On the thermodynamical interpretation of perfect fluid solutions of the Einstein equations with no symmetry

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The Gibbs–Duhem equation $dU+pdV=TdS$ imposes restrictions on the perfect fluid solutions of Einstein equations that have a one-dimensional symmetry group or no symmetry at all. In this paper, we investigate the restrictions imposed on the Stephani Universe and on the two classes of models found by Szafron. Upon the Stephani Universe and the $\beta'\neq 0$ class of Szafron symmetries are forced. We find the most general subcases of the $\beta'=0$ model of Szafron that are consistent with the Gibbs–Duhem equation and have no symmetry. © *1997 American Institute of Physics.* [S0022-2488(97)02012-9]

I. INTRODUCTION

In relativity, a perfect fluid is defined as a continuous medium whose state is determined by the energy–density (ϵ), the pressure (p) and the four-velocity (u^α) fields, and whose energy-momentum tensor has the form

$$T_{\alpha\beta}=(\epsilon+p)u_\alpha u_\beta - p g_{\alpha\beta} \tag{1.1}$$

[we will use the signature $(+,-,-,-)$, Greek indices running through the values 0, 1, 2, 3 and Latin indices running through the values 1, 2, 3]. Indeed, this form of the energy–momentum tensor guarantees that energy transport occurs only by means of mass-flow. However, a single-component perfect fluid must also obey the Gibbs–Duhem equation:

$$dU+pdV=TdS, \tag{1.2}$$

which forms a part of the second law of thermodynamics, where U is the internal energy, p is the pressure, V is the volume, T is the temperature and S is the entropy of a thermally isolated portion of a perfect fluid.

In relativity, and in particular in cosmology, we require that Eq. (1.2) applies when the internal energy, volume and entropy are referred to one particle of the fluid. If we assume that there exists a function n (the particle number density) which is conserved,

$$(nu^\alpha)_{;\alpha}=0, \tag{1.3}$$

then $U=\epsilon/n$, $V=1/n$ and $S=s/n$, where s is the entropy–density, and the relativistic version of Eq. (1.2) for a perfect fluid with the energy–momentum tensor (1.1) takes the form

$$d(\epsilon/n)+pd(1/n)=TdS. \tag{1.4}$$

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It is usually taken for granted that this equation applies. However, most of the perfect fluid solutions of the Einstein equations considered in the literature are thermodynamically trivial in one way or another. For dust, $p=0$, $\epsilon/n=\text{const}$ and $S=\text{const}$, so (1.4) is no limitation. For perfect fluids with a barotropic equation of state $\epsilon=\epsilon(p)$, Eq. (1.4) with $dS=0$ is a part of the definition of a solution. For cosmological models with a Robertson–Walker, Kantowski–Sachs or Bianchi-type geometries, all the functions ϵ, p and n depend only on the comoving time t of the fluid, so any equation of state can be imposed on them, and then (1.4) simply defines entropy. For solutions that have a symmetry group with two-dimensional orbits (for example, the spherically symmetric ones), the functions ϵ, p and n depend on two variables only. In this case, the left-hand side of (1.4) is a differential form in two variables and is guaranteed to have an integrating factor, hence T and S obeying (1.4) are guaranteed to exist.

Problems appear when the perfect fluid solution in question has a one-dimensional symmetry group or no symmetry at all. For such solutions, ϵ, p and n depend on three or four variables, and the existence of an integrating factor for the left-hand side of (1.4) is an additional limitation on the state functions. This problem has received only fleeting attention in the literature (see below). In this paper, we shall consider the consequences of (1.4) for a few perfect fluid cosmological models with no symmetry. If $[d(\epsilon/n)+pd(1/n)]$ has no integrating factor, then the solution can be interpreted as a mixture of perfect fluids (possibly interacting through reversible chemical reactions), but not as a single-component perfect fluid. If (1.3) and (1.4) can be imposed simultaneously, then, for brevity, we shall say that the model allows for a thermodynamical scheme. Note that (1.3) merely defines the function $n(x)$ and is no limitation on any model; the limitations all result from (1.4).

