SYMMETRIES OF MANIFOLDS AND TENSOR FIELDS AND THE BIANCHI CLASSIFICATION

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Andrzej Krasiński
N. Copernicus Astronomical Center, Polish Academy of Sciences
Bartycka 18, 00 716 Warszawa, Poland

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1. Definition of symmetry.

The reader will be assumed to be familiar with the notion of a differentiable manifold.

Part of the definition of a manifold M is a 1–1 mapping between the neighbourhood of each point of M and an open subset $C \subset \mathbb{R}^n$ (Fig. 1). Usually, several maps are required to cover the whole of M, but in this text we shall work within the domain of a single map.

Suppose M is mapped into itself by a mapping F (Fig. 2). A point $P \in M$ is then sent "somewhere else", its displacement being described on C by

$$x'^{\alpha} = F^{\alpha}(\{x\}), \tag{1}$$

where $\alpha = 1, \ldots, n$, $n = \dim M$, x'^{α} are the coordinates of the image of P, $\{x\}$ is the collection of coordinates of P itself, and the mapping F is represented by the collection of all the functions F^{α}. The mapping can be interpreted as a transformation of the coordinate system on M.

Let us assume a tensor field T is defined on M, for definiteness let it be a doubly covariant field with coordinate components $T_{\alpha\beta}$. At each point $P \in M$, the values of $T_{\alpha\beta}(P)$ change under the mapping F according to

$$T'_{\alpha\beta}(P') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}(P')\frac{\partial x'^{\nu}}{\partial x^{\beta}}(P')\left(T_{\mu\nu}(P)|_{x=F^{-1}(x')}\right) \tag{2}$$

(see Fig. 3) where $P' = F(P)$. Eq. (2) expresses the components of T after the transformation at the image of P through the components of T before the transformation at P. We may thus say that F sent P into P' and it sent the tensor $T(P)$ from P to P'. However, at the same time some other point $Q \in M$ was sent into P, and the corresponding tensor $T(Q)$ landed at P (see Fig. 4), with components $T'_{\alpha\beta}(P)$ that can be calculated in terms of $T_{\mu\nu}(Q)$ according to (2) if Q is substituted for P and P for P'. Since $T(P)$ and $T'(P)$ are two (in general different) tensors defined at the same point P, we may now ask the question: how are the coordinates $T'_{\alpha\beta}(P)$ related to $T_{\alpha\beta}(P)$? It may happen that

$$T'_{\alpha\beta}(P) = T_{\alpha\beta}(P) \qquad \text{for every } P \in M. \tag{3}$$

If it so happens, then we call the tensor field T invariant under the mapping F, and we call F an invariance transformation of T. If g, the metric tensor of M (assuming it exists), is invariant under F, then we call F a symmetry of the manifold M (because then the local geometry of M does not change under F).

2. The Killing equations.

Let Γ be a 1-parameter family of mappings of M into itself such that to every value of the parameter t from the range $[t_1, t_2] \stackrel{\text{def}}{=} B \subset \mathbb{R}^1$ there corresponds a mapping f_t:

$$\mathbb{R}^1 \supset B \stackrel{\text{def}}{=} [t_1, t_2] \ni t \to x'^{\alpha} = f^{\alpha}(t, \{x\}),$$

where f_t is the collection of all the functions f^{α} at a given t and Γ is the collection of all the mappings f_t for every value of $t \in B$. Let us also assume that for $t = t_0$ where $t_1 \leq t_0 \leq t_2$ the mapping f_{t_0} is an identity, i. e.

$$f^{\alpha}(t_0, \{x\}) = x^{\alpha}.$$

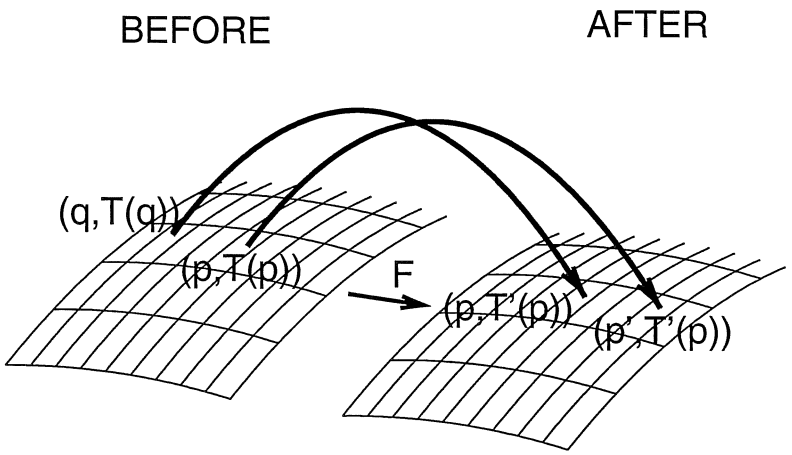

Figure 4: The same mapping sends some other point $Q \in M$ into the place occupied before by P, and the tensor $T(Q)$ is also sent to P where it becomes $T'(P)$. How is $T'(P)$ related to $T(P)$? It is a different question than the one answered by eq. (2). Arches connect points before the transformation with their images after the transformation.

Example: let $B = [0, 2\pi]$, $M = \mathbb{R}^3$, and let f_t be the rotation of $\mathbb{R}^3$ around a fixed axis A by the angle t. Γ is then the collection of rotations of $\mathbb{R}^3$ around A by all angles in the range $0 \leq t < 2\pi$ and $t_0 = 0$.

Now let $P \in M$, and apply to P the mappings f_t corresponding to all $t \in B$. The collection of all images of P will then be an arc of a curve in M passing through $P = f_{t_0}(P)$ (Fig. 5), and each $P \in M$ may be used to generate such an arc. The arc is called the orbit of P under the action of Γ (in short: Γ-orbit of P), and P will be called the initial point of the orbit (although in all "practical" instances the orbits are closed or infinite curves with no endpoints).

We shall assume that (1) the functions $f^\alpha(t, \{x\})$ are of class C^2 with respect to t (i. e. have continuous second derivatives by t); (2) that each f_t is invertible, its inverse (denoted f_t^{-1}) (3) being also of class C^2. The first assumption implies that along each orbit the field of vectors tangent to the orbit exists and is continuously differentiable. The second assumption implies that the mappings of the family Γ and their inverses form a group G. The group multiplication is the superposition of the mappings: $(f_{t_2} \circ f_{t_1})(x) = f_{t_2}(f_{t_1}(x))$, where f_{t_1} is represented by $f^\alpha(t_1, x)$ and f_{t_2} is represented by $f^\alpha(t_2, f_{t_1}(x))$. The third assumption guarantees that the orbits generated by G will have a continuously differentiable field of tangent vectors. For each mapping f_t we may then write (from Taylor's expansion):

$$x'^\alpha = x^\alpha + \left.\frac{\partial f^\alpha}{\partial t}\right|_{t=t_0} (t - t_0) + O\left(\epsilon^2\right), \tag{4}$$

where $x'^\alpha = f^\alpha(t, \{x\})$, $x^\alpha = f^\alpha(t_0, \{x\})$, $\epsilon \stackrel{\text{def}}{=} t - t_0$ and $O(\epsilon^2)$ has the property $\epsilon^{-1} O(\epsilon^2) \underset{\epsilon \to 0}{\longrightarrow} 0$. Note that we are not making any approximation here; eq. (4) is exact, but the form of

$O(\epsilon^2)$ will in the end turn out to be irrelevant. All such irrelevant terms will be denoted by the same symbol $O(\epsilon^2)$ even though they may not be identical to each other. The quantities

$$k^\alpha = \left.\frac{\partial f^\alpha}{\partial t}\right|_{t=t_0} \tag{5}$$

are components of the vectors tangent to the Γ-orbits at their initial points and are called the generators of the group G.

Suppose now that T is invariant under all the transformations in Γ , i. e. that

$$T'(P) = T(P) \qquad \text{for all } P \in M \text{ and } t \in B. \tag{6}$$

What analytic condition must $T_{\alpha\beta}$ fulfil then? Again from Taylor's formula we have

$$T'_{\alpha\beta}(P') = T'_{\alpha\beta}(P) + \epsilon T'_{\alpha\beta,\mu}(P)k^\mu + O(\epsilon^2), \tag{7}$$

where P' has the coordinates $x'^\alpha = f^\alpha(t, \{x\})$ and P has the coordinates $x^\alpha = f^\alpha(t_0, \{x\})$. ¿From (4) and (5) we have

$$\begin{aligned}\frac{\partial x^\mu}{\partial x'^\alpha} &= \frac{\partial}{\partial x'^\alpha}\left[x'^\mu - \epsilon k^\mu - O\left(\epsilon^2\right)\right] = \delta^\mu{}_\alpha - \epsilon k^\mu{}_{,\rho}\frac{\partial x^\rho}{\partial x'^\alpha} - O\left(\epsilon^2\right) \\ &= \delta^\mu{}_\alpha - \epsilon k^\mu{}_{,\rho}\left(\delta^\rho{}_\alpha - \epsilon k^\rho{}_{,\sigma}\frac{\partial x^\sigma}{\partial x'^\alpha} - O(\epsilon^2)\right) - O(\epsilon^2) = \delta^\mu{}_\alpha - \epsilon k^\mu{}_{,\alpha} + O(\epsilon^2). \end{aligned} \tag{8}$$

