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Preprint No 62

NONSTATIONARY CYLINDRICALLY SYMMETRIC ROTATING UNIVERSES

by

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/Submitted to International Journal of Theoretical Physics/

Warsaw, January 1976

Abstract. The cylindrically symmetric nonstationary perfect fluid spacetimes are proposed as possible models of a rotating Universe. It is argued that they are the simplest rotating generalizations of the Robertson-Walker spacetimes which contain no geometrical structures except those connected with matter. The formalism for treating spacetimes filled with an isentropic rotating perfect fluid or dust is recalled. Cylindrical symmetry is defined in terms of properties of the Killing vectors. Then the Killing equations together with some of the field equations are used to reduce the metric to the simplest possible form. Some hints for future investigation of such models are given.

1. INTRODUCTION

Of all the cosmological solutions found up to now those based on isotropic and homogeneous metrics of Robertson (1929, 1932 and 1935) and Walker (1935) have proven most useful. They describe rough properties of the real Universe very well. Even the singularity inherent in them has found much observational support, i.e. our observed Universe really must have been extremely hot and dense some time ago (see e.g. Peebles 1971, Weinberg 1972, Sciama 1971). However, they contain many unrealistic idealizations. For example, in an ideally isotropic and homogeneous Universe no galaxies may form. And even if they are created by some perturbations, the question appears: what is the origin of their rotation? Also, the observational evidence for isotropy and homogenity of the Universe is rather weak, since possibly much of the existing matter is not even registered by astronomers (Peebles 1971). This means that future more realistic cosmological models should be obtained by gradually abandoning those too idealized assumptions. It is a natural first step to look for such solutions which violate the conditions of homogeneity and isotropy as little as possible. Consequently, much work has been done on homogeneous anisotropic Bianchi-type universes (see e.g. Collins and Hawking 1973a and b, and references cited therein). Here we shall rather follow the suggestion of Gamow (1946) which says: the rotation of stars is explained by the fact that they form of matter in a rotating galaxy. But why do the galaxies rotate themselves? Perhaps, analogously, it is because they formed of matter in the rotating Universe. Today there is no distinct evidence of rotation of the Universe because in the course of expansion the rotation, if it were present, must have slowed down on account of the angular momentum conservation. It might

however be quite rapid in the far past.
It is possible to investigate rotating Universes in the class of the Bianchi-type homogeneous models. It might seem the right trace to follow if we want to keep close to Robertson-Walker models. However, one has to be more critical about the term "homogeneity". The homogeneity of the Bianchi models is of a different physical nature in the case of rotating matter than in the case of Friedman models. In the latter case the Universe looks homogeneous for the observer comoving with matter, and no geometrical structures except the velocity field of matter are needed to describe the splitting up of spacetime into homogeneous 3-spaces.
Now, the distinguished vector field e^0 existing in all the Bianchi-type models is hypersurface-orthogonal, and must not be colinear with the velocity field if matter is to rotate. This means that in the Bianchi-type models with rotating matter the field e^0 is an additional, independent of matter geometrical structure. It means, there are God-given preferred observers who see the Universe to be especially simple. It is not a decisive argument against those models, but it shows that two different things were named by the same word "homogeneity". If we take a Robertson-Walker portion of matter and set it into rotation we will not necessarily obtain a Bianchi-type model.
Here we propose to start the investigation of cylindrically symmetric nonstationary rotating Universes. They are free of the above disadvantage and, on the other hand, also keep close to Robertson-Walker models. However, their bad feature is that they do not encompass the closed model, as it appears in sec. 2.
General isotropy must be lost in rotating matter, since in every point of spacetime there exists the distinguished direction of the vortex-vector. The only remainder of isotropy then may be the axial symmetry about this direction. Homogeneity in the direction of vortex may be assumed as a remainder of 3-dimensional homogeneity of the Robertson-Walker models. In consequence, only two Killing vector fields survive. This approach is complementary to that based on Bianchi-type models, and may lead to new results.
We do not expect to obtain a fully satisfactory model of the Universe in this way because the galaxies formed in such a model would have their axes of rotation aligned while real galaxies have not. However, it is

better to have a model with galaxies rotating paralelly than a model which forbids rotation at all.

