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Space-Times with Spherically Symmetric Hypersurfaces

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Abstract

When discussing spherically symmetric gravitational fields one usually assumes that the whole space-time is invariant under the $O(3)$ group of transformations. In this paper, the Einstein field equations are investigated under the weaker assumption that only the 3-spaces t = const are $O(3)$ symmetric. The following further assumptions are made: (1) The t lines are orthogonal to the spaces t = const. (2) The source in the field equations in a perfect fluid, or dust, or the Λ term, or the empty space. (3) With respect to the center of symmetry the fluid source may move only radially if at all. Under these assumptions one solution with a perfect fluid source, found previously by Stephani, is recovered and interpreted geometrically, and it is shown that it is the sole solution which is not spherically symmetric in the traditional sense. The paper ends with a general discussion of cosmological models whose 3-spaces t = const are the same as in the Robertson-Walker models. No new solutions were explicitly found, but it is shown that such models exist in which the sign of curvature is not fixed in time.

§ (1): *Introduction*

When discussing spherically symmetric gravitational fields in general relativity one usually assumes that the whole of space-time is spherically symmetric, i.e., that the metric is invariant under the $O(3)$ group of transformations of coordinates. This assumption seems more restrictive than necessary because what one has in mind while doing observations is the geometry of the 3-spaces t = const whatever the choice of time is. The geometry of the space-time can then be

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recognized by indirect investigations, and it might be interesting to see what conclusions can be drawn if we assume that only each of the 3-spaces t = const is $O(3)$ symmetric while for the whole space-time it is not necessarily so.

In this paper I want to imagine a space-time as made of $O(3)$-symmetric three-dimensional spacelike hypersurfaces strung onto a timelike line orthogonal to them all, and to investigate the properties of such a space-time if the Einstein field equations are fulfilled. We shall also assume that the velocity field of matter, u^α (if any matter is present), has only the u^t and u^r components, i.e., moves only radially if at all with respect to the line joining the centers of symmetry of the 3-spaces. This assumption is justified by the fact that transversal motions of matter could be easily revealed by any observer, and would thus constitute a too-obvious evidence for the lack of ordinary spherical symmetry in the space-time. However, one could go on without the assumption of purely radial motions and see what follows. This problem still awaits investigation.

The most general source in the field equations considered here will be a perfect fluid, whose special cases (in the mathematical sense) are dustlike matter (pressure = 0), the Λ term ("pressure" = const $\stackrel{\text{def}}{=} \Lambda \neq 0$, "energy density" = $-\Lambda$), and the pure empty space (pressure = energy density = 0).

In the case of the Λ term and pure empty space no nonspherical solutions were found, i.e., the space-times considered here are forced by the field equations to be spherically symmetric in the well-known traditional sense. However, in the case of a nontrivial perfect fluid one solution is found which is not spherically symmetric as a space-time. The reason for its nonsphericity is found to be the curvature of the lines onto which the 3-spaces are strung (see Section 5). It is the solution found in 1967 by Stephani [1].

The physical difference between the space-time which is $O(3)$ symmetric as a whole and one that has only $O(3)$-symmetric hypersurfaces can be described in the following way. If we are given the 3-spaces t = const without any device to measure the time in different points, then we can reveal only the spherical symmetry of each 3-space by purely geometrical measurements. If, in addition, we attach a clock to each point of the space, then we can say: the space-time considered here is spherically symmetric as a whole if its 3-spaces t = const are spherically symmetric and all the clocks placed on one sphere go at the same rate.[1] The solution from Section 5 fits this definition.

The plan of the paper is as follows. In Section 2 the problem is posed by writing a metric form concordant with all our assumptions. In Section 3 we discuss the case which is analogous to the solution of Nariai [2] of ordinary spherical symmetry, and in fact we only recover the Nariai solution. In Section 4, we discuss the case strictly analogous to the standard spherically symmetric space-time and we find one nontraditional solution: the one of Stephani [1]. The geometrical properties of the Stephani solution are investigated in Section 5. In

[1] This statement is due to N. Salie.

Section 6 we discuss the most general space-time, obeying all the aforementioned assumptions, and we show that Sections 3 and 4 actually exhausted the problem.