(We do not differentiate ϵ because we have advanced along the orbits by the fixed parameter distance $\epsilon = t - t_0$ which is coordinate-independent). Hence, from (8) and the transformation law for $T_{\alpha\beta}$

$$\begin{aligned} T'_{\alpha\beta}(P') &= \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}T_{\mu\nu}(P) \\ &= T_{\alpha\beta}(P) - \epsilon k^\mu{}_{,\alpha}\,T_{\mu\beta}(P) - \epsilon k^\nu{}_{,\beta}\,T_{\alpha\nu}(P) + O(\epsilon^2). \end{aligned} \tag{9}$$

Comparing (7) with (9) and using the invariance condition (6) we have

$$\epsilon\left(k^\mu T_{\alpha\beta,\mu} + k^\mu{}_{,\alpha}\,T_{\mu\beta} + k^\nu{}_{,\beta}\,T_{\alpha\nu}\right) + O(\epsilon^2) = 0. \tag{10}$$

We now divide (10) by ϵ, let $\epsilon \to 0$ and recall the definition of $O(\epsilon^2)$. The result is

$$k^\mu T_{\alpha\beta,\mu} + k^\mu{}_{,\alpha}\,T_{\mu\beta} + k^\mu{}_{,\beta}\,T_{\alpha\mu} = 0. \tag{11}$$

These equations are called the Killing equations, and their solutions $k^\mu(\{x\})$ are called Killing vector fields. Every field of tangent vectors to orbits of invariance transformations of $T_{\alpha\beta}$ must fulfil (11), and every solution of (11) generates an invariance group of $T_{\alpha\beta}$. How to find the invariance transformations given k^α and vice versa will be shown later.

The equations (11) can be rewritten in an equivalent form which shows explicitly their covariance:

$$k^\mu T_{\alpha\beta;\mu} + k^\mu{}_{;\alpha}\,T_{\mu\beta} + k^\mu{}_{;\beta}\,T_{\alpha\mu} = 0. \tag{12}$$

If $T = g$ (the metric tensor), then, in view of $g_{\alpha\beta;\gamma} = 0$, eq. (12) may be rewritten as

$$k_{\alpha;\beta} + k_{\beta;\alpha} \stackrel{\text{def}}{=} 2k_{(\alpha;\beta)} = 0. \tag{13}$$

In this form, the Killing equations are most easy to remember, but less convenient to work with (and apply only to g).

Note that eq. (11) applies only to a doubly covariant tensor field, and only in this case are the generators of invariances called Killing vector fields. We shall deal with invariances of other tensor fields in sec. 5 and following.

3. The connection between generators and the invariance transformations.

¿From (4) and (5) it is seen that if $x'^{\alpha} = f^{\alpha}(t, \{x\})$ is a family of invariance transformations for a certain tensor field, then the corresponding generator is given by

$$k^{\alpha} = \left.\frac{\partial f^{\alpha}}{\partial t}\right|_{t=t_0}, \tag{14}$$

where $t = t_0$ corresponds to the identity transformation.

Finding the family of invariances given k^{α} is less straightforward. It is easy to see that any orbit $B \ni t \to x'^{\alpha}(t) = f^{\alpha}(t, \{x\})$ is tangent to $k(P')$ at its every point (where $\{x'\}$ are the coordinates of P'). Hence the orbits must obey the equations

$$\frac{\mathrm{d}x'^{\alpha}}{\mathrm{d}t} = k^{\alpha}(\{x'\}) \tag{15}$$

with the initial conditions $x'^{\alpha}(t_0) = x^{\alpha}$ (identity transformation). Thus, given $k^{\alpha}(\{x\})$, one must solve eqs. (15) with the above initial conditions, and the solutions will be the invariance transformations. It is useful to verify this on the (hopefully well known to the reader) example of the rotation group $O(3)$. Other useful examples are:

1. If the parameter t is chosen as one of the coordinates on M so that, say, $x^1 = t$ and $k^{\alpha} = \delta^{\alpha}{}_1$, then the Killing equations imply $T_{\alpha\beta,1} = 0$. Conversely, if $T_{\alpha\beta,1} = 0$, then $k^{\alpha} = \delta^{\alpha}{}_1$.

2. If $T_{\alpha\beta}$ has two Killing fields k and l (linearly independent), then the corresponding orbit parameters t and τ can be chosen as coordinates on M if and only if $[k,l]^{\alpha} \stackrel{\text{def}}{=} k^{\rho} l^{\alpha}{}_{,\rho} - l^{\rho} k^{\alpha}{}_{,\rho} = 0$. In that case $\partial T_{\alpha\beta}/\partial t = \partial T_{\alpha\beta}/\partial \tau = 0$. *Question*: what happens when $[k,l]^{\alpha} \neq 0$, i.e. why is it not possible to take the integral lines of k and l as coordinates in this case? A hint at the answer will be given in sec. 5.

4. The basis of Killing vector fields.

Note that the Killing equations are linear and homogeneous with respect to k^{α}. Hence, if k and l are Killing vector fields, then so is $(Ak + Bl)$, where A and B are constants. The general solution of the Killing equations, $K^{\alpha}(\{x\})$, should thus be a linear combination of basis solutions. How do we know the basis is finite? The answer is: we know it, provided K generate symmetries of M. The proof given below (taken from [1]) does not work if K generate an invariance group of a tensor field other than the metric tensor (and examples of infinite bases are known).

For K generating symmetries of M we have from (13)

$$K_{\alpha;\beta} = -K_{\beta;\alpha}, \tag{16}$$

and from the Ricci identity

$$K_{\alpha;\beta\gamma} - K_{\alpha;\gamma\beta} = R^{\rho}{}_{\alpha\beta\gamma}K_{\rho}. \tag{17}$$

Because $R^{\rho}{}_{[\alpha\beta\gamma]} \equiv 0$, we have from the above

$$\left(K_{\alpha;\beta} - K_{\beta;\alpha}\right);_{\gamma} + \left(K_{\gamma;\alpha} - K_{\alpha;\gamma}\right);_{\beta} + \left(K_{\beta;\gamma} - K_{\gamma;\beta}\right);_{\alpha} = 0.$$

Using now (16), the above reduces to

$$K_{\alpha;\beta\gamma} + K_{\gamma;\alpha\beta} + K_{\beta;\gamma\alpha} = 0,$$

and again from (16) this yields

$$K_{\gamma;\alpha\beta} = K_{\beta;\alpha\gamma} - K_{\beta;\gamma\alpha} = R^{\rho}{}_{\beta\alpha\gamma}K_{\rho} \tag{18}$$

(in the last step, eq. (17) was used). Let us note the following: in a given Riemannian manifold (where $g_{\alpha\beta}$ and, consequently, $R^{\alpha}{}_{\alpha\beta\gamma}$ are given as functions of $\{x\}$ on open neighbourhoods of any nonsingular point), eq. (18) allows to calculate **algebraically** $K_{\gamma;\alpha\beta}(P_0)$ if $K_{\gamma}(P_0)$ is given. If $K_{\gamma;\delta}(P_0)$ is given as well, then from the derivative of eq. (18) we can **algebraically** calculate $K_{\gamma;\alpha\beta\epsilon}(P_0)$. Differentiating eq. (18) consecutively, we can then calculate all covariant derivatives of K_{γ} at P_0. Further, having all these derivatives (and so, equivalently, all partial derivatives of K_{γ} at P_0), we can calculate $K_{\gamma}(P)$ where $P \in M$ lies in such a neighbourhood of P_0 in which the Taylor series for $K_{\alpha}(P)$ is convergent. Note however that after each differentiation a new derivative of the Riemann tensor appears, so, in order that the series is convergent, $R^{\rho}{}_{\alpha\beta\gamma}$ must be analytic in that neighbourhood. From this argument we see that $K_{\gamma}(P_0)$ and $K_{\gamma;\delta}(P_0)$ at any chosen $P_0 \in M$ are the data which are needed to determine $K_{\gamma}(P)$ uniquely (if $R^{\rho}{}_{\alpha\beta\gamma}(P_0)$ are not sufficiently differentiable, then simply another initial point is needed, not more data). But $K_{\gamma;\delta}$ obey (16), so $K_{\gamma;\delta}(P_0)$ are $\frac{1}{2}n(n-1)$ constants, and $K_{\gamma}(P_0)$ are n constants in an n-dimensional manifold. Thus, the Taylor series for $K_{\gamma}(P)$ will contain at most $\frac{1}{2}n(n+1)$ arbitrary constants multiplying various functions of $\{x\}$. The multipliers of those constants will be the basis solutions, and hence their number cannot exceed $\frac{1}{2}n(n+1)$.□

The prescription for finding the basis of the Killing vector fields for the metric tensor is therefore the following:

1. Solve the Killing equations. The general solution will then depend on $N \leq \frac{1}{2}n(n+1)$ arbitrary constants, $k^{\mu} = K^{\mu}(A_1, \ldots, A_N, \{x\})$.

2. Calculate $\underset{(i)}{k^{\mu}} \overset{\text{def}}{=} \frac{\partial K^{\mu}}{\partial A_i}, i = 1, \ldots, N$ – the basis. Each $\underset{(i)}{k^{\mu}}$ generates a one-parameter subgroup of symmetries discussed in section 2.

For tensor fields other than the metric tensor a finite basis may not exist, i. e. the general solution of the invariance equations will contain arbitrary functions rather than arbitrary constants. This is the case e.g. for the invariance group of $R^{\alpha\beta}{}_{\gamma\delta}$ (note the

positions of indices!) in a space of constant curvature: any arbitrary vector field l has the property that $R'^{\alpha\beta}{}_{\gamma\delta}(\{x\}) = R^{\alpha\beta}{}_{\gamma\delta}(\{x\})$ for transformations generated by l (see [2] for less trivial examples).