The present paper starts with a short review of previously obtained results for a rotating isentropic perfect fluid (Krasiński 1974, 1975a). Then cylindrical symmetry is defined in terms of properties of the Killing vectors. These properties, together with the Killing equations, are used to introduce the most convenient coordinate system, and to establish the simplest form of the metric tensor. Then some of the field equations are used for further simplification of the metric. The paper ends in the place where new simplifying assumptions become necessary to continue the integration of the field equations. Some such assumptions are suggested as reasonable.

2. PRELIMINARIES

We start with a short repetition (Krasiński 1974 and 1975a). For an isentropic rotating perfect fluid there exist the dynamically distinguished coordinates (x^0, x^1, x^2, x^3) characterized by the following equations:

$$u^\alpha = H\delta^\alpha_0 \tag{2.1}$$

$$w^\alpha = \varrho H^{-1}\delta^\alpha_3 \tag{2.2}$$

$$\begin{aligned} g_{00} &= H^{-2} \\ g_{01} &= x^2 H^{-2} \\ g_{02} &= g_{03} = 0 \end{aligned} \tag{2.3}$$

$$g \stackrel{\text{def}}{=} \det[g_{\alpha\beta}] = -\varrho^{-2}H^{-2} \tag{2.4}$$

where ϱ is the density of the rest-mass of particles, $\varrho = m_0 \cdot n$, m_0 being the mean mass of a particle of the fluid, and n being the density of number of particles. The function H is the enthalpy per unit rest energy defined by $H = (\epsilon + p)/\varrho c^2$ where ϵ is the energy density and p is the pressure. All these thermodynamical quantities are one-variable functions of p, $\varrho(p)$ being an unknown function, and H being given in terms of p and $\varrho(p)$ by:

$$H = H_0 + \frac{1}{c^2}\int_0^p \frac{dp'}{\varrho(p')}, \qquad H_0 = \text{const} \tag{2.5}$$

The vectors u^α and w^α are velocity and vorticity fields respectively.

The energy-momentum tensor has the following form:

$$T_{\alpha\beta} = \rho c^2 H u_\alpha u_\beta - p g_{\alpha\beta} \qquad (2.6)$$

These distinguished coordinates are defined exact to the transformations:

$$\begin{aligned} x^0 &= x^{0'} - S(x^{1'}, x^{2'}) \\ x^1 &= F(x^{1'}, x^{2'}) \\ x^2 &= G(x^{1'}, x^{2'}) \\ x^3 &= x^{3'} + T(x^{1'}, x^{2'}) \end{aligned} \qquad (2.7)$$

where T is an arbitrary function of two variables while F and G obey the equation:

$$F_{,1'}G_{,2'} - F_{,2'}G_{,1'} = 1 \qquad (2.8)$$

The function S is fixed exact to an additive constant by the equations:

$$\begin{aligned} S_{,1'} &= GF_{,1'} - x^{2'} \\ S_{,2'} &= GF_{,2'} \end{aligned} \qquad (2.9)$$

(Equation (2.8) guarantees the integrability of the set (2.9)).
This representation of the metric was invented by Plebański (1970) by an ingenious investigation of the equations of motion $T^{\alpha\beta}{}_{;\beta} = 0$ and the equation of continuity $(nu^\alpha)_{;\alpha} = 0$. It is easy to verify that in this coordinate system all these equations are fulfilled identically. We recall after Krasiński (1974 and 1975a) that dust may be considered as the special case corresponding to $H \equiv 1$ above.

3. CYLINDRICALLY SYMMETRIC SPACETIMES

We assume that local cylindrical symmetry is fully characterized by the following four properties:

A. There exist two Killing vector fields $\underset{(\varphi)}{k}$ and $\underset{(z)}{k}$ of which $\underset{(\varphi)}{k}$ corresponds to axial symmetry and $\underset{(z)}{k}$ generates translations along the symmetry axis.