The final sections of the paper are devoted to an analogous problem with homogeneity. Here it is assumed that the 3-spaces t = const are homogeneous with respect to a three-parameter group acting transitively, but the whole space-time is not necessarily invariant with respect to this group. The problem here is considerably more complicated, so it is assumed for simplicity that the 3-spaces are spherically symmetric in addition to being homogeneous, that they are strung onto a line which is orthogonal to them all, and, as before, that matter displays no transversal motions to the distinguished observers.

One class of solutions, found previously by Stephani [1], was reobtained. Another class was also investigated, in which no new solutions were found explicitly. In both classes the geometry of each of the subspaces t = const is the same as in the Robertson-Walker (RW) metrics, but the curvature of the 3-spaces t = const is varying in time in a different way, so that it can change its sign.

The present paper is one of the possible specifications of the idea of C. B. Collins [3], who proposed to investigate space-times having subspaces with definite symmetry groups. In fact, the paper was motivated by the recent beautiful and enlightening criticism of standard cosmology by G. F. R. Ellis, expressed particularly in [4].

The calculations for this paper were carried out with use of the symbolic formula-manipulation computer system ORTOCARTAN [5, 6].

§ (2): *Definition and Statement of the Purpose*

We shall deal with such space-times, whose subspaces t = const are all spherically symmetric in the normal sense. Thus, it should be possible to choose the coordinates in the space-time so that in every space t = const the metric form is

$$ds_3{}^2 = \tilde{\gamma}(r)\, dr^2 + \tilde{\delta}(r)(d\vartheta^2 + \sin^2 \vartheta\, d\varphi^2) \tag{1}$$

where $\tilde{\gamma}(r)$ and $\tilde{\delta}(r)$ are arbitrary functions of the coordinate r [7], implicitly understood to be the values, at a fixed t, of some arbitrary functions of two variables, $\gamma^2(t, r)$ and $\delta^2(t, r)$, respectively.

The most general such space-time has the metric form

$$\begin{aligned} ds^2 = {} & D^2(t, r, \vartheta, \varphi)\, dt^2 + 2\alpha_1(t, r, \vartheta, \varphi)\, dt\, dr + 2\alpha_2(t, r, \vartheta, \varphi)\, dt\, d\vartheta \\ & + 2\alpha_3(t, r, \vartheta, \varphi)\, dt\, d\varphi - \gamma^2(t, r)\, dr^2 - \delta^2(t, r)(d\vartheta^2 + \sin^2 \vartheta\, d\varphi^2) \end{aligned} \tag{2}$$

where D, α_1, α_2, and α_3 are arbitrary functions of four variables. For simplicity, just to gain an insight into a new kind of geometry which seems not to have been considered before,[2] we shall assume throughout the paper that the spaces (1)

[2] An exhaustive search through the *Physics Abstracts* from the 1926 volume till the present volumes, conducted partially in connection with Ref. 7, did not reveal any attempt of the kind considered in the present paper.

form the space-time (2) by being strung onto a timelike congruence which is orthogonal to them all, i.e., the t lines are orthogonal to the spaces t = const. Consequently we assume that

$$\alpha_1 = \alpha_2 = \alpha_3 = 0 \tag{3}$$

We shall also denote

$$\gamma \stackrel{\text{def}}{=} e^{\mu(t,r)} \tag{4}$$

The paper will be mostly devoted to the question: What kind of a source can generate the metric (2)-(3) through the Einstein field equations if the metric is not to be spherically symmetric in the traditional sense (i.e., the function D is to depend on at least one of the variables ϑ and φ)?

The components of the Ricci and Einstein tensors, referred to in the paper, will be all with respect to the simplest orthonormal tetrad of differential forms connected with (2)-(3).

§ (3): *The Nariai-like Case*

In standard textbooks one usually chooses one of the coordinates (t, r) so that the function $\delta(t, r)$ in (2) assumes a prespecified shape. Clearly, this is impossible when δ = const, and we shall consider this case first. Thus we denote $\delta = L$ and consider the field equations for the metric

$$ds^2 = (D\,dt)^2 - (e^{\mu}dr)^2 - (L\,d\vartheta)^2 - (L \sin\vartheta\, d\varphi)^2 \tag{5}$$