5. Invariance of other tensor fields.

Eq. (11) was derived from the transformation law (7). Consequently, the form of the condition of invariance depends on the transformation properties of a tensorial object. Repeating the reasoning which led to (11) for different objects, we would obtain the following invariance equations:

For a scalar ϕ, the requirement $\phi'(P) = \phi(P)$ implies

$$k^\alpha \phi_{,\alpha} = 0.$$

For a contravariant vector field, $V'^\alpha(P) = V^\alpha(P)$ implies

$$k^\rho V^\alpha{}_{,\rho} - V^\rho k^\alpha{}_{,\rho} = 0.$$

(We can now answer the question posed at the end of sec. 3: the parameters on the integral lines of two Killing fields can be chosen as coordinates on M only if each Killing field is invariant under the transformations generated by the other).

For a covariant vector field (i.e. a form), $\omega'_\alpha(P) = \omega_\alpha(P)$ implies

$$k^\rho \omega_{\alpha,\rho} + k^\rho{}_{,\alpha}\, \omega_\rho = 0.$$

We can also require that a certain tensor field is not invariant, but undergoes a specific change during a mapping. For example, the metric tensor may change as follows

$$g'_{\alpha\beta}(P) = \lambda g_{\alpha\beta}(P). \tag{19}$$

The transformations that result from (19) are called conformal symmetries of the manifold. Their generators obey

$$k_{(\mu;\nu)} = \phi g_{\mu\nu} \tag{20}$$

To find the relation between λ and ϕ is an easy exercise for the readers.

6. The algebra of Killing vector fields.

For each quantity mentioned above it may be verified that if k and l are generators of invariances, then so is

$$[k, l]^\alpha \stackrel{\text{def}}{=} k^\rho l^\alpha{}_{,\rho} - l^\rho k^\alpha{}_{,\rho}\,.$$

(This fact is true for invariances of every tensorial quantity, not only for those mentioned in sec. 5, see next section). Since for the Killing fields generating symmetries of M a finite basis exists, we conclude that there exist such constants $C^l{}_{ij}$ that, for the basis fields

$$[\underset{(i)}{k}, \underset{(j)}{k}]^\alpha = C^l{}_{ij} \underset{(l)}{k}{}^\alpha \qquad \text{(sum over } l\text{)}. \tag{21}$$

The constants $C^l{}_{ij}$ are called structure constants of the symmetry group. We see thus that the set of Killing fields for a given manifold M is a Lie algebra. For generators of invariances of other tensor fields, the coefficients $C^l{}_{ij}$ will not necessarily be constant.

7. The Lie derivative.

If we carefully trace the procedure that led to the Killing equations (11), and also the procedures leading to the other equations listed in sec. 5, then we will notice that the invariance equations are obtained in the following steps:

1. Take the tensor field T (arbitrary indices) to be $T(t_0)$, where t_0 is the value of the orbit parameter corresponding to the identity transformation.

2. Using the transformation law for T under coordinate changes calculate $T(t)$ up to terms linear in $t - t_0$ (the remaining terms are not neglected, but left in implicit form).

3. Calculate the quantity

$$\lim_{t\to t_0} \frac{T(t) - T(t_0)}{t - t_0} \stackrel{\text{def}}{=} \underset{k}{\pounds} T, \tag{22}$$

where $k^\alpha = \mathrm{d}x^\alpha/\mathrm{d}t$, and equate the result to zero.

It does not take a great effort to see that in this way we are calculating the derivative of the tensor field along the Γ-orbits at their initial points. The quantity $\underset{k}{\pounds} T$ is called the Lie derivative [1] of the tensor field T along the vector field k and can be calculated also if T is not invariant under the action of Γ. We have thus:

$$\left(\underset{k}{\pounds} T = 0\right) \Longleftrightarrow \left(T'(P) \equiv T(P)\right). \tag{23}$$

Being a derivative indeed, the Lie derivative has all the algebraic properties of ordinary differentiation, i. e.:

$$\underset{k}{\pounds} \left(aT_1 + bT_2\right) = a \underset{k}{\pounds} T_1 + b \underset{k}{\pounds} T_2 \tag{24}$$

for arbitrary k, T_1 and T_2 and constant a, b;

$$\underset{k}{\pounds} a = 0 \tag{25}$$

for any k and constant a, and for a tensor product it gives

$$\underset{k}{\pounds} \left(T_1 \otimes T_2\right) = \left(\underset{k}{\pounds} T_1\right) \otimes T_2 + T_1 \otimes \left(\underset{k}{\pounds} T_2\right). \tag{26}$$

These properties allow us to derive the general formula for the components of the Lie derivative of any tensor field $T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l}$ along any vector field k^α using the results of sec. 5 :

$$\underset{k}{\pounds} T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l} = k^\rho T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l,\rho} - \sum_{i=1}^{k} k^{\alpha_i}{}_{,\rho_i} T^{\alpha_1\ldots\rho_i\ldots\alpha_k}_{\beta_1\ldots\beta_l} + \sum_{j=1}^{k} k^{\rho_j}{}_{,\beta_j} T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\rho_j\ldots\beta_l}, \tag{27}$$

[1] The notion of the Lie derivative was introduced by the Polish mathematician W. Ślebodziński in 1931 [3], while the name was proposed by van Dantzig and made popular by Schouten [4].

where sums over i and j extend over all positions of ρ_i and ρ_j, respectively, in the series of indices. This formula shows that the Lie derivative acts very similarly to a directional covariant derivative along k, the factors $k^\mu{}_{,\nu}$ playing the role of the Christoffel symbols projected onto k, but with an opposite sign, i. e. $-k^\mu{}_{,\nu} \longleftrightarrow \left\{ \begin{matrix} \mu \\ \nu\rho \end{matrix} \right\} k^\rho$. With the help of (27) one can now verify that for any tensor field T and any vector fields k and l we have

$$\underset{[k,l]}{\pounds} T \equiv [\underset{k}{\pounds}, \underset{l}{\pounds}]T, \tag{28}$$

and so $[k, l]$ always generates an invariance of T if k and l do.

8. The Bianchi classification of 3-dimensional Lie algebras.

As already mentioned, the vector fields which generate symmetries of a manifold form a Lie algebra. For reasons to be explained further, the case when this algebra is 3-dimensional was found physically important. The aim of the Bianchi classification is to sort out the 3-dimensional algebras of vector fields which are inequivalent in the following sense.

A basis of vector fields is not defined uniquely. If $\underset{(i)}{k}$ is a basis, then

$$\underset{(i')}{\widetilde{k}}{}^\alpha \overset{\text{def}}{=} A_{i'}{}^j \underset{(j)}{k}{}^\alpha \qquad \text{(sum over } j\text{)} \tag{29}$$

is also a basis, provided the matrix $A_{i'}{}^j$ (of constant elements) is nonsingular. Such a change of basis is accompanied by the following change of the structure constants

$$\widetilde{C}^{l'}_{i'j'} = \left(A^{-1}\right)^{l'}{}_l A_{i'}{}^i A_{j'}{}^j C^l{}_{ij}. \tag{30}$$

Thus, two sets of structure constants which are related by (30) correspond to two bases of vector fields which are equivalent, i. e. generate the same group. How can one recognize whether any matrix A obeying (30) exists for two given sets of $C^l{}_{ij}$? The answer is provided by the Bianchi classification. Note that no reference will be made in this section to the way in which the corresponding group acts on the manifold. Hence, this is in fact a classification of quite arbitrary 3-dimensional algebras. The presentation is based on [5] (the authors of [5] give credit for their presentation to C. Behr, but no reference is given).[2]

¿From the definition (21) it follows that $C^l{}_{ij} = -C^l{}_{ji}$, so for an N-dimensional group (whose basis of generators contains N elements) the number of structure constants cannot exceed $N \cdot \frac{1}{2}N(N-1)$. Actually, through the Jacobi identity

$$[[\underset{(i)}{k}, \underset{(j)}{k}], \underset{(l)}{k}] + [[\underset{(j)}{k}, \underset{(l)}{k}], \underset{(i)}{k}] + [[\underset{(l)}{k}, \underset{(i)}{k}], \underset{(j)}{k}] = 0, \tag{31}$$

the definition (21) implies a further limitation on $C^l{}_{ij}$:

$$C^m{}_{ij} C^n{}_{ml} + C^m{}_{jl} C^n{}_{mi} + C^m{}_{li} C^n{}_{mj} = 0, \tag{32}$$

[2]The mystery of the original presentation of this approach was revealed to the public only recently, see [20, 21]. The approach was invented by E. Schücking and first presented at a seminar in Hamburg. It was handed down to other researchers via W. Kundt's notes taken at that seminar and by personal communication by C. G. Behr. The earliest papers that used this approach were those by Estabrook, Wahlquist and Behr [22] and by Ellis and MacCallum [7, 8].

and so the number of independent $C^l{}_{ij}$ is in fact smaller than $\frac{1}{2}N^2(N-1)$. We shall take (32) into account later.