B. $\underset{(z)}{k}{}^\mu = \delta^\mu_3$

C. $[\underset{(z)}{k}, \underset{(\varphi)}{k}] = 0$

D. $g_{\mu\nu} \underset{(z)}{k}{}^\mu \underset{(\varphi)}{k}{}^\nu = 0$

The properties A, C and D need no justification because they just reflect the symmetries of a cylindrical surface in an euclidean space.

The property B, in virtue of (2.2), means that the axis of symmetry coincides with the axis of rotation.

Our definition of cylindrical symmetry is purely formal. The coordinates given by (2.1) - (2.4) have been adjusted to physics, not to geometry, and their geometrical meaning is a bit vague. Therefore the symmetry group will have to be investigated for each solution separately when it is found, to check whether it really generates cylindres or e.g. planes. The proper globally cylindrically symmetric spacetimes are anyway contained in the above defined collection.

It is easy to observe that those Bianchi-type Universes which contain a two-parametric abelian subgroup of symmetries may be special cases of the class considered here [1]. Of all Bianchi models only types VIII and IX do not posess this property (see e.g. Appendix C to Belinskii, Khalatnikow and Lifschitz 1970). There is no reason to worry about type VIII as it does not contain any of Robertson-Walker models.

Unfortunately, the closed model is a type IX metric, and it is rejected from the present investigation by the assumption C (Collins and Hawking 1973b). With explicit use of the Killing equations and the properties B and C one can prove that it is possible to specialize the coordinates given by (2.1) - (2.4) so as to obtain:

$$\underset{(4)}{k}^{\mu} = \delta^{\mu}_{1} \qquad (3.1)$$

This might seem trivial, as it is always possible to adjust the coordinates to the commuting Killing vectors in such a way that they generate the translations along the respective coordinate lines. Here it is important that such "Killing coordinates" are contained <u>within the class of section 2.</u> The proof is given in Appendix A. In this special coordinate system the metric simply does not depend on x^3 and x^1, while property D means:

$$g_{13} = 0 \qquad (3.2)$$

Together with (2.1) - (2.4) this implies that the metric is of the form:

$$ds^2 = H^{-2}(dx^0)^2 + 2x^2H^{-2}dx^0dx^1 + g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + $$
$$+ g_{22}(dx^2)^2 + 2g_{23}dx^2dx^3 + g_{33}(dx^3)^2 \qquad (3.3)$$

where all the functions depend only on x^0 and x^2.

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[1] They are if they fulfil the additional requirement D.

The coordinate system, in which the metric (3.3) is represented, is not unique, it is defined exact to the transformations:

$$
\begin{aligned}
x^0 &= x^{0'} - C\int x^{2'} F_{,2'} dx^{2'} \\
x^1 &= C^{-1} x^{1'} + F(x^{2'}) \\
x^2 &= C x^{2'} \\
x^3 &= x^{3'} + T(x^{2'})
\end{aligned}
\qquad (3.4)
$$

where $C \neq 0$ is an arbitrary constant, and $F(x^2)$, $T(x^2)$ are arbitrary functions of one variable. This simply follows from Appendix A.

4. CANONICAL FORM OF THE METRIC

We introduce now the following notation:

$$
\begin{aligned}
g_{11} &= (x^2)^2 H^{-2} - h \\
g_{12} &= -(hk)^{1/2} \\
g_{22} &= -k - l - j \\
g_{23} &= -(jw)^{1/2} \\
g_{33} &= -w
\end{aligned}
\qquad (4.1)
$$

where h, k, l, j, w will be the new unknown functions, all positive.
Th; equation (2.4) now reads:

$$hlw = \rho^{-2} \qquad (4.2)$$

We also introduce the base of differential forms:

$$
\begin{aligned}
e^0 &\equiv e^0{}_\alpha dx^\alpha = H^{-1} dx^0 + x^2 H^{-1} dx^1 \\
e^1 &\equiv e^1{}_\alpha dx^\alpha = h^{1/2} dx^1 + k^{1/2} dx^2 \\
e^2 &\equiv e^2{}_\alpha dx^\alpha = l^{1/2} dx^2 \\
e^3 &\equiv e^3{}_\alpha dx^\alpha = j^{1/2} dx^2 + w^{1/2} dx^3
\end{aligned}
\qquad (4.3)
$$