If, as we assumed, the source in the field equations is no more general than a perfect fluid, then the energy-momentum tensor is of the form

$$T^{ij} = (\epsilon + p)\, u^i u^j - p g^{ij} \tag{6}$$

where indices refer to the orthonormal tetrad. With our additional assumption, that the velocity field may have only the u^t and u^r components, the Einstein field equations imply for the Ricci tensor

$$R_{03} = R_{03} = R_{12} = R_{13} = R_{23} = R_{22} - R_{33} = 0 \tag{7}$$

However, in the case of (5), $R_{01} = 0$ identically, which means that the fluid must be moving along the t lines, i.e., $u^r = 0$. The equations $R_{02} = R_{03} = 0$ say, respectively,

$$\begin{aligned} \mu_{,t} D_{,\vartheta}/LD^2 &= 0 \\ \mu_{,t} D_{,\varphi}/LD^2 \sin\vartheta &= 0 \end{aligned} \tag{8}$$

Consequently, either $D_{,\vartheta} = D_{,\varphi} = 0$, or $\mu_{,t} = 0$. The first case leads to an ordinary spherically symmetric solution. Moreover, if $D_{,\vartheta} = D_{,\varphi} = 0$, then

$R_{22} = R_{33} = L^{-2} = \text{const}, R_{00} + R_{11} = 0$, and so the source may only be the Λ term, $\Lambda = L^{-2}$. This case was shown in [7] to lead to the solution of Nariai [2], and so we can expect no new result along this line.

When we follow the other case in (8) we arrive again at the result $D = D(t, r)$ by another route. So finally we could not escape the full spherical symmetry, and we have reobtained the solution of Nariai [2] here.

§ (4): *The Schwarzschild-like Case*

From now on we shall consider only metrics (2)-(3) in which δ is not constant. In this section we shall investigate the (apparently) special case when the function δ may be chosen to be the r coordinate. In the standard spherically symmetric case the necessary and sufficient condition for this was the gradient of δ being a spacelike vector. There, however, we could employ transformations mixing t with r, i.e., $t = t(t', r'), r = r(t', r')$. Not so here: if D depends on ϑ or φ, then to preserve the form (2) with $\alpha_i = 0$ we may employ only the nonmixing transformations $t = t(t'), r = r(r')$. Consequently, δ may be chosen as the new coordinate r only when $\delta = \delta(r)$ (i.e., $\delta_{,t} = 0$) in the old coordinates.

The equations $R_{02} = R_{03} = 0$ are here nearly identical with (8) (only r replaces L) and so pose the same dilemma: either we return to the well-investigated case of ordinary spherical symmetry (see, e.g. [8][3]), or $\mu_{,t} = 0$. We shall consider only the second case. Since, however,

$$R_{01} = 2\mu_{,t}/rDe^{\mu} \tag{9}$$

we see that with $\mu_{,t} = 0$ we have $R_{01} = 0$, and so if the source is a perfect fluid, then it must move along the t lines. The source-free components of the field equations yield now $A = B = C = 0$, or

$$e^{-2\mu}(\mu_{,r} + 1/r) - 1/r = 0 \tag{10}$$

Again, the first case is not interesting, being just spherically symmetric, so we shall consider only (10). Then we obtain

$$e^{-2\mu} = 1 + Kr^2 \tag{11}$$

where $K = \text{const}$, and

$$\begin{aligned} D = r\,[A(t) \sin\vartheta \cos\varphi + B(t) \sin\vartheta \sin\varphi + C(t) \cos\vartheta] \\ + E(t)(1 + Kr^2)^{1/2} + s \end{aligned} \tag{12}$$

where E is an arbitrary function of t. The quantity s, though in general being an arbitrary function of t, if it is not zero, may always be set equal to 1 by the

[3]This reference is a beginning of a long and multiply branching chain of other references, leading into past times.

coordinate transformation $\int s(t)\,dt = t'$. Therefore we shall assume that $s = 1$ or $s = 0$.

The remaining field equations just define the energy density ϵ and the pressure p:

$$\begin{aligned} \varkappa\epsilon &= -3K \\ \varkappa p &= 3K(1 - 2s/3D) \end{aligned} \tag{13}$$

where $\varkappa = 8\pi G/c^4$. Thus ϵ = const, and so the fluid is incompressible. Moreover, the metric corresponding to (11)-(12) is conformally flat, which altogether resembles the interior Schwarzschild solution [9]. This solution was found by Stephani [1] as one of the metrics which can be imbedded in a flat five-dimensional space.