For $N=3$, $\frac{1}{2}N^2(N-1)=9$ and equals the number of elements of a general 3×3 matrix. Thus for a 3-dimensional algebra all structure constants can be packed into a single 3×3 matrix, let us call it H^{ab}. The 1–1 correspondence between $C^l{}_{ij}$ and the elements of H^{ab} is

$$H^{ab}=\frac{1}{2}\epsilon^{akl}C^b{}_{kl} \Longleftrightarrow C^i{}_{jk}=\epsilon_{sjk}H^{si}, \tag{33}$$

where ϵ^{akl} and ϵ_{sjk} are the antisymmetric symbols of Levi-Civita. The matrix H^{ab} can now be split into its symmetric part

$$n^{ab}\overset{\text{def}}{=}H^{(ab)}=\frac{1}{2}\left(H^{ab}+H^{ba}\right) \tag{34}$$

and its antisymmetric part $H^{[ab]}$. As usually, in a 3-dimensional space there is a 1–1 correspondence between antisymmetric matrices and vectors, so $H^{[ab]}$ can be represented by the vector a defined by

$$a_i=-\frac{1}{2}\epsilon_{ijk}H^{[jk]} \Longleftrightarrow H^{[ij]}=-\epsilon^{ijm}a_m. \tag{35}$$

¿From (33), (34) and (35) we have

$$C^l{}_{jk}=\epsilon_{sjk}n^{sl}-\delta^{lm}_{jk}a_m, \tag{36}$$

where $\delta^{lm}_{jk}\overset{\text{def}}{=}\delta^l{}_j\delta^m{}_k-\delta^l{}_k\delta^m{}_j$. The identity (32) for $N=3$ may be written as

$$C^m{}_{ij}C^n{}_{ml}\epsilon^{ijl}=0, \tag{37}$$

and with use of (36) this implies

$$n^{is}a_s=0, \tag{38}$$

i. e. either $a_i=0$ or else n^{is} has at least one zero eigenvalue.

¿From the transformation laws (29) and (30) it is seen that n^{ab} will transform, under the change of basis (29), like a twice contravariant tensor density

$$\tilde{n}^{i'j'}=(\det A)^{-1}\left(A^{-1}\right)^{i'}{}_i\left(A^{-1}\right)^{j'}{}_j n^{ij}, \tag{39}$$

while a_i will transform like a covariant vector

$$\tilde{a}_{i'}=A_{i'}{}^i a_i. \tag{40}$$

The transformations (39) can be used to diagonalise n^{ij}. Let us then assume that the basis $\overset{(i)}{k}$ was chosen appropriately and n^{ij} is of the form

$$n^{ij}=\begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{pmatrix}. \tag{41}$$

This does not yet fix the basis uniquely, transformations which permute (n_1, n_2, n_3) are still allowed. Therefore if any of the n_i is zero, it can always be moved by transformations of the basis to the upper left corner in (41), i. e. we can always assume that

$$n_1 = 0 \qquad \text{if} \qquad a_i \neq 0. \tag{42}$$

If $n_1 = 0$ is the single zero eigenvalue, then in view of (38) the vector a will assume, after such a choice of basis, the form

$$a_i = [a, 0, 0]. \tag{43}$$

If $n_1 = 0$ is a multiple eigenvalue, then the choice (42) still leaves us with the freedom to rotate a (by the transformations (40)) within the eigenspace of the $n_1 = 0$ eigenvalue. We can then rotate a so that it will assume the form (43). Consequently, (43) may be assumed always, also if $a_i = 0$. In such a basis, (38) reduces to

$$an_1 = 0. \tag{44}$$

Now let us insert all the information about $C^l{}_{ij}$ into the formulae for the commutators:

$$[\underset{(1)}{k}, \underset{(2)}{k}] = a \underset{(2)}{k} + n_3 \underset{(3)}{k}, \tag{45}$$

$$[\underset{(2)}{k}, \underset{(3)}{k}] = n_1 \underset{(1)}{k}, \tag{46}$$

$$[\underset{(3)}{k}, \underset{(1)}{k}] = n_2 \underset{(2)}{k} - a \underset{(3)}{k}. \tag{47}$$

This form of the commutation relations was obtained using only rotations of the basis vector fields $\underset{(i)}{k}$. From now on, no further rotations are allowed, but we may still scale $\underset{(i)}{k}$ without changing their directions, i. e.

$$\underset{(i)}{k} = C_i \underset{(i)}{k'} \qquad \text{(no sum)}. \tag{48}$$

After such a scaling, (45) – (47) change to

$$[\underset{(1)}{k}, \underset{(2)}{k}] = \frac{a}{C_1} \underset{(2)}{k} + \frac{C_3}{C_1 C_2} n_3 \underset{(3)}{k}, \tag{49}$$

$$[\underset{(2)}{k}, \underset{(3)}{k}] = \frac{C_1}{C_2 C_3} n_1 \underset{(1)}{k}, \tag{50}$$

$$[\underset{(3)}{k}, \underset{(1)}{k}] = \frac{C_2}{C_1 C_3} n_2 \underset{(2)}{k} - \frac{a}{C_1} \underset{(3)}{k} \tag{51}$$

(primes were dropped for clarity). We now want to use these scalings to make a, n_1, n_2, n_3 as simple as possible. It is seen that by choosing C_1, C_2, C_3 appropriately we can scale those of the parameters (a, n_1, n_2, n_3) that are nonzero and that a nonzero value can never be made zero by a basis transformation (within the class of bases already defined). Hence, the preliminary classification into different cases should be as shown in Table 1, where S stands for "something" (different from zero).

TABLE 1

a	0	0	0	0	0	0	0	0	S	S	S	S
n_1	0	0	0	0	S	S	S	S	0	0	0	0
n_2	0	0	S	S	0	0	S	S	0	0	S	S
n_3	0	S	0	S	0	S	0	S	0	S	0	S
CASE NUMBER	1	/	/	/	2	/	3	4	5	6	/	7

However, not all the entries in Table 1 must really be considered separately. Permutations of the basis vectors that do not result in violating (44) are still allowed, they permute n_1, n_2 and n_3. Clearly, any such permutation is allowed when $a = 0$, and with $a \neq 0$ we are still allowed to permute n_2 and n_3. Hence, only the cases indicated in the last line of the table have a chance to be inequivalent.

We will see, however, that the classification of the algebras into inequivalent types does not match the columns of Table 1 – the table will only serve to make the presentation orderly. The classification was done for the first time by L. Bianchi in 1898 [6]. He assigned the Roman numbers I to IX to different cases and called them types. By tradition, his numbering is still in use, although it does not seem natural in the more modern derivation presented below. Bianchi himself ordered the types by the properties of the derived algebras of the full 3-dimensional algebra. The derivation by his method is lengthy and complicated.

Let us consider the consecutive columns of Table 1.

(1) $a = n_1 = n_2 = n_3 = 0$.

This is Bianchi type I where all commutators are zero.

(2) $a = n_2 = n_3 = 0$, $n_1 \neq 0$.

Taking $C_1 = C_2C_3/n_1$ we obtain $n_1' = 1$. This is Bianchi type II.

(3) $a = n_3 = 0$, $n_1 \neq 0 \neq n_2$.

Taking $C_1 = C_2C_3/n_1$ we obtain $n_1' = 1$. However, as seen from (49) – (51), we have then $n_2' = n_1n_2/{C_3}^2$ and by choosing C_3 we will not be able to change the sign of n_2'. Therefore we must consider two cases separately:

(3a) $n_1n_2 > 0$.

Then we take $C_3 = (n_1n_2)^{1/2}$ and obtain $n_2' = 1$. This is a subcase of Bianchi's type VII. Bianchi himself called it type VII_1, today it is called type VII_0.

(3b) $n_1n_2 < 0$.

Then we take $C_3 = (-n_1n_2)^{1/2}$ and obtain $n_2' = -1$. This is a subcase of Bianchi's type VI, called today VI_0. Bianchi noted that the derivation of commutation relations for this case requires a separate treatment, but the final result fits well within the general type VI, so he did not give it any special name.

(4) $a = 0$, $n_1n_2n_3 \neq 0$.

We take $C_1 = C_2C_3/n_1$ and obtain $n_1' = 1$. But then, as before, $n_2' = n_1n_2/{C_3}^2$, and the two possible signs of n_1n_2 have to be considered separately.

(4a) $n_1 n_2 > 0$.

We take $C_3 = (n_1 n_2)^{1/2}$ and obtain $n_2' = 1$. However, $n_3' = n_1 n_3 / {C_2}^2$, and two further subcases arise:

($4a_1$) $n_1 n_3 > 0$.

Then we take $C_2 = (n_1 n_3)^{1/2}$ and obtain $n_3' = 1$. Thus finally $n_1' = n_2' = n_3' = 1$, $a = 0$. This is Bianchi type IX.

($4a_2$) $n_1 n_3 < 0$.

Then we take $C_2 = (-n_1 n_3)^{1/2}$ and obtain $n_3' = -1$. Hence finally $n_1' = n_2' = 1 = -n_3'$. This is Bianchi type VIII.

(4b) $n_1 n_2 < 0$.

We then take $C_3 = (-n_1 n_2)^{1/2}$ obtaining $n_2' = -1$. But then we must again consider the same two subcases as before for C_2:

($4b_1$) $n_1 n_3 > 0$.

Then we take $C_2 = (n_1 n_3)^{1/2}$ and obtain $n_3' = 1$. Through the basis change (still allowed!)

$$\underset{(3)}{k} = \underset{(2)}{\widetilde{k}}, \qquad \underset{(2)}{k} = -\underset{(3)}{\widetilde{k}}$$

we then obtain the same parameter values as in case ($4a_2$).

($4b_2$) $n_1 n_3 < 0$.

Then we take $C_2 = (-n_1 n_3)^{1/2}$ obtaining $n_3' = -1$. Again through the basis change:

$$\underset{(1)}{k} = -\underset{(3)}{\widetilde{k}}, \qquad \underset{(3)}{k} = -\underset{(2)}{\widetilde{k}}, \qquad \underset{(2)}{k} = -\underset{(12)}{\widetilde{k}}$$

we arrive back at the case ($4a_2$).