Bona and Coll[1] were apparently the first to observe that (1.4) may restrict a metric: they showed that the Stephani Universe[2–5] allows for a thermodynamical scheme only if the metric acquires a 3-dimensional symmetry group acting on two-dimensional orbits. Relativistic thermodynamics of perfect fluids was discussed at length by Coll and Ferrando[6] without invoking explicit examples. Quevedo and Sussman[7] gave an example of the Szarfron $\beta'=0$ model[8] that has no symmetry and allows for a thermodynamical scheme, and showed that the parabolic Szafron $\beta'\neq 0$ model does not allow for it unless it has a symmetry. Quevedo and Sussman[9] also analyzed the conditions for the existence of a thermodynamical scheme in the Stephani Universe, and derived the corresponding nonbarotropic equation of state. In this case the equation of state does not allow any plausible physical interpretation. The Gibbs–Duhem equation (1.4) together with the continuity equation (1.3) were also discussed by Goode[10] and by Coley.[11] Goode discussed them as an element of thermodynamical interpretation of a solution with a heat-conducting dust source. Goode's solution, before thermodynamics is imposed on it, is a generalization of the $\beta'=0$ solution of Szekeres.[12] After imposing the thermodynamical relations, the solution simplifies. Whether the simplification necessarily involves symmetries is not known. Coley[11] emphasized the importance of considering (1.3)–(1.4) as a necessary part of physical interpretation of any cosmological model. This seems to be the whole body of literature on the subject published so far.

In this paper, we identify the most general Szafron models of the $\beta'=0$ family that allow for a thermodynamical scheme (Sec. IV) even though they have no symmetry; we also verified that the general Szafron models of the $\beta'\neq 0$ family do not allow a thermodynamical scheme unless they have symmetries (Sec. V). These are the main results of the paper. In addition, in Sec. II we describe the necessary conditions for the existence of the thermodynamical scheme, and in Sec. III we give some additional details of the result of Bona and Coll[1] for the Stephani Universe.

The models of Stephani[2] and of Szafron[8] are so far the only known exact solutions of the Einstein equations that have no symmetry, can be considered to be cosmological models (because they generalize the Robertson–Walker class of solutions) and allow for nontrivial thermodynamics; see also Ref. 13. The other class of solutions found by Stephani[14] has no symmetry as well, and has some cosmological relevance,[13] but it has constant pressure. Therefore its source is in fact dust in a spacetime with cosmological constant, and (1.4) is trivially satisfied for it. The well-known Szekeres solutions[12] with no symmetry are the dust limit of the Szafron models considered here, and so pose no thermodynamical problems either.

II. THE NECESSARY CONDITIONS FOR THE EXISTENCE OF THE THERMODYNAMICAL SCHEME

Let us write (1.4) in the form

$$\omega := (1/n)\,d\epsilon - (1/n^2)(\epsilon + p)\,dn = T\,dS. \tag{2.1}$$

In general, the quantities ϵ, and p are obtained from Einstein's equations as functions of the coordinates. Equation (1.3) can always be integrated yielding a function n in terms of the coordinates, and so ω will be a given differential form in four variables (although the form is spanned on just two differentials, $d\epsilon$ and dn, the function p will in general be functionally independent of ϵ and n). Equation (2.1) can be solved for T and S if ω has an integrating factor, i.e. if $\omega \wedge d\omega = 0$. This may be written equivalently as

$$d\epsilon \wedge dp \wedge dn = 0, \tag{2.2}$$

which means that a functional dependence (an equation of state) connects ϵ, p and n.

Note that Eq. (1.3) is a necessary, but not a sufficient condition for n to be interpreted as a particle number density. The physical particle number density must obey a thermodynamically meaningful equation of state, which should be derived from Eq. (2.2). In this paper, we shall not impose any condition on n apart from (1.3) and (2.2). Therefore, only our negative results will be conclusive: if (1.3) and (2.2) imply additional symmetry, then the model does not allow for a thermodynamical scheme in general. If (1.3) and (2.2) can be imposed without introducing a symmetry, then additional work on the interpretation of n is required. This we postpone to a separate paper.

The result of this paper is that for the Szafron models with $\beta' \neq 0$ a nontrivial thermodynamical scheme imposes symmetries, while the Szafron models with $\beta' = 0$ are restricted by the thermodynamical scheme in a different way which not necessarily implies a symmetry. (We call the thermodynamical scheme trivial if it implies $p = \text{const}$, and in particular $p = 0$.)