§ (5): *Geometrical Properties of the Stephani Solution*

Let us display the solution explicitly:

$$\begin{aligned} ds^2 &= D^2 dt^2 - (1 + Kr^2)^{-1}\, dr^2 - r^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \\ D &= r[A(t) \sin\vartheta \cos\varphi + B(t) \sin\vartheta \sin\varphi + C(t) \cos\vartheta] \\ &\quad + E(t)(1 + Kr^2)^{1/2} + s \end{aligned} \tag{14}$$

$s = 1$ or 0.

We see from (13) that the pressure differs from $(-\epsilon)$ (i.e., from the cosmological constant) by a term proportional to s, and so with $s = 0$ the source of the metric (14) reduces to the Λ term, $\Lambda = 3K$. Then (14) seems to be a highly nonsymmetric generalization of the de Sitter solution [10] which results from (14) in its standard from when $A = B = C = s = 0, E = 1$. However, as stated above, the Weyl tensor of (14) is equal to zero. Therefore, if in addition the Ricci tensor of (14) happens to be equal to the Λ term, the space-time becomes just a space of constant curvature with the Riemann tensor given by

$$R^{ij}{}_{kl} = K(\delta^i{}_k \delta^j{}_l - \delta^i{}_l \delta^j{}_k) \tag{15}$$

where the deltas are the Kronecker symbols. Thus in every invariant respect the metric (14) with $s = 0$ is equivalent to the de Sitter solution having the largest possible symmetry group. Moreover, with $K = 0$ the metric (14) is just flat. To demonstrate these facts by a direct coordinate transformation is a challenging exercise which the author did not undertake.

With $s \neq 0 \neq K$, and the functions A, B, C, E completely arbitrary the solution (14) has no symmetries.

The actual (when $s \neq 0 \neq K$) and spurious (when $s = 0$) lack of symmetry in (14) can be easily interpreted geometrically. Let us calculate the expression

$$\frac{dt^\alpha}{d\lambda} + \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} t^\beta t^\gamma \overset{\text{def}}{=} a^\alpha \tag{16}$$

for the metric (14), where $\{^{\alpha}_{\beta\gamma}\}$ are Christoffel symbols and $t^\alpha = \delta^\alpha{}_0$, i.e., t^α is a vector field tangent to the t lines, constant in the λ parametrization and in the coordinates presently used. It appears that $a^\alpha \propto t^\alpha$ if and only if $A = B = C = 0$ (when $K = 0$) or $A = B = C = E = 0$ (when $K \neq 0$). But if $a^\alpha \propto t^\alpha$, then t^α is a geodesic vector, i.e., the t lines are then geodesics. Thus the functions $A(t), B(t)$, $C(t)$, and $E(t)$ simply measure the geodesic curvature of the t lines, and taking the limit $(A^2 + B^2 + C^2 + E^2) \to 0$ corresponds to "straightening out" the t lines onto which the spherical 3-spaces are strung. In the case of the flat ($K = 0$) or de Sitter ($K \neq 0 = s$) space-time, by straightening the t-lines we still remain within the same space-time, while in the general case ($K \neq 0 \neq s$) straightening the t lines means changing the space-time from the solution (14) to the interior Schwarzchild solution [9] which results from (14) when $A = B = C = 0$, $E = \text{const}$, $s \neq 0 \neq K$.

To summarize: we can string spherical 3-spaces onto orthogonal t-lines which are either: (I) "straight" (i.e., geodesic) or (II) "curved" (nongeodesic). If the space-time so obtained appears empty (or endowed with the Λ term only), then in both cases it is flat (or, respectively, of constant curvature). If there is any matter in the space-time, then in case (I) the space-time is fully spherically symmetric while in case (II) it is not.

§ (6): *The General Case with $\delta \neq$ const*

Space limitations do not allow us to present here the details of the calculations, so let us state the result only. If δ is permitted to depend on t, then the field equations show that either $D = D(t, r)$ in (2), i.e., the solution is spherically symmetric in the traditional sense, or the solution (14) reemerges.

In this way we conclude that all the solutions of our problem were exhausted in Sections 3 and 4. Incidentally, this means that no generality was lost in Section 4 on assuming $\delta = r$. Here, however, this fact is a hard-calculation result of the field equations, as opposed to the ordinary spherically symmetric case, where $\delta = r$ (if δ is not constant) was merely a choice of coordinates.