The metric (3.3) may be written:

$$ds^2 = (e^0)^2 - (e^1)^2 - (e^2)^2 - (e^3)^2 = \eta_{ij} e^i e^j \qquad (4.4)$$

where the indices i, j = 0, 1, 2, 3 label forms.
The set of coefficients $\{e^i{}_\alpha\}$ given by (4.3) defines the set of reciprocal coefficients $\{e^\alpha{}_i\}$ by

$$e^i{}_\alpha e^\alpha{}_j = \delta^i{}_j \qquad (4.5)$$

The coefficients e^i_α and e^α_i may be used to represent all the tensors by their scalar components, e.g. $R^\alpha{}_\beta \longrightarrow R^i{}_j = e^i{}_\alpha e^\beta{}_j R^\alpha{}_\beta$.
Proceeding the standard methods of computing the curvature tensor (see e.g. Misner, Thorne and Wheeler 1973) we obtain the components of the Ricci tensor $R^i{}_j$ and insert them in the field equations:

$$R^i{}_j = (\varkappa/c^2)(T^i{}_j - \tfrac{1}{2}\delta^i_j T) + \Lambda\delta^i_j \qquad (4.6)$$

where $\varkappa = 8\pi G/c^2$, G - the gravitational constant, Λ - the cosmological constant, $T = T^i{}_i$ and $T^i{}_j$ are the scalar components of the energy-momentum tensor $T_{\alpha\beta}$ given by (2.6).
Consider the equation $-R^1{}_2 + x^2 H^{-1} h^{-1/2} R^0{}_2 = 0$ (all indices refer to scalar components). It can be written in the form:

$$-H h^{-1} w^{-1/2} \zeta_{,0} = 0 \qquad (4.7)$$

where:

$$\zeta \overset{def}{=} w^{1/2}\Big\{ \frac{(x^2)^2 k^{1/2} H_{,0}}{H^2 l^{1/2}} + \frac{H k^{1/2}}{4 h l^{1/2}}\left[(x^2)^2 H^{-2} + h\right] h_{,0} + \frac{H}{4(kl)^{1/2}}\left[(x^2)^2 H^{-2} - h\right] k_{,0} +$$
$$+ \frac{x^2 h^{1/2} H_{,2}}{H^2 l^{1/2}} + \frac{x^2 h_{,2}}{2H(hl)^{1/2}} - \frac{h^{1/2}}{2Hl^{1/2}} - \frac{(x^2)^2}{2H^3(hl)^{1/2}}\Big\} \qquad (4.8)$$

Next consider the equation $R^0{}_0 - R^1{}_1 + (H/x^2 h^{1/2})\left[(x^2)^2 H^{-2} + h\right] R^0{}_1 = \varkappa\rho H$.
With the help of (4.7), and after a long calculation it may be put in the form:

$$(x^2)^{-1}(hlw)^{-1/2} H \zeta_{,2} = -\varkappa\rho H. \qquad (4.9)$$

This shows that $\zeta \neq 0$, unless $\rho = 0$ what corresponds to empty space solutions.
In virtue of (4.2) and (4.7) the equation (4.9) yields $\zeta_{,2} = -\varkappa x^2$, i.e.:

$$\zeta = -\tfrac{1}{2}\varkappa (x^2)^2 + A \qquad (4.10)$$

where A = const.
Finally let us consider the equation $-R^1{}_3 + x^2 H^{-1} h^{-1/2} R^0{}_3 = 0$. It is easily seen to be:

$$\tfrac{1}{2} H h^{-1} (lwj)^{-1/2} \zeta \left(j^{-1} j_{,0} - w^{-1} w_{,0}\right) = 0 \qquad (4.11)$$