III. THE THERMODYNAMICAL SCHEME IN THE STEPHANI UNIVERSE

The metric of the Stephani Universe is

$$ds^2 = D^2 dt^2 - V^{-2}(t,x,y,z)(dx^2 + dy^2 + dz^2), \tag{3.1}$$

where

$$V = R^{-1}\{1 + \tfrac{1}{4}\,k(t)[(x - x_0(t))^2 + (y - y_0(t))^2 + (z - z_0(t))^2]\}, \quad D = F(t)V_{,t}/V, \tag{3.2}$$

and $F(t)$, $R(t)$, $k(t)$, $x_0(t)$, $y_0(t)$ and $z_0(t)$ are arbitrary functions of time. The source in the Einstein equations is a perfect fluid with the velocity field $u^{\alpha} = D^{-1}\delta^{\alpha}_{0}$ and energy–density ϵ and pressure p given by

$$\kappa\epsilon = 3C^2(t), \quad \kappa p = -3C^2 + 2CC_{,t}V/V_{,t}, \tag{3.3}$$

where $\kappa = 8\pi G/c^4$ and $C(t)$ is connected with the other functions of time by

$$k(t) = [C^2(t) - 1/F^2(t)]R^2(t). \tag{3.4}$$

The Stephani Universe has in general no symmetry and is the most general conformally flat perfect fluid solution with nonzero expansion (see theorem 32.15 in Ref. 15). A particle number density function obeying (1.3) has here the form $n = N(x,y,z)V^3$, where N is an arbitrary function.

The problem of existence of a thermodynamical scheme in this model was solved by Bona and Coll.[1] The scheme exists when V has the special form

$$V = \frac{k}{4R}\,(x^2 + y^2 + z^2 - 2x_0 x + 4Ax_0 + 4B), \tag{3.5}$$

where A and B are arbitrary constants. Only the rotational symmetry in the (y,z) plane is evident here, but in fact this subcase of the Stephani Universe has a 3-dimensional symmetry group acting on 2-dimensional orbits; see Ref. 1. The generators of the group, found from the Killing equations, are

$$\begin{aligned} k_1 &= \left(-\frac{1}{2}\,xy + Ay\right)\frac{\partial}{\partial x} + \left[\frac{1}{4}\,(x^2 - y^2 + z^2) - Ax - B\right]\frac{\partial}{\partial y} - \frac{1}{2}\,yz\,\frac{\partial}{\partial z}, \\ k_2 &= \left(-\frac{1}{2}\,xz + Az\right)\frac{\partial}{\partial x} - \frac{1}{2}\,yz\,\frac{\partial}{\partial y} + \left[\frac{1}{4}\,(x^2 + y^2 - z^2) - Ax - B\right]\frac{\partial}{\partial z}, \\ k_3 &= z\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial z}, \end{aligned} \tag{3.6}$$

and the commutators among them are $[k_1,k_2] = (A^2 + B)k_3$, $[k_2,k_3] = k_1$ and $[k_3,k_1] = k_2$. From here, it is seen that with $B > -A^2$ the solution is spherically symmetric, with $B < -A^2$ it is hyperbolically symmetric, and with $B = -A^2$ it is plane symmetric. This result is equivalent to the one by Bona and Coll[1] except that Bona and Coll obtained it by postulating invariance of n with respect to (3.6), while we have found that it is a general solution of the conditions for a thermodynamical scheme, and invariance of n necessarily follows. The $N = n/V^3$ is restricted by (2.2) to the form

$$N = CG(w)/(x - 2A)^3, \tag{3.7}$$

where C is an arbitrary constant, G is an arbitrary function and

$$w := x + (y^2 + z^2 + 4A^2 + 4B)/(x - 2A). \tag{3.8}$$

In Ref. 9, the thermodynamical scheme conditions for a special case of the solution (3.5) were investigated using the work by Bona and Coll.[16] The result of Ref. 9 is erroneous; that special solution admits in fact a 3-dimensional group of isometries.

IV. THE THERMODYNAMICAL SCHEME IN THE SZAFRON MODELS WITH $\beta' = 0$

The metric of the Szafron models with $\beta' = 0$ is

$$ds^2 = dt^2 - e^{2\alpha}dz^2 - e^{2\beta}(dx^2 + dy^2), \tag{4.1}$$

where

$$e^{\beta} = \Phi(t)/[1 + \tfrac{1}{4}\,k(x^2 + y^2)], \tag{4.2}$$

$$e^{\alpha} = \lambda(t,z) + Se^{\beta}, \tag{4.3}$$

$$S=[\tfrac{1}{2}\,U(z)(x^2+y^2)+V_1(z)x+V_2(z)y+2W(z)], \tag{4.4}$$

k is an arbitrary constant, $U(z)$, $V_1(z)$, $V_2(z)$ and $W(z)$ are arbitrary functions, the function $\Phi(t)$ is determined by the equation