§ (7): *Space-Times with Homogeneous Hypersurfaces*

In analogy with the foregoing part of the paper we can consider space-times which are composed of homogeneous 3-spaces $t = \text{const}$ orthogonally stacked, but which are not themselves invariant under the groups of symmetry of the 3-spaces. An ambitious project here would be to consider all the possible Bianchi types of transitive groups [11]. For simplicity we shall consider only the space-times which directly correspond to the standard Robertson-Walker universes.

We shall thus deal with space-times in which the 3-spaces t = const are homogeneous and isotropic, i.e., are the same as in the RW models.

Two inequivalent extensions of this kind were considered, corresponding to two different representations of the RW metrics. (The number of inequivalent extensions may be even larger corresponding to the different coordinates used to represent the RW metrics. Because each 3-space t = const is isotropic and homogeneous, there is no way to identify single points of it, and consequently there exists a multitude of correspondences between points of different 3-spaces, established by the family of the t lines. Quite a different thing occurred in Sections 1-6: a 3-space which is spherically symmetric but inhomogeneous has a well-defined center, and so it is most natural to assume, as we did, that one of the t lines joins the centers of all the 3-spaces.) We shall begin with the space-time in which the 3-spaces t = const have the metric[4]:

$$ds_3{}^2 = R^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2) \right] \tag{17}$$

where R and k are constants, understood to be momentary values of certain functions $R(t)$ and $k(t)$ at a fixed t. For simplicity we shall again assume that the lines of the time coordinate are orthogonal to the spaces t = const given by (17); and that the fluid source of the metric is moving, with respect to the t lines, only radially if at all. Consequently, we assume the metric of the space-time to be of the form

$$ds^2 = D^2(t, r, \vartheta, \varphi)\, dt^2 - R^2(t) \left[\frac{dr^2}{1 - k(t) r^2} + r^2 (d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2) \right] \tag{18}$$

where D is an arbitrary function of four variables while R and k are functions of t.

Since we have shown in Sections 4 and 6 that δ (equal to rR here) may depend on t only when $D_{,\vartheta} = D_{,\varphi} = 0$, we will not be surprised to find in the next section that $D_{,\vartheta} = D_{,\varphi} = 0$ unless R = const, k = const. However, one interesting new possibility arises here: if k indeed depends on t, and is positive for some t and negative for others, then the curvature of the 3-spaces t = const, equal to k/R^2, may change its sign for a certain t. Thus, potentially, the closed model could evolve here into an open one or vice versa. In itself, the idea of making k a function of t is so simple that the question arises why nobody has thought of it before. The answer is: whenever spatially homogeneous cosmological models were considered, it was always assumed that the group acting transitively in the spaces t = const is at the same time the symmetry group (or subgroup) of the

[4]See p. 210 in Ref. 10.

whole space-time. Under this assumption one concludes (see, e.g., [12]) that the metric components may depend on t only through linear combinations of one-argument functions of t, and so if (18) is to be homogeneous in the traditional Bianchi sense, then only R may depend on time while k must be a constant.

§ (8): *General Constraints from the Field Equations*

Here we shall be interested only in the perfect fluid or dust as possible sources, and again the velocity field of the source, u^α, will be such that $u^\vartheta = u^\varphi = 0$. Consequently, the tetrad components of the Einstein tensor G_{ij}, in the orthonormal tetrad connected with (18), must fulfill the equations

$$G_{02} = G_{03} = G_{12} = G_{13} = G_{23} = G_{22} - G_{33} = 0 \tag{19}$$

The equations $G_{02} = G_{03} = 0$ yield, respectively,

$$\begin{aligned} &\tfrac{1}{2}\alpha D_{,\vartheta}/rD^2 R = 0 \\ &\tfrac{1}{2}\alpha D_{,\varphi}/rD^2 \sin\vartheta = 0 \end{aligned} \tag{20}$$

where

$$\alpha = r^2 k_{,t}/(1 - kr^2) + 4R_{,t}/R \tag{21}$$

Thus either $D_{,\vartheta} = D_{,\varphi} = 0$ or $\alpha = 0$. We would tend to discard the first case as spherically symmetric and thus uninteresting. However, it is easy to see that $\alpha = 0$ implies $k_{,t} = R_{,t} = 0$, and so, on rescaling r by $r' = Rr$ and redefining k by $k' = -k/R^2$ we recover precisely the case considered in Section 4. On the other hand, when $D_{,\vartheta} = D_{,\varphi} = 0$ we still retain the possibility of varying sign of curvature which seems interesting.