(5) $a \neq 0$, $n_1 = n_2 = n_3 = 0$.

Taking $C_1 = a$ we obtain $a' = 1$. This is Bianchi type V.

(6) $a \neq 0 \neq n_3$, $n_1 = n_2 = 0$.

We take $a = C_1$, $C_3 = aC_2/n_3$ and obtain $a' = 1 = n_3'$. This is Bianchi type IV.

(7) $an_2 n_3 \neq 0$, $n_1 = 0$.

Taking $C_2 = C_1 C_3 / n_2$ we obtain $n_2' = 1$. We lose again the possibility to change the sign of $n_3' = n_2 n_3 / {C_1}^2$ and must consider two cases separately:

(7a) $n_2 n_3 > 0$.

Then we take $C_1 = (n_2 n_3)^{1/2}$ and obtain $n_3' = 1$. Since however we fix C_1 in this way, we have no possibility left to scale a. Hence, with $n_3' = 1$, the value of a remains arbitrary. Note that then the algebras corresponding to $n_2 = n_3 = 1$ but different values of a **are not** equivalent. Bianchi called this case type VII_2, but, as opposed to previous types, this is a one-parameter family of inequivalent types. With $a = 0$, an algebra equivalent to the one obtained in (3a) results. This is seen if, with $a = 0$, we perform the basis change

$$\underset{(1)}{k} = \underset{(3)}{\widetilde{k}}, \qquad \underset{(3)}{k} = -\underset{(1)}{\widetilde{k}}$$

The general type VII, with $a \neq 0$, is denoted today VII_h.

(7b) $n_2 n_3 < 0$.

Then we take $C_1 = (-n_2 n_3)^{1/2}$ and obtain $n_3' 1 = -1$. Just as before, we are left then with no possibility to change a. Again, this is a one-parameter family of inequivalent types. Bianchi denoted it VI, today it is denoted VI_h. With $a = 0$, the algebra obtained in (3b) results, i. e. type VI_0. To see this, the same basis change as in (7a) is necessary.

In Bianchi's scheme the subcase of type VI corresponding to $a = 1$ emerged as a separate type which he called type III. In it:

$$a = n_2 = 1 = -n_3, \qquad n_1 = 0.$$

The final classification of 3-dimensional algebras into inequivalent cases is shown in Table 2.

TABLE 2

a	0	0	0	0	0	0	1	1	a	a	1
n_1	0	1	1	1	1	1	0	0	0	0	0
n_2	0	0	1	-1	1	1	0	0	1	1	1
n_3	0	0	0	0	1	-1	0	1	1	-1	-1
BIANCHI TYPE	I	II	VII_0	VI_0	IX	VIII	V	IV	VII_h	VI_h	III

9. Dimension of the group vs. dimension of the orbit.

In. sec. 2 we defined the orbits of 1-parameter families of mappings that were 1-dimensional. At this point it should be clear for the reader that the notion of an orbit can be defined as well for a family (or group) of mappings that has several parameters; such orbits can be multidimensional curved spaces.

Let us recall that the Bianchi classification was done without any reference to the way in which the generators act on the manifold M or on the fields defined on M. Two situations are possible:

1. The 3 generators (which are by definition linearly independent as **vector fields**) may also be linearly independent as **vectors** at each single point of M. This happens e.g. for generators of translations in $\mathbb{R}^3$.

2. The 3 generators may be at each single point of M linearly dependent. This happens e.g. for the generators of the group $O(3)$ which is a 3-dimensional group, but has 2-dimensional orbits.

It is rather easy to show that a 3-dimensional invariance group of a tensor cannot have 1-dimensional orbits (from the invariance equations, if the orbits are 1-dimensional, then any two generators will be proportional to each other with a constant factor).

If the orbits are 2-dimensional, then their curvature is characterized by just 1 scalar which must be invariant under the action of the group, i. e. constant. The constant

curvature may then be either positive (the orbits are then 2-dimensional spheres) or zero (the orbits are then 2-planes) or negative (such a surface has the metric $ds^2 = (1 + a^2r^2)^{-1}dr^2 + r^2d\phi^2$, but cannot be embedded in the $\mathbb{R}^3$ with a positive-definite Euclidean metric). It may then be verified by direct calculation that the 3 respective algebras of generators are of Bianchi types IX, VII_0 and VIII respectively. Only these Bianchi types are possible for groups which have 2-dimensional orbits.

If a 3-dimensional group has 3-dimensional orbits, then the scalar curvature of the orbits must also be constant. However, in 3 dimensions the scalar curvature does not fully characterize the curvature of space, there exists also the Ricci tensor. Therefore more geometries are possible, they will be briefly discussed in the next section.

10. The spatially homogeneous Bianchi-type spacetimes.

If a 3-dimensional group of mappings of M into itself has 3-dimensional orbits, then several situations are possible, e. g.:

1. The mappings may or may not be symmetries of M (they could be for example conformal symmetries of M).

2. The orbits might be timelike, spacelike or even null hypersurfaces in M.

In each situation one could consider the various Bianchi types. Examples of solutions of Einstein's equations are known where the orbits are timelike (e. g. [9] for Bianchi type I), and a systematical investigation of all the spacetimes with 3-dimensional timelike orbits of the Bianchi groups was done by Harness [19].[3] Examples are also known where 3 dimensional groups act as groups of symmetries on certain preferred 3-dimensional submanifolds of M, but are not symmetries of the whole M (these are the so called spacetimes with intrinsic symmetries, the term was introduced by Collins [10], for examples see [11] and [12]). They have not been systematically investigated, even though some of them do seem to be interesting as models of our Universe. A brief discussion of general properties of spacetimes where a 3-dimensional symmetry group has 3-dimensional null orbits is given in [13], but no examples of solutions of Einstein's equations are given.

Most effort went into investigating the case when the 3-dimensional orbits are spacelike and the group in question is a group of symmetries of M (but examples are known where the orbits are spacelike in one part of M and timelike elsewhere, see [14] and [15]). Before we consider this case in more detail, a few definitions must be given.

Let $\mathrm{Orb}(P, G)$ denote the G-orbit of the point $P \in M$. The group G **acts transitively** on a manifold S when for every $Q \in S$, $\mathrm{Orb}(Q, G) = S$ (in further considerations S will be a submanifold of M). *Examples* : the group $O(3)$ acts transitively on the surface of a sphere, the group of arbitrary translations in $\mathbb{R}^n$ acts transitively on $\mathbb{R}^n$. The space S on which G acts transitively is called **homogeneous with respect to** G.

If, for every $Q \in S$ (assumed homogeneous with respect to G) there exists a subgroup $H \subset G$ such that $HQ = Q$, then G is said to act **multiply transitively** on S. A group acting transitively, but not multiply, acts **simply transitively**. *Examples* : The

[3]Long after these notes were first written up, several more examples of non-spacelike orbits appeared, see the series of papers by this author [23, 24]. In those papers, rotating dust models with 3-dimensional transitive symmetry groups were considered; in several of them the orbits are timelike or null.

group of arbitrary translations in $\mathbb{R}^n$ acts simply transitively on $\mathbb{R}^n$; the group $O(3)$ acts multiply transitively on a 2-sphere since each point p of the sphere remains unchanged by all rotations around the axis which passes through p.

A 4-dimensional manifold M with the metric tensor g of signature $(+---)$ is called a **spatially homogeneous spacetime of Bianchi type** if it has the following properties:

1. It has a 3-dimensional symmetry group G.

2. The orbits of the group are spacelike hypersurfaces in M on which G acts simply transitively.

Such spacetimes were often considered as cosmological models of "second approximation" to reality, the first approximation being the Friedman-Lemaitre-Robertson-Walker (FLRW) models. The Bianchi type models are homogeneous (i. e. invariant with respect to shifting along the orbits of G) , but not isotropic, i. e. not invariant with respect to rotations. It must be said, however, that this "second approximation" does not go very far because in such a spacetime all scalars that can be calculated from the metric tensor via the field equations inherit the symmetries of the metric and can depend only on time (see sec. 5), just like in the FLRW models.

It is important to remember that a Bianchi type spacetime may have a larger symmetry group, the definition only requires then that the full symmetry group has a 3-dimensional subgroup which acts simply transitively on certain spacelike hypersurfaces.

11. Invariant vector fields.

According to the definition in sec. 5, the vector field X is invariant under the group of transformations generated by the vector field k if

$$\underset{k}{\pounds} X \equiv [k, X] = 0. \tag{52}$$

Now let X be a vector field on S, let $\underset{(i)}{k}$, $i = 1, \ldots, n$ be a set of generators of invariances of X such that at every point $P \in S$, $\left\{ \underset{(i)}{k}(P) \right\}_{i=1}^{n}$ is a basis in the tangent space to S at P (i. e. the orbits of the group generated by $\underset{(i)}{k}$ are n-dimensional, and so is S). In that case the matrix K where $K_i{}^{\alpha} = \underset{(i)}{k}{}^{\alpha}(P)$ is nonsingular at every $P \in S$, and so defines the inverse matrix κ. Let us write (52) for all the fields $\underset{(i)}{k}$:

$$\underset{(i)}{k}{}^{\rho} X^{\alpha}{}_{,\rho} - X^{\rho} \underset{(i)}{k}{}^{\alpha}{}_{,\rho} = 0 \tag{53}$$

and multiply this set of equations by the matrix κ. The result will be

$$X^{\alpha}{}_{,\beta} - \kappa^{i}{}_{\beta} \underset{(i)}{k}{}^{\alpha}{}_{,\rho} X^{\rho} = 0 \qquad \text{(sum over } i\text{)}. \tag{54}$$

This formula is similar to the condition of covariant constancy of a vector field, where the quantity

$$G^{\alpha}{}_{\beta\gamma} \overset{\text{def}}{=} - \kappa^{i}{}_{\beta} \underset{(i)}{k}{}^{\alpha}{}_{,\gamma} \qquad \text{(sum over } i\text{)} \tag{55}$$

plays the role of the affine connection. Let us follow this analogy and calculate the curvature tensor defined by $G^{\alpha}{}_{\beta\gamma}$. It is a simple exercise to show that such a curvature tensor is equal to zero. Therefore eq. (54) defines a transport of the vector field X along the vector fields $\underset{(i)}{k}$ that is formally analogous to the parallel transport on S except that the transport (54) has vanishing curvature, and so will be path-independent. Consequently, if $X(P)$ is defined at any single point $P \in S$, then (54) will uniquely define X at any other point of S.