$$2\Phi,_{tt}/\Phi+\Phi,_t^2/\Phi^2+k/\Phi^2+\kappa p(t)=0, \tag{4.5}$$

where $p(t)$ (the pressure in the perfect fluid source) is an arbitrary function, and the function $\lambda(t,z)$ is determined by the equation

$$\Phi\lambda,_{tt}+\Phi,_t\lambda,_t-(\Phi,_{tt}+\Phi,_t^2/\Phi+k/\Phi)\lambda=U(z)+kW(z). \tag{4.6}$$

The source in the Einstein equations is a geodesically and irrotationally moving perfect fluid with the pressure $p(t)$, the velocity field $u^\alpha=\delta_0^\alpha$ and the energy-density given by

$$\kappa\epsilon=2E(t,z)e^{-\alpha}+3(\Phi,_t^2+k)/\Phi^2, \tag{4.7}$$

where

$$E(t,z):=\lambda\Phi,_{tt}/\Phi-\lambda,_{tt}\equiv\Phi,_t\lambda,_t/\Phi-(\Phi,_t^2+k)\lambda/\Phi^2-(U+kW)/\Phi. \tag{4.8}$$

The $\beta'=0$ means that β does not depend on z; this case has to be considered separately because the limit $\beta,_z\to 0$ of the corresponding solutions with $\beta,_z\neq 0$ is singular; see Sec. V. An overview of properties of these solutions, along with a complete list of literature about them, is given in Ref. 13. The $\beta'=0$ solutions simultaneously generalize the Robertson–Walker (R–W) metrics (which result when $\lambda=0$ and $U=-kW$, k is the spatial curvature index in the limit) and metrics with the Kantowski–Sachs (K–S) symmetry[13,17,18] (which result when $U=V_1=V_2=W=\lambda,_z=0$).

With no loss of generality, we can assume $W(z)=0$. This specialization amounts to just redefining U and λ (see Ref. 13), and we shall do so in the following. (After such a specialization the R–W limit changes to $\{U=-ku(z),\ \lambda=\Phi u\}$). Note that the coordinate z is not defined uniquely. All the formulae given are covariant with the transformations $z=f(z')$, where f is an arbitrary function. The particle number density function obeying (1.3) has here the form $n=N(x,y,z)e^{-\alpha-2\beta}$, where $N(x,y,z)$ is an arbitrary function.

Equation (2.2) implies here $N=e^{2B}F(Se^B,z)$, where $e^B=e^\beta/\Phi$, and

$$[(\lambda/\Phi+X)E,_Z-E\lambda,_Z/\Phi]F,_X/F+EF,_Z/F=E,_Z, \tag{4.9}$$

where $X=Se^B$, $Z=z$. This is a quasi-linear partial differential equation determining $F(X,Z)$. However, the coefficients in (4.9) do depend on time, while F should be, by the definition of N, independent of t. We first solve (4.9) as if F were allowed to depend on t, and then we impose the condition $F,_t=0$. The general solution of (4.9) is $F=EG(\mathscr{U})$, where G is an arbitrary function and $\mathscr{U}:=(\lambda/\Phi+X)/E+f(t)$, with $f(t)$ being another arbitrary function. The condition $F,_t=0$ reads now as

$$E,_tG+G,_{\mathscr{U}}[(\lambda/\Phi),_t-(\lambda/\Phi+X)E,_t/E+f,_tE]=0. \tag{4.10}$$

Three cases arise here: I. $E,_t=0=G,_{\mathscr{U}}$, this will turn out to be included in the case III below and does not require a separate treatment; II. $E,_t=0$, $(\lambda/\Phi),_t=-f,_tE$, this one will be considered separately further on; III. $E,_t\neq 0\neq G,_{\mathscr{U}}$, this is the generic case that we will consider first.