Therefore we consider the first case, thus returning to spherical symmetry rather soon. In this case the other equations of (19) are fulfilled identically.

§ (9): *The Fluid Moving along the t Lines*

In this case the equation $G_{01} = 0$ must be fulfilled which says

$$[rk_{,t}D/(1 - kr^2) + 2D_{,r}R_{,t}/R](1 - kr^2)^{1/2}/RD^2 = 0 \tag{22}$$

If $R_{,t} = 0$ then $k_{,t} = 0$, and so the case of Section 4 is recovered. We shall then assume that $R_{,t} \neq 0$. Then, if $k_{,t} = 0$ we have from (22) $D_{,r} = 0$, i.e., $D = D(t)$. In this case D may be set equal to 1 by a coordinate transformation, and the standard Robertson-Walker metric is recovered. We shall be interested only in the new situation which results when $k_{,t} \neq 0 \neq R_{,t}D_{,r}$. Then, from (22)

$$D = \phi(t) \exp\{[Rk_{,t} \ln(1 - kr^2)]/4kR_{,t}\} \tag{23}$$

where $\phi(t)$ is an arbitrary function which may be set equal to 1 by a coordinate transformation. Unfortunately, this is a blind alley. Further, rather involved equations which result on substituting (23) into the Einstein tensor run into a contradiction unless $k_{,t} = 0$. Thus, only standard Robertson-Walker solutions are recovered here.

§(10): *The General Fluid*

If we allow the fluid to move in any direction in the (t, r) space, then we have the following:

(1) If it is a genuine perfect fluid, then the only equation to be obeyed by D is

$$(G_{00} + G_{22})(G_{11} - G_{22}) - G_{01}{}^2 = 0 \tag{24}$$

After this is fulfilled, the pressure, energy density, and velocity components are algebraically determined through G_{ij}.

(2) If it is a dust, then the two equations

$$G_{00}G_{11} - G_{01}^2 = 0 \tag{25}$$

$$G_{22} = 0 \tag{26}$$

must be fulfilled (note that, owing to spherical symmetry, $G_{22} = G_{33}$ is an identity).

No answer has been reached whether (25)-(26) can be fulfilled. The provisional answer should be "no" since then the function $D(t, r)$ must simultaneously obey two differential equations. The equation (24) on the other hand, *can* be fulfilled, being just one equation for one function $D(t, r)$, with $k(t)$ and $R(t)$ being arbitrary parameters. The prospects of obtaining a general integral of that equation are, however, rather dim owing to its complexity.

§(11): *A Different Extension of the RW Models*

Let us now repeat the reasoning of Sections 7-10 for a space-time which is composed of these same 3-spaces in a different way. Let us take a 3-space t = const from a Robertson-Walker metric in a spherical conformally flat representation [13]:

$$ds_3{}^2 = \frac{R^2}{(1 + \frac{1}{4}kr^2)^2} \left[dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right] \tag{27}$$

where again R and k are understood to be momentary values of certain functions $R(t)$ and $k(t)$ at a certain value of t. If the 3-spaces (27) are strung onto a congruence of t lines which are orthogonal to them, then the space-time metric is

$$ds^2 = D^2(t, r, \vartheta, \varphi)\, dt^2 - \{R^2(t)/[1 + \tfrac{1}{4}k(t)\, r^2]^2\} \times [dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)] \tag{28}$$

With k = const, (28) is just a different coordinate representation of the metric (18). With $k_{,t} \neq 0$, (28) in general *is not* a transform of (18) even if $D = D(t, r)$.

So in (28) we have a new class of space-times. We shall investigate them with the same assumptions concerning the sources as in previous cases.

The field equations $G_{02} = G_{03} = 0$ give the results

$$\phi D_{,\vartheta} = \phi D_{,\varphi} = 0 \tag{29}$$

where

$$\phi := [-\tfrac{1}{2} r\dot{k}R + 2\dot{R}(1 + \tfrac{1}{4}kr^2)/r]\,/R^2D^2 \tag{30}$$

Thus either $D_{,\vartheta} = D_{,\varphi} = 0$ or $\phi = 0$. We see easily that $\phi = 0$ implies $\dot{k} = \dot{R} = 0$, and in this case the solution of Section 4 will be reobtained. Consequently, only the case $D_{,\vartheta} = D_{,\varphi} = 0$ is worth investigating.