We can then define a vector **basis** $\underset{(i)}{X}(P)$ at $P \in S, i = 1, \ldots, n$, and use eq. (54) (called the **Lie transport**) to transplant the basis to all other point of S. The vector fields $\underset{(i)}{X}$ on S thus obtained will be automatically invariant with respect to the transformations generated by the fields $\underset{(i)}{k}$. We have thus proven the

Theorem 1 *If a set of vector fields $\underset{(i)}{k}$ on S exists such that $\underset{(i)}{k}(P)$ form a basis of the tangent space to S at each $P \in S$, then there exists a set of vector fields $\underset{(i)}{X}$ on S that are invariant under the group of transformations generated by $\underset{(i)}{k}$ and also form a basis at each $P \in S$.*

Now let us assume in addition that all the fields $\underset{(i)}{k}$ are Killing fields, and let us calculate the Lie derivatives along $\underset{(i)}{k}$ of the quantities

$$g_{(j)(k)} \overset{\text{def}}{=} \underset{(j)}{X}{}^{\alpha} \underset{(k)}{X}{}^{\beta} g_{\alpha\beta}, \tag{56}$$

where $g_{\alpha\beta}$ is the metric tensor on S. Since $\underset{(i)}{X}$ are invariant, we have $\underset{\underset{(i)}{k}}{\pounds} \underset{(j)}{X} = 0$, and since $\underset{(i)}{k}$ are now Killing fields, we have $\underset{\underset{(i)}{k}}{\pounds} g_{\alpha\beta} = 0$, and so $\underset{\underset{(i)}{k}}{\pounds} g_{(j)(k)} = 0$. But $g_{(j)(k)}$ are scalars, so from sec. 5 it follows that

$$0 = \underset{\underset{(i)}{k}}{\pounds} g_{(j)(k)} = \underset{(i)}{k}{}^{\rho} g_{(j)(k),\rho} \tag{57}$$

and since $\underset{(i)}{k}$ form a basis, this means that $g_{(j)(k)}$ are constants.

If $\underset{(i)}{X}$ and $\underset{(j)}{X}$ are invariant, then so is $[\underset{(i)}{X}, \underset{(j)}{X}]$, this can be verified. Since we showed that $\underset{(i)}{X}(P)$ form a basis at every $P \in S$, it follows that $[\underset{(i)}{X}, \underset{(j)}{X}]$ can be decomposed in this basis,

$$[\underset{(i)}{X}, \underset{(j)}{X}] = D^{l}{}_{ij} \underset{(l)}{X}, \tag{58}$$

where the coefficients $D^{l}{}_{ij}$ (which are scalars) could be expected to depend on the point P. However, calculating the Lie derivative of (58) along $\underset{(i)}{k}$, using the invariance of $[\underset{(i)}{X}, \underset{(j)}{X}]$ and of $\underset{(l)}{X}$, and using the fact that $\underset{(l)}{X}$ form a basis at each $P \in S$ we conclude that $D^{l}{}_{ij}$ are in fact constants.

Let us recall now that the basis $\underset{(i)}{X}$ was defined by choosing $\underset{(i)}{X}(P_0)$ arbitrarily at a certain $P_0 \in S$ and transporting it off P_0 by (54). Let us assume then that at P_0 we have

$$\underset{(i)}{X}^{\alpha}(P_0) = \underset{(i)}{k}^{\alpha}(P_0). \tag{59}$$

In this case it can be proven that

$$D^{l}{}_{ij} = -C^{l}{}_{ij}, \tag{60}$$

where $C^{l}{}_{ij}$ are the structure constants defined by the commutators of $\underset{(i)}{k}$. Hint for the proof: since both $\underset{(i)}{k}$ and $\underset{(i)}{X}$ are bases at every $P \in S$ and $\underset{(i)}{k} = \underset{(i)}{X}$ at P_0, then there exists a matrix $M(\{x\})$ (point-dependent!) such that $\underset{(i)}{X}^{\alpha} = M_i{}^{j}\underset{(j)}{k}^{\alpha}$ (sum over j) and $M_i{}^{j}(P_0) = \delta_i{}^{j}$. Knowing this, play with the commutators at P_0.

12. The metrics of the Bianchi-type spacetimes.

Let $\underset{(i)}{k}$ be the 3 Killing fields required by the definition, and let x^I be the coordinates in the homogeneous hypersurfaces $(i, I = 1, 2, 3)$.

The homogeneous hypersurfaces are tangent to $\underset{(i)}{k}$ (by definition) and uniquely define the vector field m orthogonal to them,

$$m^{\alpha} = |g|^{-1/2}\epsilon^{\alpha\beta\gamma\delta}\underset{(1)}{k}_{\beta}\underset{(2)}{k}_{\gamma}\underset{(3)}{k}_{\delta} \tag{61}$$

where $g = \det(g_{\alpha\beta})$. Let us choose the parameter on the integral lines of m as the t-coordinate in M:

$$m^{\alpha} = \frac{\partial x^{\alpha}}{\partial t}. \tag{62}$$

Since m is orthogonal to all $\underset{(i)}{k}$, in such coordinates we have

$$g_{0I} = 0 \tag{63}$$

(because $\underset{(i)}{k}^{0} = 0$ and the $\underset{(i)}{k}$ are linearly independent). From the (00) component of the Killing equations, using (63) and $\underset{(i)}{k}^{0} = 0$, we further obtain $\underset{(i)}{k}^{I}g_{00,I} = 0$, i. e. $g_{00} = g_{00}(t)$. By the next coordinate transformation, $t' = \int [g_{00}(t)]^{1/2}\,\mathrm{d}t$, we obtain (dropping the prime)

$$\mathrm{d}s^2 = \mathrm{d}t^2 - g_{IJ}\mathrm{d}x^I\mathrm{d}x^J. \tag{64}$$

¿From the other Killing equations we then conclude that $\underset{(i)}{k}^{I}{}_{,0} = 0$ and that $\underset{(i)}{k}$ are Killing fields also for the 3-metric with components g_{IJ}, not only for the 4-metric $g_{\alpha\beta}$. Since for the homogeneous hypersurfaces with the 3-metric g_{IJ} the fields $\underset{(i)}{k}$ form a basis at each point, we can apply here the results of section 11 to each single hypersurface. Here, however, eq. (57) will read

$$\underset{(i)}{k}^{R}g_{(j)(k),R} = 0, \tag{65}$$

but $\frac{\partial}{\partial t}g_{(j)(k)}$ is not determined. Consequently, in the Bianchi type spacetimes the $g_{(j)(k)}$ will be functions of t. Let us denote by $\omega^{(i)}{}_I$ the matrix inverse to $\underset{(i)}{X}{}^I$ (where $\underset{(i)}{X}$ are the vector fields invariant with respect to the transformations generated by $\underset{(i)}{k}$), then

$$g_{IJ} = g_{(j)(k)}(t)\omega^{(j)}{}_I\omega^{(k)}{}_J \tag{66}$$

and

$$ds^2 = dt^2 - g_{(j)(k)}(t)\omega^{(j)}{}_I\omega^{(k)}{}_J\mathrm{d}x^I\mathrm{d}x^J. \tag{67}$$

In order to find $\omega^{(i)}{}_I$ one has to construct the Killing fields and their invariant fields for each Bianchi type separately. The resulting formulae can be found e. g. in [16].

13. The isotropic Bianchi type spacetimes.

A model of the Universe which is simultaneously spatially homogeneous and isotropic must belong to the FLRW class. We now wish to investigate the relation of the FLRW models to the Bianchi type models. The calculations underlying this investigation are simple but lengthy, so we will omit them as far as possible and present only the results.