In case III, Eq. (4.10) implies $G,_{\mathscr{U}\mathscr{U}}=0$, i.e., $F=A(\lambda/\Phi+X)+\psi(t)E$, where A is an arbitrary constant and $\psi(t)$ is an arbitrary function. Now it is seen that the case $E,_t=0=G,_{\mathscr{U}}$ is contained here as the subcase $A=0=E,_t=\psi,_t$. We can assume $\psi\neq 0$ because with $\psi=0$ the condition $F,_t=0$

implies either $A=F=0$, i.e., $n=0-a$ thermodynamical nonsense, or $(\lambda/\Phi),_t=0$, i.e., a R–W metric. The condition $F,_t=0$ implies now $A\lambda/\Phi+\psi E=H(z)$, where $H(z)$ is an arbitrary function. Substituting the definition of E into this we obtain

$$\lambda,_t=K(t)\lambda+L(t,z), \tag{4.11}$$

where

$$K(t):=\Phi,_t/\Phi+k/(\Phi\Phi,_t)-A/(\psi\Phi,_t), \quad L(t,z):=U(z)/\Phi,_t+H(z)\Phi/(\psi\Phi,_t). \tag{4.12}$$

(We may assume $\Phi,_t\neq 0$ because otherwise $p=$const.) The solution of Eq. (4.11) is

$$\lambda=J(z)e^{\int Kdt}+e^{\int Kdt}\int Le^{-\int Kdt}dt, \tag{4.13}$$

where $J(z)$ is another arbitrary function of z. Substituting (4.13) into (4.6) we obtain an equation of the following form:

$$J(z)F_1(t)+U(z)F_2(t)+H(z)F_3(t)=0, \tag{4.14}$$

where F_1, F_2 and F_3 are functions of t composed of Φ,ψ and their derivatives (see Appendix A). Three possibilities now arise.

(α) All the three functions J, U and H are linearly independent. Then (4.14) implies $F_1=F_2=F_3=0$, and it can be shown from (4.5) and (4.6) that $p=$const, so this case is thermodynamically trivial.

(β) Two of the functions $\{J,U,H\}$ are linearly independent, while the third one is their linear combination. In each of these cases, two linear combinations of the functions $\{F_1,F_2,F_3\}$ must vanish, which leads to a set of two differential equations to be obeyed by $\psi(t)$ and $\Phi(t)$. Some of the resulting solutions are nontrivial, but not all of them. For example, the trivial solution from point (α) reappears in the two cases: $H=0$, with $\{J,U\}$ being linearly independent, and $U=0$, $k\neq 0$ with $\{J,H\}$ being linearly independent. However, with $U=k=0$ and $\{J,H\}$ being linearly independent, the functions Φ and ψ have to obey only one equation:

$$\Phi,_{tt}=-A/\psi+3\Phi,_t^2/\Phi-\psi,_t\Phi,_t/\psi, \tag{4.15}$$

which means that $\Phi(t)$ can be arbitrary, $p(t)$ (in general nontrivial) is determined by (4.5) with $k=0$, and $\psi(t)$ is determined by (4.15). It can be shown (by careful retracing and adaptation of the reasoning in Ref. 19 to the case $p\neq 0$) that with $U=k=0$, $(\lambda/\Phi),_t\neq 0$ and generic V_1 and V_2 the model has no symmetry. Hence, the case we have just identified is an example of a Szafron $\beta'=0$ model with no symmetry that allows for a thermodynamical scheme.

(γ) Each pair in the set $\{J,U,H\}$ is linearly dependent. This means that there is only one function of z in this set and that Φ and ψ are connected by (4.14) with J, U, and H replaced by arbitrary constants, i.e., $\Phi(t)$ is again arbitrary. It can be shown again that the Szafron model has in general no symmetry also in this case, so we have here another example of a model with a thermodynamical scheme and with no symmetries. The case discussed by Quevedo and Sussman[7] is contained in the class (γ) as the case $J=H=F_2=0$.

It remains now to investigate the case II of the three cases listed after Eq. (4.10). We have then $E,_t=0$ and

$$\lambda=-E(z)f(t)\Phi(t)+H(z)\Phi(t), \tag{4.16}$$

where $E(z)$ and $H(z)$ are arbitrary functions. This λ must obey (4.6) with $W=0$ and (4.8) with $E=E(z)$. The condition $F,_t=0$ is identically satisfied. This subcase has nonzero shear as long as

$f,_t \neq 0$ (if $f,_t=0$, then the thermodynamically trivial solution with $\Phi,_t=0$ results). Equations (4.16), (4.6) and (4.8) lead to the following two equations determining Φ and f:

$$f,_t = kf/(\Phi\Phi,_t) - 1/\Phi,_t - C/(\Phi\Phi,_t), \tag{4.17}$$

$$\Phi(\Phi+C)\Phi,_{tt} - k(\Phi+C) - (3\Phi+C)\Phi,_t^2 = (-k+\Phi\Phi,_{tt} - \Phi,_t^2)kf, \tag{4.18}$$

where C is an arbitrary constant. The functions E, H and U are connected by $kH+U=CE$. When $k=0$, Eq. (4.18) decouples from (4.17). Even in that special case, p is not constant and the solution has in general no symmetry. Hence, this is another example of a Szafron $\beta'=0$ model without a symmetry and with a nontrivial thermodynamical scheme.