If $D_{,\vartheta} = D_{,\varphi} = 0$, then the equations $G_{12} = G_{13} = G_{23} = G_{22} - G_{33} = 0$ are all fulfilled identically. However, we have to consider separately the case of the fluid moving along the t lines and the case of the fluid moving off the t lines.

If the fluid moves along the t lines, then the equation $G_{01} = 0$ must be fulfilled, which reads

$$(r - \tfrac{1}{4}kr^3/V)\,\dot{k}/RD + (2V\dot{R}/R^2 - \tfrac{1}{2}\dot{k}r^2/R)\,D_{,r}/D^2 = 0 \tag{31}$$

where

$$V := 1 + \tfrac{1}{4}k(t)\, r^2 \tag{32}$$

The equation (31) is easily integrated to yield

$$D = F(t)\left(\frac{\dot{V}}{V} - \frac{\dot{R}}{R}\right) \tag{33}$$

where F is an arbitrary function of t. With such D, the equation $G_{11} = G_{22}$ is obeyed automatically, and so the energy-momentum tensor defined by the field equations has the algebraic form necessary for a perfect fluid. The density ϵ and the pressure p are given by

$$\kappa\epsilon = 3\left(\frac{k}{R^2} + \frac{1}{F^2}\right) \tag{34}$$

$$\kappa p = -\frac{k}{R^2} - \frac{3}{F^2} - 2\frac{\dot{F}}{F^2D} + \frac{F\dot{k}}{R^2D}\left(1 - \frac{1}{2}\frac{kr^2}{V}\right) \tag{35}$$

More elegant formulas result if one parametrizes the functions after Stephani [1] as

$$F = 1/\alpha, \qquad V = v/a, \qquad k = (C^2 - \alpha^2)/a^2, \qquad R = 1/a \tag{36}$$

where α, a, and C are new functions of t. Then

$$D = \dot{v}/\alpha v \tag{37}$$

$$\kappa\epsilon = 3C^2 \tag{38}$$

$$\kappa p = -3C^2 + 2C\dot{C}\, v/\dot{v} \tag{39}$$

The metric in this case is

$$ds^2 = (\dot{v}/\alpha v)^2\, dt^2 - v^{-2}\,[dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)] \tag{40}$$

with

$$v = a + \tfrac{1}{4}(C^2 - \alpha^2)\, r^2/a \tag{41}$$

The solution given by (38)-(41) is a special case of another solution found by Stephani [1] as embeddable in a five-dimensional flat space. We shall deal with that solution more closely in the next section. Let us note here only that all the functions of t are completely arbitrary, and so k may change its sign as many times as one wishes. So the sign of spatial curvature is not a fixed property of the models of this class, just as we have argued in Section 7. The solution reduces to ordinary Robertson-Walker metrics in the special cases when $C^2 - \alpha^2 = \epsilon a^2$ where $\epsilon = +1, 0$, or -1.

If the fluid is not moving along the t lines, then, in addition to the equations $G_{02} = G_{03} = G_{12} = G_{13} = G_{22} - G_{33} = 0$, used up to (30), the single equation (24) has to be fulfilled. Its solution should be a generalization of the Stephani model. We leave this for separate investigation in a future work.

§ (12): *Generalized Friedman Models with Wandering Center of Symmetry*

We shall present here the farthest extension of the Friedman-Robertson-Walker models along the line of Section 11. In the previous section some rudimentary regularity of the space-time was preserved because the t lines mapping each 3-space into other 3-spaces were assumed to join the points which were "corresponding" in the following sense. Each of the 3-spaces t = const was homogeneous and so had no center of symmetry defined geometrically. However, the coordinate systems used in the 3-spaces had all their centers (origins), and it was tacitly assumed that if a t line passes through the origin of coordinates in one 3-space, then it contains all the other origins of the other 3-spaces. This can be seen from the form (28). Such an assumption limited the class of metrics considered, and as a result the field equations forced the whole space-time to be spherically symmetric. This limitation can be, however, relaxed and we shall do it here. We will now assume that the (coordinate) centers of symmetry of different 3-spaces are arbitrarily shifted with respect to each of the t lines.