A spherically symmetric spacetime is shown by a standard textbook argument to have the following metric form

$$\mathrm{d}s^2 = \alpha(t,r)\mathrm{d}t^2 + 2\beta(t,r)\mathrm{d}t\mathrm{d}r + \gamma(t,r)\mathrm{d}r^2 + \delta(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2\right), \tag{68}$$

where α, β, γ and δ are arbitrary functions of t and r. This form of the metric is preserved by the transformations $t = f(t',r')$, $r = g(t',r')$ which can be used to simplify (68) further. In order to transform (68) into the form (67) we can choose f and g so that the new $\beta = 0$. In the new form, only such transformations will be permissible which preserve the hypersurfaces $t = \text{const}$ (they are determined geometrically by the symmetry groups existing in the Bianchi models). The transformations are $t = f(t')$, $r = g(r')$. In agreement with (64), α should then depend only on t and be transformable to 1. So finally, a spherically symmetric metric can possibly be homogeneous in the Bianchi sense only if it can be put in the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \gamma(t,r)\mathrm{d}r^2 + \delta(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2\right). \tag{69}$$

With such a metric one should now solve the Killing equations. This is the lengthy procedure that we shall omit. In the process of integrating the equations, several alternatives show up, each one connected with a certain possible property of the functions γ and δ. In the generic case, when $\gamma(t,r)$ and $\delta(t,r)$ are quite arbitrary, no additional symmetries exist. Among the special cases will be the spherically symmetric Kantowski–Sachs spacetime [17] in which the symmetry group is 4-dimensional and acts on the homogeneous hypersurfaces multiply transitively (the other, hyperbolically symmetric Kantowski–Sachs spacetime **has** a simply transitive subgroup, contrary to what [17] says, but this case does not show up here). Other alternatives are e.g. the de Sitter spacetime (with a symmetry group of 10 parameters) and static spacetimes (with 4 parameters).

Some properties of the alternative that is of interest to us can be guessed in advance. The spacetime with the metric (69) has $O(3)$ as a symmetry group. Since $O(3)$ has 2-dimensional orbits, it cannot be the homogeneity group existing in the Bianchi type spacetimes. Hence, the full symmetry group in a homogeneous and isotropic spacetime must contain $O(3)$ and the homogeneity group H as two subgroups. $O(3)$ cannot be a subgroup of H because it acts multiply transitively. $O(3)$ and H cannot have any common 2 dimensional subgroup because $O(3)$ has no 2-dimensional subgroups at all. All 1-dimensional subgroups of $O(3)$ are rotations around an axis which act multiply transitively and so cannot be subgroups of H . The conclusion is, $O(3)$ and H cannot have any common subgroup apart from the identity transformation. Hence, the full symmetry group will have **at least** 6 parameters, 3 of them connected with $O(3)$ and 3 with H. On the other hand, we showed in sec. 3 that an n-dimensional manifold can have a symmetry group of **at most** $\frac{1}{2}n(n+1)$ parameters, i. e. at most 6 when $n = 3$. Consequently, the spatially homogeneous and isotropic spacetimes have symmetry groups with exactly 6 parameters.

After the Killing equations for (69) are solved and the relevant case is chosen from among the solutions, the metric assumes the familiar form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\left(1+\frac{1}{4}kr^2\right)^{-2}\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2\right)\right], \tag{70}$$

where $R(t)$ is an arbitrary function and k is an arbitrary constant which, if nonzero, can be scaled to $+1$ or -1 by the transformations of r (eq. (70) is obtained with use of the transformations $r = g(r')$, the remaining freedom is $g = Cr'$, $C = \mathrm{const}$). The resulting basis of the Killing fields is

$$\begin{aligned}
\underset{(1)}{k} &= -V\cos\vartheta\frac{\partial}{\partial r} + W\sin\vartheta\frac{\partial}{\partial\vartheta},\\
\underset{(2)}{k} &= V\sin\vartheta\cos\phi\frac{\partial}{\partial r} + W\cos\vartheta\cos\phi\frac{\partial}{\partial\vartheta} - W\frac{\sin\phi}{\sin\vartheta}\frac{\partial}{\partial\phi},\\
\underset{(3)}{k} &= V\sin\vartheta\sin\phi\frac{\partial}{\partial r} + W\cos\vartheta\sin\phi\frac{\partial}{\partial\vartheta} + W\frac{\cos\phi}{\sin\vartheta}\frac{\partial}{\partial\phi},\\
\underset{(4)}{k} &= \cos\phi\frac{\partial}{\partial\vartheta} - \cot\vartheta\sin\phi\frac{\partial}{\partial\phi},\\
\underset{(5)}{k} &= \sin\phi\frac{\partial}{\partial\vartheta} + \cot\vartheta\cos\phi\frac{\partial}{\partial\phi},\\
\underset{(6)}{k} &= -\frac{\partial}{\partial\phi},
\end{aligned} \tag{71}$$

where

$$V(r)\overset{\mathrm{def}}{=}1+\frac{1}{4}kr^2, \qquad W(r)\overset{\mathrm{def}}{=}\frac{1}{r}-\frac{1}{4}kr. \tag{72}$$

The last 3 generators are easily recognized as those of $O(3)$. For writing out the 15 commutators of the 6 generators we shall use the abbreviation $[i,j]$ for $[\underset{(i)}{k},\underset{(j)}{k}]$. The commutators are

$$
\begin{array}{lll}
[1,2] = k\,\underset{(4)}{k} & & \\
[1,3] = k\,\underset{(5)}{k} & [2,3] = k\,\underset{(6)}{k} & \\
[1,4] = -\underset{(2)}{k} & [2,4] = \underset{(1)}{k} & [3,4] = 0 \\
[1,5] = -\underset{(3)}{k} & [2,5] = 0 & [3,5] = \underset{(1)}{k} \\
[1,6] = 0 & [2,6] = -\underset{(3)}{k} & [3,6] = \underset{(2)}{k} \\
[4,5] = \underset{(6)}{k} & & \\
[4,6] = -\underset{(5)}{k} & [5,6] = \underset{(4)}{k}\,. &
\end{array}
\tag{73}
$$

The last 3 commutators are those of the algebra of $O(3)$. In (73) such relations should be found which correspond to some of the Bianchi algebras. The task is more complicated than it seems, since the Bianchi classification introduced a certain basis, and there is no guarantee that (71) contains any Bianchi sub-basis in its standard form. Indeed, some of the Bianchi algebras are linear combinations of the vectors (71).

For $k = +1$ (the "closed" FLRW model), the Bianchi basis is

$$
\begin{aligned}
\underset{(1)}{L} &= \tfrac{1}{2}\left(\underset{(1)}{k} + \underset{(6)}{k}\right), \\
\underset{(2)}{L} &= \tfrac{1}{2}\left(\underset{(2)}{k} - \underset{(5)}{k}\right), \\
\underset{(3)}{L} &= \tfrac{1}{2}\left(\underset{(3)}{k} + \underset{(4)}{k}\right).
\end{aligned}
\tag{74}
$$

This is an algebra of Bianchi type IX.

For $k = -1$ (the "open" FLRW model) two Bianchi algebras can be found in (73). They are

$$
\begin{aligned}
\underset{(1)}{L} &= -\underset{(1)}{k}, \\
\underset{(2)}{L} &= \underset{(2)}{k} + \underset{(4)}{k}, \\
\underset{(3)}{L} &= \underset{(3)}{k} + \underset{(5)}{k},
\end{aligned}
\tag{75}
$$

which is of Bianchi type V, and

$$
\begin{aligned}
\underset{(1)}{l} &= -a\,\underset{(1)}{k} + \underset{(6)}{k}, \\
\underset{(2)}{l} &= \underset{(2)}{k} + \underset{(4)}{k}, \\
\underset{(3)}{l} &= \underset{(3)}{k} + \underset{(5)}{k}.
\end{aligned}
\tag{76}
$$

which is of Bianchi type VII_h.

Also for $k = 0$ two Bianchi algebras are found in (73), in their standard bases. One of them is the Bianchi type I algebra consisting of the vectors $\underset{(1)}{k}$, $\underset{(2)}{k}$ and $\underset{(3)}{k}$ (this is easy to see, since all their commutators simply vanish), the other is the Bianchi type VII_0 algebra consisting of the vectors $\underset{(1)}{k}$, $\underset{(2)}{k}$ and $\underset{(4)}{k}$.

It is easy to verify that the examples of bases shown are indeed the standard Bianchi bases of the types indicated. It is, however, more difficult to prove that only these Bianchi algebras are contained in (71), this fact was apparently first discovered by L. P. Grishchuk [18]. The relation between the symmetries of (70) and the possible Bianchi groups is discussed in more detail in [7].

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FIGURE CAPTIONS

Fig. 1. There exists a 1–1 correspondence between points of an open subset of the manifold M and points of the coordinate chart $C \subset \mathbb{R}^n$. Usually, the mappings between M and $\mathbb{R}^n$ develop singularities which cause that different neighbourhoods of M require separate coordinate charts. The operations considered in sec. 1 take place within a single coordinate chart.

Fig. 2. A mapping F of M into itself, accompanied by a mapping of C into itself. The thick line encircles the area of M which is depicted in Fig. 3, as viewed from the left.

Fig. 3. The mapping F sends the point $P \in M$ and the tensor $T(P)$ to $P' \in M$, where $T(P)$ changes into $T'(P')$. On C, the coordinates x'^{α} of P' are functions of the coordinates x^{α} of P; the set of functions $x'^{\alpha} = F^{\alpha}(\{x\})$ represents F. The components $T'_{\alpha\beta}(P')$ are calculated in terms of $T_{\alpha\beta}(P)$ according to eq. (2).

Fig. 5. Take a few points of M (here P_1, P_2 and P_3) and apply to each all the mappings of a 1-parameter family Γ (see text). For each point P_i, the collection of all its images will be an arc of a curve in M called the orbit of P_i under the action of Γ.