V. THE THERMODYNAMICAL SCHEME IN THE SZAFRON MODELS WITH $\beta' \neq 0$

The metric of these models is of the same form as in (4.1) but here (in a notation adapted from Szafron[8]):

$$e^\beta = \Phi(t,z)/S(x,y,z), \quad e^\alpha = h(z)S\cdot(e^\beta),_z, \tag{5.1}$$

$$S := A(z)(x^2+y^2) + 2B_1(z)x + 2B_2(z)y + C(z), \tag{5.2}$$

where h, A, B_1, B_2 and C are arbitrary functions of z, the function $\Phi(t,z)$ is determined by Eq. (4.5) with k being a function of z that obeys the relationship:

$$AC - B_1^2 - B_2^2 = \tfrac{1}{4}[h^{-2}(z) + k(z)]. \tag{5.3}$$

Note that the limit $\beta,_z \to 0$ of this solution is singular. Therefore the case $\beta,_z=0$ discussed in Sec. IV has to be derived separately from the Einstein equations. The source in the Einstein equations is again a geodesically and irrotationally moving perfect fluid with the velocity field $u^\alpha = \delta^\alpha_0$, the arbitrary pressure $p(t)$, and the energy–density:

$$\kappa\epsilon = (h/\Phi^2)E(t,z)e^{-\alpha} + 3(\Phi,_t^2+k)/\Phi^2, \quad \text{with} \;\; E(t,z) = \Phi(\Phi,_t^2+k),_z - 2\Phi,_z(\Phi,_t^2+k). \tag{5.4}$$

The R–W limit results when $\Phi = zR(t)$, and $k = k_0 z^2$, where k_0 is a constant (the spatial curvature index of the R–W metric).[13,20] The particle number density function defined by (1.3) is here $n = N(x,y,z)e^{-\alpha-2\beta}$. In this case, Eq. (2.2) can be shown (by a rather long and tedious calculation) to either reduce the solution (5.1)–(5.4) to dust or to impose a symmetry group on it. The group has at least 3 dimensions, and its orbits are at least two-dimensional. Hence, for this class of Szafron models a nontrivial thermodynamical scheme can exist only if there are symmetries in the spacetime.

VI. CONCLUSIONS

We have verified the following

1. For the Stephani Universe and the Szafron models with $\beta' \neq 0$ a nontrivial thermodynamical scheme (that is, one in which $p \neq 0$) can exist only in those subcases in which the spacetime acquires an at least 3-dimensional symmetry group acting on at least 2-dimensional orbits.

2. The Szafron model with $\beta'=0$ does have subcases that have no symmetry and allow for a nontrivial thermodynamical scheme. In the subcase of class (γ) in Sec. IV, the scale factor $\Phi(t)$ remains arbitrary, but the form of the function λ is limited. In the nontrivial subcases of class (β), the evolution of the scale factor is fixed, while the generality of λ is limited to a lesser degree.

Like we stated in the Introduction, only the first result is conclusive. For the Szafron model with $\beta'=0$ it remains to be verified whether among the functions n there are any which obey an interpretable equation of state. Also, as stated in the Introduction, the negative result of point 1 above only means that these models cannot be interpreted as single-component perfect fluids. It remains to be seen whether they can be interpreted as noninteracting mixtures of perfect fluids or mixtures in which reversible chemical reactions occur. Our results show that there is no simple connection between the existence of a thermodynamical scheme and symmetries.

The hope that motivated this paper was that the Gibbs–Duhem equation (1.2) would force a definite form upon the arbitrary functions of time in the models, and thus would play a similar role as the equation of state does. This happens indeed in the class (β) models of Sec. IV, but it is not a general rule. In the class (γ) models of Sec. IV, the one arbitrary function of time, $p(t)$, survives intact after the integrability of (1.2) is ensured.