It will be more convenient now to represent the metric form of the 3-space in such coordinates which explicitly exhibit the arbitrary position of the origin of the coordinate system. One possible form is

$$ds_3^2 = (R/V)^2 (dx^2 + dy^2 + dz^2) \tag{42}$$

where

$$V = 1 + \tfrac{1}{4}k[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2] \tag{43}$$

Here R, k, x_0, y_0, and z_0 are arbitrary constants. In Section 11 we have assumed that R and k were values of the functions $R(t)$ and $k(t)$ at a fixed moment of time, while x_0, y_0, and z_0 were genuine constants removable by a coordinate transformation describing a simultaneous shift of the origin in all the 3-spaces. This assumption is by no means necessary: x_0, y_0, and z_0 may be as well assumed to be functions of time. The space-time metric is then

$$ds^2 = D^2(t,x,y,z)\,dt^2 - [R^2(t)/V^2(t,y,x,z)](dx^2 + dy^2 + dz^2) \tag{44}$$

where

$$V(t,x,y,z) = 1 + \tfrac{1}{4}k(t)\{[x - x_0(t)]^2 + [y - y_0(t)]^2 + [z - z_0(t)]^2\} \tag{45}$$

This space-time is composed of the same 3-spaces as the space-time (28), but in general, with arbitrary $x_0(t), y_0(t)$, and $z_0(t)$, the coordinate origins of two different 3-spaces do not belong to the same t line. As one proceeds from one 3-space to another, the origin wanders with respect to the t lines. This characterization seems coordinate dependent, but the space-time (44) is different from (28) because the metric (44)-(45) is not spherically symmetric [in fact, with the functions x_0, y_0, z_0 being completely arbitrary, (44) has no symmetries at all, as the Killing equations show].

The field equations for the metric (44)-(45) were solved by Stephani [1], and we shall call the solution the Stephani universe. It is obtained by assuming that the source in the Einstein equations is a perfect fluid moving along the t lines. The field equations $G_{01} = G_{02} = G_{03} = 0$ imply then again (33), and with (33) the equations $G_{12} = G_{13} = G_{23} = G_{11} - G_{22} = G_{22} - G_{33} = 0$ are fulfilled identically. With Stephani's parametrization (36) the same formulas (38)-(39) for the density and pressure result.

§ (13): *Some Properties of the Stephani Universe*

From the formula (39) one can conclude that $p = 0$ implies that k, x_0, y_0, and z_0 are constants, and so then $D = D(t)$ [from (33) and (43)]. In this case a coordinate transformation makes D equal to 1, and so the old Friedman models are reobtained. Note that this means that vanishing pressure forbids the sign of spatial curvature to vary with time. (The same conclusion was hinted at, by counting the number of equations and the number of unknown functions, in Section 10.) This statement can be supported by the following intuitive argument. In the normal RW cosmology, the behavior of the closed and the open universe can be explained in purely Newtonian terms. The universe is closed when its mass-density relative to the rate of expansion is large enough to halt the expansion by its gravitational field, and is open otherwise. Therefore, to have the closed universe change into an open one a mechanism for the decay of mass would have to be invoked. In principle, one mechanism is conceivable. It is well known that the pressure in a gas or fluid, which in Newtonian physics can only tend to expand the volume of the medium, gives, in the general-relativity theory, a positive contribution to the energy-density [14], and so, at large values, exhibits the opposite tendency to enhance the self-gravitation.[5] In an expanding universe the pressure would gradually diminish. It is therefore possible, in principle, that initially the contribution of pressure to self-gravitation would be large enough to close the universe, while in the later stages of expansion this effect of pressure would become negligible and thus, with suitable amount of rest mass, the universe would open up. The opposite change would occur in a contracting universe.

This intuitive argument does not explain how several changes, from positive curvature to negative and back to positive, and so forth, could occur during the evolution of the model. Such changes are mathematically evidently possible, as is seen from (28) and (45): the Stephani universe exists with an arbitrary function $k(t)$. The physics and astronomy in such a universe will be investigated in separate papers. Let us note here only the obvious conclusion that, since the energy density is a function of time only, and the pressure depends also on the other coordinates, no equation of state of the simple form $\epsilon = \epsilon(p)$ is admissible.

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