The only term that can vanish is the second one in braces. Its vanishing means $j^{1/2} = f(x^2)\, w^{1/2}$ where f is a function of x^2 only. This means that:

$$g_{23} = f(x^2)\, g_{33} \qquad (4.12)$$

If we perform the transformation (3.4) with $T = -\int f(x^2)dx^2$ we obtain $g'_{23} = 0$, i.e. $j' = 0$ in the new coordinates. The coordinates in which $g_{23} = 0$ are given exact to the transformations (3.4) with T being an arbitrary constant.
Thus we have proven that the most general cylindrically symmetric spacetime with rotating isentropic perfect fluid has the following metric form:

$$ds^2 = H^{-2}(dx^0 + x^2 dx^1)^2 - h(dx^1)^2 - 2(hk)^{1/2} dx^1 dx^2 - (k+l)(dx^2)^2 - w(dx^3)^2 \qquad (4.13)$$

with additional equations (4.2), (4.8) and (4.10).
A typical argument would be: there must exist such coordinates in which $g_{23} = 0$ for otherwise the metric (3.3) would not posess the reflectional symmetry across the z = 0 plane, as it should being cylindrically symmetric. However, it is more nice to have such a result obtained from the field equations.

5. SOME HINTS FOR FURTHER INVESTIGATION

Let us notice that the function $F(x^2)$ in (3.4) allows for significant simplification of those metrics (4.13) for which $(k/h)_{,0} = 0$, i.e. $k = \alpha(x^2)h$. Then it is enough to take $F = -\int \alpha^{1/2} dx^2$ to obtain k = 0 in the new coordinates. It is tempting to think that every metric (4.13) has this property in virtue of the field equations. This is not the case: Ozsváth's solutions of class II (with $k = 2^{-1/2}$) and of class III are counterexamples (Ozsváth 1970). There $(k/h)_{,0} \neq 0$ when the metric is put in the form (4.13) (consult Krasiński 1975c for the appropriate expression of Ozsváth's solutions).
With $(k/h)_{,0} \neq 0$ the remaining field equations are so complex that it is rather hopeless to integrate them without any new simplifying assumptions.
The most obvious simplification would be to deal with shearfree solutions. Detailed investigation of the field equations reveals however that all shearfree solutions are stationary being thus of no interest in cosmology. The proof of this statement is sketched in Appendix B.
The stationary cylindrically symmetric solutions were all found and

discussed in Krasiński 1974, 1975a and 1975b. They must be special or limitting cases of the time-dependent ones. Second way of proceeding would then be to try to generalize those solutions by inserting functions of time in place of constants, and feeding everything back into the field equations. Still another way is to assume that some scalar equations like $\omega^2/\varrho H$ = const (where ω is the scalar of rotation) which obtained in the stationary case still hold, with constants perhaps generalized to functions of time.
An explicit model of a rotating and expanding Universe, not found as yet, would be of a great value in connection with the problem of the initial singularity. The author hopes that the present work will be a step towards this final goal.

APPENDIX A

Special coordinates for $\underset{(\varphi)}{k}^{\mu}$.

The assumptions A - D from section 3 give us no direct information about the form of the Killing vector $\underset{(\varphi)}{k}$. Therefore we assume it in the most general form:

$$\underset{(\varphi)}{k}^{\mu} = V(x)\delta^{\mu}_{0} + X(x)\delta^{\mu}_{1} + Y(x)\delta^{\mu}_{2} + Z(x)\delta^{\mu}_{3} \qquad (A.1)$$

where V, X, Y, Z are unknown functions of all the four coordinates. The properties B and C imply at once:

$$V_{,3} = X_{,3} = Y_{,3} = Z_{,3} = 0 \qquad (A.2)$$

All the scalar and vector fields defined by the metric tensor are invariant under the symmetry transformations. The function H is a scalar by definition, and it is easy to see that H is defined by the metric through the equations $\det[g_{\mu\nu}] = -\varrho^2 H^2$ and $T = \frac{c^2}{\varkappa}(4\Lambda - R) = \varrho c^2 H - p(H)$ (the last equation was obtained by contraction of the field equations (4.6)). Therefore H should be invariant under the action of $\underset{(z)}{k}$ and $\underset{(\varphi)}{k}$ what means, respectively:

$$H_{,3} = 0 \qquad (A.3)$$

$$VH_{,0} + XH_{,1} + YH_{,2} = 0 \qquad (A.4)$$

Also, u^{α} is defined by $g_{\alpha\beta}$ as the timelike eigenvector of the energy-

momentum tensor which in turn is given as a function of $g_{\alpha\beta}$ and its derivatives through the Einstein field equations. Consequently, ω^α should also be invariant, i.e. it should commute with $\underset{(z)}{k}$ and $\underset{(\varphi)}{k}$. In virtue of (2.1) and (A.3) it commutes with $\underset{(z)}{k}$ identically while commutation with $\underset{(\varphi)}{k}$, in virtue of (A.3) and (A.4), means:

$$V_{,0} = X_{,0} = Y_{,0} = Z_{,0} = 0 \qquad \text{(A.5)}$$

So V, X, Y, Z depend only on x^1 and x^2. This property is preserved by the transformations (2.7). Now we shall show that by a suitable choice of F, G and T in (2.7) such coordinates may be obtained, in which:

$$Y = Z = 0 \qquad \text{(A.6)}$$

The equation (2.8) means that the Jacobian of the transformation (2.7) equals 1. This makes the computation of the inverse Jacobi matrix $[x^{\alpha'}{}_{,\alpha}]$ very easy:

$$\begin{aligned} &x^{0'}{}_{,0} = 1, \quad x^{0'}{}_{,1} = G - x^{2'}G_{,2'}, \quad x^{0'}{}_{,2} = x^{2'}F_{,2'}, \\ &x^{1'}{}_{,1} = G_{,2'}, \quad x^{1'}{}_{,2} = -F_{,2'}, \\ &x^{2'}{}_{,1} = -G_{,1'}, \quad x^{2'}{}_{,2} = F_{,1'}, \\ &x^{3'}{}_{,1} = G_{,1'}T_{,2'} - G_{,2'}T_{,1'}, \quad x^{3'}{}_{,2} = -F_{,1'}T_{,2'} + F_{,2'}T_{,1'}, \quad x^{3'}{}_{,3} = 1 \end{aligned} \qquad \text{(A.7)}$$

unspecified $x^{\alpha'}{}_{,\alpha}$ being zero. This allows us to find the components of the transformed $\underset{(\varphi)}{k}$:

$$\begin{aligned} V' &= V + (G - x^{2'}G_{,2'})X + x^{2'}F_{,2'}Y \\ X' &= G_{,2'}X - F_{,2'}Y \\ Y' &= -G_{,1'}X + F_{,1'}Y \\ Z' &= (G_{,1'}T_{,2'} - G_{,2'}T_{,1'})X - (F_{,1'}T_{,2'} - F_{,2'}T_{,1'})Y + Z \end{aligned} \qquad \text{(A.8)}$$

If we wish that $Y' = 0$ we impose on F and G an equation:

$$-G_{,1'}X + F_{,1'}Y = 0 \qquad \text{(A.9)}$$

in addition to (2.8). The problem is to prove that the set of equations (2.8) - (A.9) does have solutions. There are four cases:

I. If X = Y = 0 then property D implies $Z^2 g_{33} = 0$. But $g_{33} \neq 0$ for otherwise $-\omega^2 = g_{\alpha\beta}\omega^\alpha\omega^\beta = 0$ and there would be no rotation. Consequently, Z = 0. The Killing equations yield then easily the result V = const, i.e. $\underset{(\varphi)}{k}{}^\mu = \delta^\mu_0$ what corresponds to the stationary solutions considered in previous works (Krasiński 1974 and 1975a).