In those classes in which arbitrary functions of time are present in spite of the lack of symmetry, an equation of state still has to be imposed on the resulting solution. We recall (see Refs. 3 and 21) that for the Szafron model the barotropic equation of state $p=p(\epsilon)$ trivializes it to a spatially homogeneous one (and in particular to an R–W one in some cases).

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The calculations for this paper were partly done with the algebraic computer systems Maple[22] and Ortocartan.[23]

APPENDIX A

The three functions appearing in Eq. (4.14) are defined as follows:

$$F_1(t)=\left[k/\Phi-k\Phi,_{tt}/\Phi,_t^2+A\Phi\psi,_t/(\psi^2\Phi,_t)+A\Phi\Phi,_{tt}/(\psi\Phi,_t^2)\right.$$
$$\left.+\left(\frac{k}{\Phi}-\frac{A}{\psi}\right)^2\Phi/\Phi,_t^2-\frac{3A}{\psi}\right]\exp\left(\int Kdt\right), \tag{A1}$$

$$F_2(t)=F_1(t)\int(1/\Phi,_t)\exp\left(-\int Kdt\right)dt+1+\frac{k}{\Phi,_t^2}-\frac{A\Phi}{\psi\Phi,_t^2}-\frac{\Phi\Phi,_{tt}}{\Phi,_t^2}, \tag{A2}$$

$$F_3(t)=F_1(t)\int\left[\frac{\Phi}{\psi\Phi,_t}\right]\exp\left(-\int Kdt\right)dt+\left(\frac{\Phi}{\psi}\right)$$
$$\times\left[1+\frac{k}{\Phi,_t^2}-\frac{A\Phi}{\psi\Phi,_t^2}-\frac{\Phi\Phi,_{tt}}{\Phi,_t^2}\right]+\frac{2\Phi}{\psi}-\frac{\Phi^2\psi,_t}{\psi^2\Phi,_t}, \tag{A3}$$

where $K(t)$ is given by (4.12).

[1] C. Bona and B. Coll, Gen. Relativ. Gravit. **20**, 297 (1988).
[2] H. Stephani, Commun. Math. Phys. **4**, 137 (1967).
[3] A. Krasiński, Gen. Relativ. Gravit. **13**, 1021 (1981).
[4] A. Krasiński, Gen. Relativ. Gravit. **15**, 673 (1983).
[5] A. Krasiński, in *The Big Bang and Georges Lemaitre*, edited by A. Berger (Reidel, Dordrecht, 1984), p. 63.
[6] B. Coll and J. J. Ferrando, J. Math. Phys. **30**, 2918 (1989).
[7] H. Quevedo and R. Sussman, Class. Quantum Grav. **12**, 859 (1995).
[8] D. Szafron, J. Math. Phys. **18**, 1673 (1977).
[9] H. Quevedo and R. A. Sussman, J. Math. Phys. **36**, 1365 (1995).
[10] S. W. Goode, Class. Quantum Grav. **3**, 1247 (1986).
[11] A. A. Coley, Phys. Lett. A **137**, 235 (1989).

[12] P. Szekeres, Commun. Math. Phys. **41**, 55 (1975); Phys. Rev. D **12**, 2941 (1975).
[13] A. Krasiński, *Inhomogeneous Cosmological Models* (Cambridge University Press, Cambridge, in press).
[14] H. Stephani, Class. Quantum Grav. **4**, 125 (1987).
[15] D. Kramer, H. Stephani, E. Herlt, and M. A. H. MacCallum, *Exact Solutions of the Einstein Field Equations* (Cambridge University Press, Cambridge, 1980).
[16] C. Bona and B. Coll, C. R. Acad. Sci. Paris **1301**, 613 (1985).
[17] R. Kantowski and R. K. Sachs, J. Math. Phys. **7**, 443 (1966).
[18] A. S. Kompaneets and A. S. Chernov, Zh. Eksp. Teor. Fiz. **47**, 1939 (1964) [Sov. Phys. JETP **20**, 1303 (1965)].
[19] W. B. Bonnor, A. H. Sulaiman, and N. Tomimura, Gen. Relativ. Gravit. **8**, 549 (1977).
[20] S. W. Goode and J. Wainwright, Phys. Rev. D **26**, 3315 (1982).
[21] A. Spero and D. A. Szafron, J. Math. Phys. **19**, 1536 (1978).
[22] Maple's Reference Manual (University of Waterloo, Canada, 1994).
[23] A. Krasiński, Gen. Relativ. Gravit. **25**, 165 (1993).