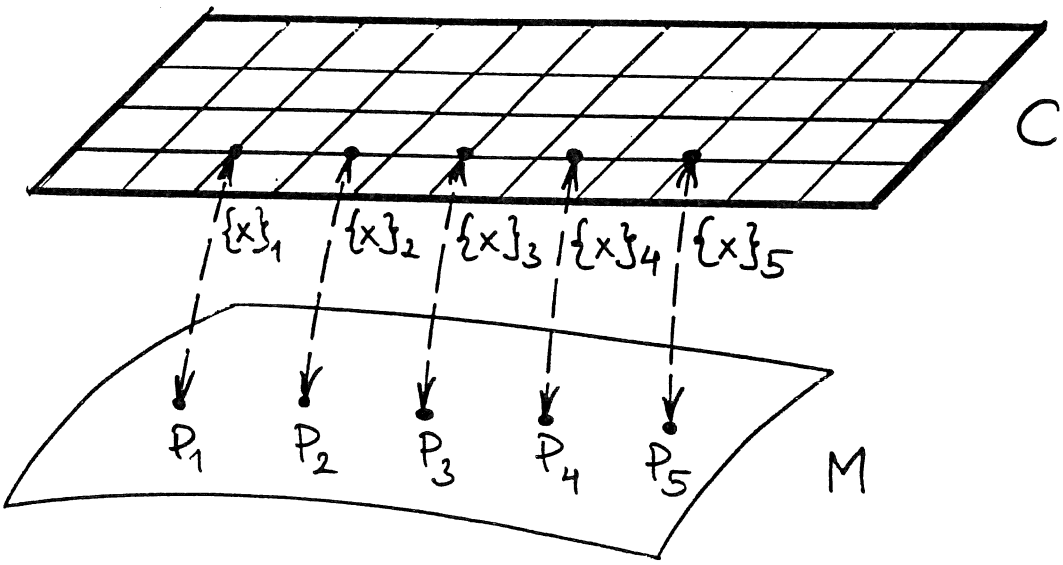
C
{x}1
{x}2
{x}3
{x}4
{x}5
P1
P2
P3
P4
P5
M

Fig. 1

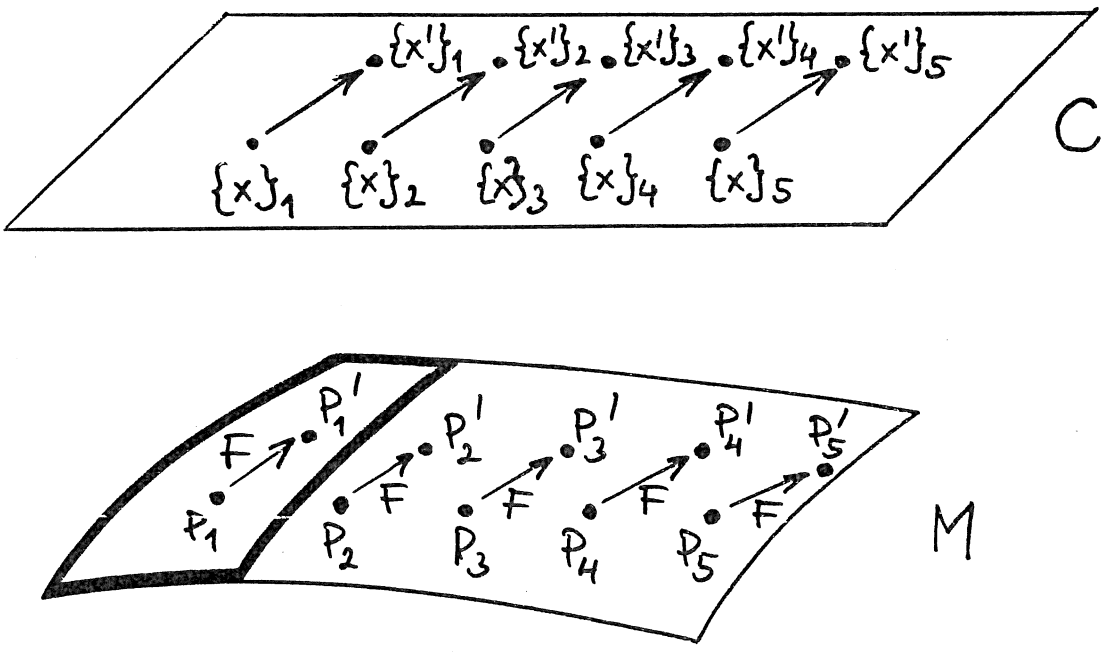
{x'}1
{x'}2
{x'}3
{x'}4
{x'}5
C
{x}1
{x}2
{x}3
{x}4
{x}5
P1'
P2'
P3'
P4'
P5'
F
F
F
F
F
P1
P2
P3
P4
P5
M

Fig. 2

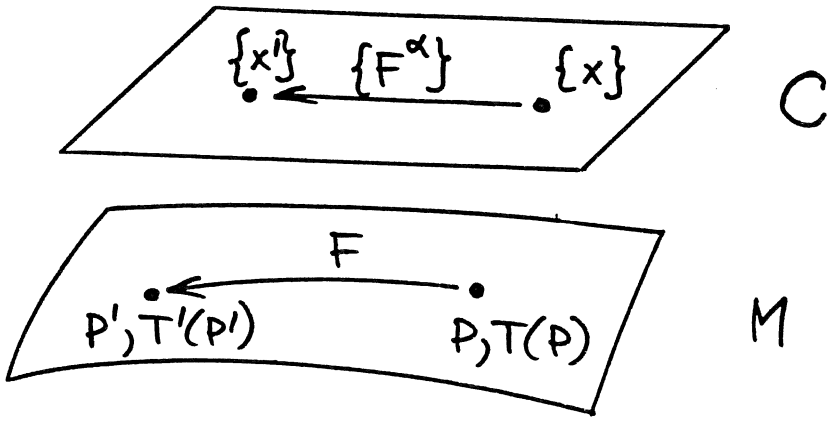

Fig. 3

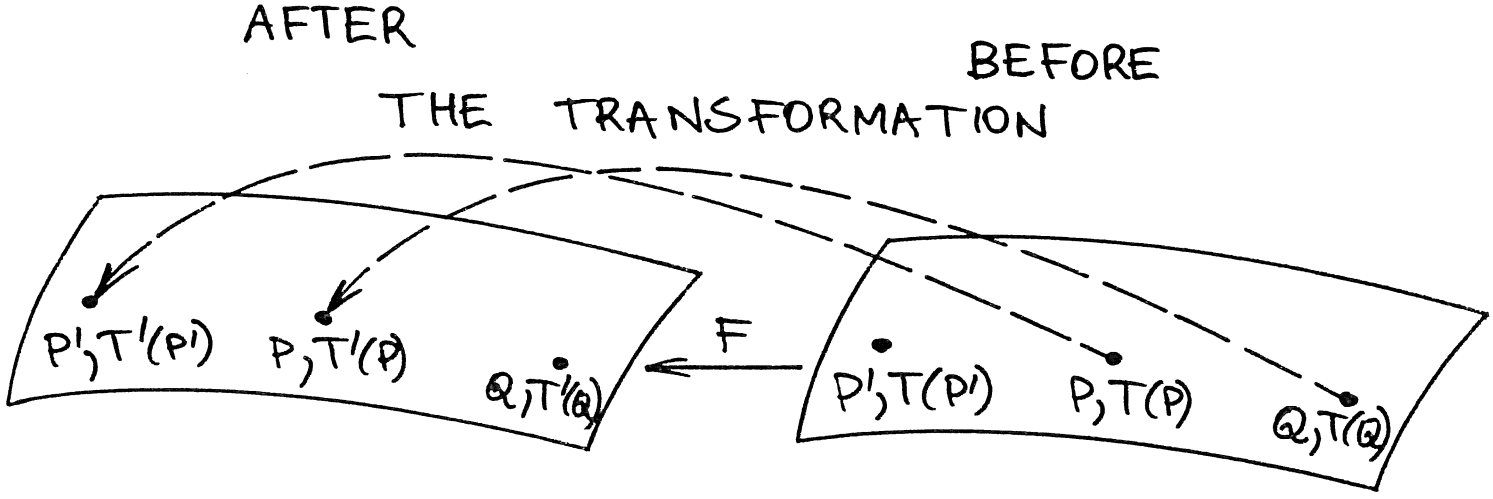

Fig. 4

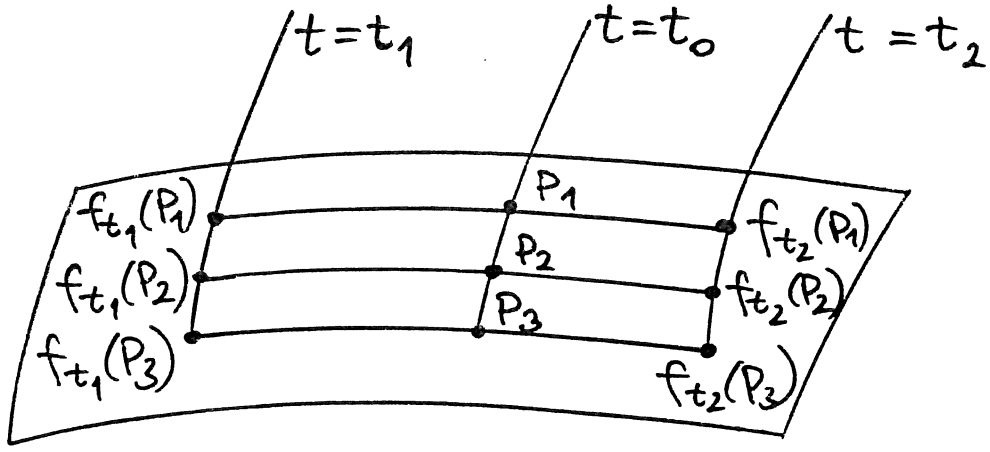

Fig. 5