This case lies beyond the scope of the present work.
II. If $X \neq 0 = Y$ then the result is ready, and (A.9) only gives the transformations which preserve this property.
III. If $X = 0 \neq Y$ then the transformation (2.7) with $F_{,1'} = 0$ yields $Y' = 0$.
IV. The only case we are left with is $X \neq 0 \neq Y$. In this case we obtain from (2.8) and (A.9):

$$G_{,1'} = (Y/X) F_{,1'} \tag{A.10}$$

$$G_{,2'} = \frac{1}{F_{,1'}} + \frac{Y}{X} F_{,2'} \tag{A.11}$$

In this place it is convenient to introduce explicitly the inverse transformation $x^{1'} = \phi(x^1, x^2)$, $x^{2'} = \psi(x^1, x^2)$.
With the help of (A.7) the equations (A.10) and (A.11) are seen to change to:

$$\psi_{,1} + (Y/X)\psi_{,2} = 0 \tag{A.12}$$

$$\phi_{,1} + (Y/X)\phi_{,2} = 1/\psi_{,2} \tag{A.13}$$

The equation (A.12) is a simple linear equation for ψ, obviously having a solution, and after ψ is found, (A.13) forms a well-defined linear inhomogeneous equation for ϕ. The existence of its solution is guaranteed by the well-known Cauchy-Kovalevska theorem (see e.g. Courant and Hilbert 1962). Since ψ and ϕ exist, the existence of F and G obeying (A.10) - (A.11) is guaranteed by $1 = \partial(F,G)/\partial(\psi,\phi)$.
This completes the proof that $Y' = 0$ may be achieved.
Now, with F and G found above the equation $Z' = 0$ becomes a linear inhomogeneous partial differential equation for T with all coefficients being well defined functions of $x^{1'}$ and $x^{2'}$. The existence of its solutions is again guaranteed by the Cauchy-Kovalevska theorem.
Inserting $Y = Y' = Z = Z' = 0$ in (A.8) we obtain $G_{,1'} = 0$ and $G_{,2'} T_{,1'} = 0$.
This, together with (2.8) and (2.9), allows to show easily that the transformations preserving the properties $Y = Z = 0$ are given by:

$$\begin{aligned} x^0 &= x^{0'} + x^{1'} \cdot x^{2'} - x^{1'} \cdot G(x^{2'})/G_{,2'} - \int G F_{,2'} dx^{2'} \\ x^1 &= x^{1'}/G_{,2'} + F(x^{2'}) \\ x^2 &= G(x^{2'}) \\ x^3 &= x^{3'} + T(x^{2'}) \end{aligned} \tag{A.14}$$

where F, G and T are arbitrary except that $G_{,2'} \neq 0$.
Now let us consider the Killing equations:

$$\underset{(\varphi)}{k^{\delta}} g_{\alpha\beta,\delta} + \underset{(\varphi)}{k^{\delta}}_{,\alpha} g_{\delta\beta} + \underset{(\varphi)}{k^{\delta}}_{,\beta} g_{\alpha\delta} = 0 \qquad (A.15)$$

with $\underset{(\varphi)}{k^{\mu}} = V(x^1,x^2)\delta^{\mu}_{0} + X(x^1,x^2)\delta^{\mu}_{1}$. Taking the components (01) and (02) of (A.15) we obtain respectively:

$$V_{,1} + x^2 X_{,1} = 0 \qquad (A.16)$$

$$V_{,2} + x^2 X_{,2} = 0 \qquad (A.17)$$

(in obtaining (A.16) we have used (A.4)). The above equations imply:

$$X = X(x^2) \qquad (A.18)$$

$$V = -\int x^2 X_{,2}\, dx^2 \qquad (A.19)$$

Now take the transformation (A.14) with F and T arbitrary, and $G(x^{2'})$ being the inverse function to ϕ:

$$x^{2'} = \exp\int \frac{dG}{V(G)/X(G) + G} \qquad (A.20)$$

Such a transformation leads to $V' = 0$ and $X' =$ const. Thus (3.1) is proven. Inserting $V = 0$ in (A.20) we obtain $x^{2'} = C^{-1}G$, $C =$ const. Then (A.14) reduces to (3.4).

