with $k=-1$ is contained among the solutions of this set, as shown in Eqs. (4.12)–(4.24). Unfortunately, no explicit example of a more general solution could be found. Attempts to follow *ad hoc* Ansatzes produced uninteresting results. The Ansatz $K=K_{33}$ led to the deSitter solution in disguise, in which the t-lines had nonzero rotation. The Ansatz $K_{11}=K/C$ ($C=$const), which is consistent with the Friedmann limit, led to such complicated equations that it could not even be verified if they are not contradictory. The assumption of zero shear implies zero expansion, in virtue of the theorem $(\sigma=0)\Rightarrow(\omega\theta=0)$ that holds for dust (see Ref. 10).

The set S was shown to have a two-dimensional group of point-symmetries, given by (5.14), and to admit no Lagrangian of the Hilbert type. It was also verified that no first integrals of the form (6.1) exist.

The progress achieved in this paper was the reduction of the problem of existence of a rotating generalization of the $k=-1$ Friedmann model to the technical problem of finding an explicit solution of the set S. The solvability of the set S may be taken for granted because the Friedmann model itself was shown to be one of its solutions. It is still unknown, though, whether a continuous family of solutions exists labeled by the parameter ω (rotation) such that the limit $\omega\rightarrow 0$ taken in the explicit solution leads to the $k=-1$ Friedmann model.

A similar analysis as done here should be done for the other promising cases identified in Ref. 3.

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APPENDIX A: NONEXISTENCE OF A HILBERT-TYPE LAGRANGIAN FOR THE SET {(4.3), (4.5), (4.6)}

Equations (4.3), (4.5) and (4.6) can be written in the form

$$\frac{\mathrm{d}^2 z^i}{\mathrm{d}t^2} = W^i{}_{jk} \frac{\mathrm{d}z^j}{\mathrm{d}t} \frac{\mathrm{d}z^k}{\mathrm{d}t} + V^i{}_j \frac{\mathrm{d}z^j}{\mathrm{d}t} + U^i, \tag{A1}$$

where $i=0,1,2$; $z^0=h$, $z^1=K$, $z^2=F$ and $W^i{}_{jk}$, $V^i{}_j$ and U^i are functions of (h,K,F) (but not of t). Note that the set (A.1) is covariant with respect to arbitrary transformations $z^i \to z'^i = f^i(\{z^j\})$: the first derivatives $\mathrm{d}z^j/\mathrm{d}t$ transform then like a contravariant vector, and so do the terms U^i, the coefficients $V^i{}_j$ transform like a mixed tensor, and the coefficients $(-W^i{}_{jk})$ transform like components of an affine connection. [The nontensorial terms in the transformed $(-W^i{}_{jk})$ arise from $\mathrm{d}^2 z^i/\mathrm{d}t^2$.] The most natural Ansatz for a Lagrangian for (A1) is

$$L = Q_{ij} \frac{\mathrm{d}z^i}{\mathrm{d}t} \frac{\mathrm{d}z^j}{\mathrm{d}t} + L_i \frac{\mathrm{d}z^i}{\mathrm{d}t} + \Phi, \tag{A2}$$

where Q_{ij}, L_i and Φ are functions of (h,K,F). Such a Lagrangian would result from the Hilbert Lagrangian by taking out a complete divergence and integrating the result with respect to the spatial variables. The Euler–Lagrange equations implied by (A2) are

$$Q_{is} \frac{\mathrm{d}^2 z^s}{\mathrm{d}t^2} = -\left(Q_{ki,l} - \frac{1}{2} Q_{kl,i} \right) \frac{\mathrm{d}z^k}{\mathrm{d}t} \frac{\mathrm{d}z^l}{\mathrm{d}t} + \frac{1}{2}(L_{k,i} - L_{i,k}) \frac{\mathrm{d}z^k}{\mathrm{d}t} + \frac{1}{2}\Phi,_i . \tag{A3}$$

If these are to be equivalent to (A1), then the following must hold:

$$Q_{is} W^s{}_{kl} = -\tfrac{1}{2}(Q_{ki,l} + Q_{li,k} - Q_{kl,i}), \tag{A4}$$

$$Q_{is} V^s{}_k = \tfrac{1}{2}(L_{k,i} - L_{i,k}), \tag{A5}$$

$$Q_{is} U^s = \tfrac{1}{2}\Phi,_i . \tag{A6}$$

Equation (A4) implies that $(-W^i{}_{jk})$ must be Christoffel symbols constructed from the metric Q_{ij}, Eq. (A5) implies that $\frac{1}{2}L_i$ must be a vector potential for the tensor field $Q_{is}V^s{}_k$, and Eq. (A6) implies that $\Phi/2$ must be a scalar potential for the vector field $Q_{is}U^s$. All of these are strong conditions and they may be impossible to fulfill in many cases.

Indeed, for our equations (4.3), (4.5), (4.6), the solution of (A4) turns out to be $Q_{ij} \equiv 0$, i.e., the Lagrangian (A2) does not exist. An outline of the proof is given in Appendix D of Ref. 5.

Since the Euler–Lagrange equations (A4) are covariant with respect to arbitrary transformations of the Lagrangian variables [in our case $h \to h'(h,K,F)$, etc.], and equations of the form (A1) are covariant, too, the conclusion that a Lagrangian of the form (A2) exists (or does not exist) is coordinate-independent, i.e., having shown that Eqs. (4.3),(4.5),(4.6) do not follow from a Lagrangian (A2) in our variables $\{h,K,F\}$, we know that no such Lagrangian will exist in any other variables.

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