APPENDIX B

Shearfree solutions.

The nonvanishing components of shear are:

$$\begin{aligned}
\sigma_{11} &= -\tfrac{1}{2}Hh_{,0} - \tfrac{1}{3}Hh\varrho^{-1}\varrho_{,0} \\
\sigma_{12} &= \tfrac{1}{2}Hg_{12,0} + \tfrac{1}{3}Hg_{12}\varrho^{-1}\varrho_{,0} \\
\sigma_{22} &= \tfrac{1}{2}Hg_{22,0} + \tfrac{1}{3}Hg_{22}\varrho^{-1}\varrho_{,0} \\
\sigma_{33} &= \tfrac{1}{2}Hg_{33,0} + \tfrac{1}{3}Hg_{33}\varrho^{-1}\varrho_{,0}
\end{aligned} \qquad (B.1)$$

The above formulas are written in the coordinate system of (4.13). We see that when $\sigma_{\alpha\beta} = 0$ then h, g_{12}, g_{22} and g_{33} are all proportional to $\varrho^{-2/3}$ with the proportionality factor independent of x^0.

Therefore here $(k/h)_{,0} = (g_{12}^2/h^2)_{,0} = 0$, and it is possible to achieve $g_{12} = 0$ by the coordinate transformation mentioned in section 5.
Consequently, the shearfree metrics (4.13) may be put in the form:

$$ds^2 = H^{-2}(dx^0 + x^2 dx^1)^2 - \varrho^{-2/3}[\alpha(x^2)(dx^1)^2 + \beta(x^2)(dx^2)^2 + \gamma(x^2)(dx^3)^2] \qquad (B.2)$$

where α, β, γ are unknown functions of x^2 obeying $\alpha \cdot \beta \cdot \gamma = 1$ (in consequence of (2.4)), and H with ϱ are the only functions allowed to depend on x^0.
We started with ten field equations. Two of them have been used to obtain (4.10), and one more to obtain (4.11). Then, after the metric was put in the form (4.13), two more field equations turned to identities. We are left with five field equations plus the equation

$$\zeta = -\frac{1}{2}\varkappa (x^2)^2 + A \qquad (B.3)$$

where ζ is given by (4.8). Some of the field equations still contain pressure terms on their right hand side, so one of them may be used to cancel the pressure in the others. Then we stay with four second-order equations, and (B.3) which is a first order one. It will be more convenient to represent (B.3) by two second order equations $\zeta_{,0} = 0, \zeta_{,2} = -\varkappa x^2$. So finally we have six equations linear in the six second derivatives $H_{,00}, H_{,02}, H_{,22}, \varrho_{,00}, \varrho_{,02}$ and $\varrho_{,22}$. They can be solved algebraically for those second derivatives, i.e. represented in the form:

$$H_{,ij} = F_{ij}(H, H_{,i}, \varrho, \varrho_{,i}, \varphi), \quad \varrho_{,ij} = G_{ij}(H, H_{,i}, \varrho, \varrho_{,i}, \varphi) \qquad (B.4)$$

where φ stands for other variables than H and ϱ. The functions F_{ij} and G_{ij} are of first order in H and ϱ.
To the equations (B.4) we apply the integrability conditions $F_{00,2} - F_{02,0} = G_{00,2} - G_{02,0} = F_{02,2} - F_{22,0} = G_{02,2} - G_{22,0} = 0$, and use (B.4) to remove all second derivatives of H and ϱ that reappeared. In this way four equations of first order in H and ϱ are obtained. Indeed, it is enough to use the last two integrability conditions to obtain the result $H_{,0} = 0 = \varrho_{,0}$ which means stationarity.
The detailed calculations are too complex to be worth presenting.

ACKNOWLEDGMENTS

This work was done and first presented in the Sternberg Astronomical Institute in Moscow in the spring of 1974. I thank Dr. Leonid P. Grishchuk for inviting me there.

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