Some solutions of the Einstein field equations for a rotating perfect fluid

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(Received 4 September 1973)

The equations of isentropic rotational motion of a perfect fluid are investigated with use of Darboux' theorem. It is shown that, together with the equation of continuity, they guarantee the existence of four scalar functions on space–time, which constitute a dynamically distinguished set of coordinates. It is assumed that in this coordinate system the metric tensor is constant along the lines tangent to velocity and vorticity fields. Under these assumptions a complete set of solutions of the field equations with $T_{ij} = (\epsilon + p)u_i u_j - pg_{ij}$ is found. They divide into three families, first of which contains six types of new solutions with nonzero pressure. The second family contains only the Gödel's solution, and the third one, only the Lanczos' solution. Symmetry groups, exterior metrics, type of conformal curvature, geometrical and physical properties of the new solutions are investigated. A short review of other models of rotating matter is given.

INTRODUCTION

It was not long after the creation of the general relativity theory that people tried to construct a solution of the Einstein field equations for rotating matter. The problem was interesting both from theoretical and observational point of view because nobody knew how to describe the rotational motion in the formalism of general relativity while many stars and galaxies exhibited visible rotation. Today even the possibility of rotation of the universe in the large is admitted.[1]

However, for quite a long time models of rotating matter were constructed under very special assumptions. The authors either used the method of "slow rotation" approximation (first paper by J. Lense and H. Thirring[2] in 1918) or assumed the energy-momentum tensor corresponding to dust (K. Lanczos[3] in 1924 and many others). It was not till 1967 that M. Trümper[4] clearly stated the problem of searching for solutions with pressure different from zero, but he has just written down the field equations and stopped after arriving at some general statements. There were a few papers whose authors went further but they left the problem behind when the equations were simplified and nearly integrated (i.e., there remained only one or two equations to be solved). They gave at most special cases of solutions which were mathematically simple (e.g., J. Stewart and G. F. R. Ellis,[5] J. Wainwright.[6])

Until 1972, in fact, just two complete results were obtained—by H. D. Wahlquist[7] in 1968 and E. Herlt[8] in 1972. The aim of the present paper was to supply new metrics of this kind. I have used the method of description of the isentropic rotational motion of the perfect fluid introduced by J. Plebański.[9] Under the assumptions, which are clearly stated in Sec. 1, the field equations were completely integrated. The resulting metrics divide into three families, the first of which contains six types of new solutions with nonzero pressure. Each of the other families contains just one solution known before.

The first family solutions are investigated in detail. Their symmetry groups, exterior metrics, type of conformal curvature, geometrical and physical properties are established and discussed. A few special cases are investigated in more detail. I also give a short review of the solutions found by other authors.

Most of the material presented in Sec. 1 is taken from J. Plebański's paper.[9]

1. THE EQUATIONS OF MOTION AND DYNAMICALLY DISTINGUISHED COORDINATES

Throughout the paper we shall use the signature $(+---)$. The equations of motion of a perfect fluid have the form:

$$T^{\alpha\beta}{}_{;\beta} = 0, \tag{1.1}$$

where

$$T^{\alpha\beta} = (\epsilon + p)\,u^\alpha u^\beta - pg^{\alpha\beta}. \tag{1.2}$$

The quantity $(\epsilon + p)$ is called the enthalpy density. Let $\mathcal{H}$ denote the enthalpy per unit mass,

$$\mathcal{H} = (\epsilon + p)/\rho, \tag{1.3}$$

where ρ is the density of the rest-mass. Independently of (1.1) the conservation of the total rest mass is postulated:

$$(\rho u^\alpha)_{;\alpha} = 0. \tag{1.4}$$

By virtue of (1.3) and (1.4) Eqs. (1.1) take the form

$$0 = T_\alpha{}^\beta{}_{;\beta} = \rho u^\beta (\mathcal{H} u_\alpha)_{;\beta} - p_{,\alpha}. \tag{1.5}$$

The enthalpy in phenomenological thermodynamics obeyed the following identity:

$$d\mathcal{H} = (1/\rho)\,dp + TdS. \tag{1.6}$$

This equation may be considered to be the definition of temperature and entropy in general relativity. Namely, only two of the state functions $(\mathcal{H}, \rho, p)$ can be independent. Therefore the form $(d\mathcal{H} - (1/\rho)\,dp)$ has an integrating factor which we denote by $1/T$ and its inverse we call the temperature. Then the form $(1/T)(d\mathcal{H} - 1/\rho)dp)$ is a total differential of a function S which we call entropy.

With the help of (1.6) we get in (1.5)

$$\rho\,[u^\beta(\mathcal{H}u_\alpha)_{;\beta} - \mathcal{H}_{,\alpha} + TS_{,\alpha}] = 0. \tag{1.7}$$

Now the identities $u^\alpha u_\alpha = 1$ and $u^\beta u_{\beta;\alpha} = 0$ allow us to write (1.7) as

$$[(\mathcal{H}u_\alpha)_{,\beta} - (\mathcal{H}u_\beta)_{,\alpha}]u^\beta + TS_{,\alpha} = 0. \tag{1.8}$$

These are the equations of motion of a perfect fluid in a form equivalent to (1.1).

We shall confine ourselves to isentropic motions, where $S_{,\alpha} = 0$. Then (1.3) and (1.6) imply

$$d[(\epsilon + p)/\rho] = (dp)/\rho. \tag{1.9}$$

We see that $d\epsilon = [(\epsilon + p)/\rho]d\rho$ and so $\epsilon = \epsilon(\rho)$, $p=p(\rho)$; in other words, $\rho = \rho(p)$ and $\epsilon = \epsilon(p)$. Thus (1.9) is an ordinary differential equation, and we can integrate it to obtain

$$\epsilon + p = \rho c^2 \left(H_0 + \frac{1}{c^2}\int_0^p \frac{dp}{\rho(p)}\right) \tag{1.10}$$

where $H_0 = \text{const}$. If we assume that $\epsilon(p=0) = \rho(0)c^2$ then $H_0 = 1$. Let us denote

$$H \overset{\text{def}}{=} H_0 + \frac{1}{c^2}\int_0^p \frac{dp}{\rho(p)}. \tag{1.11}$$

Then Eqs. (1.8) with $S_{,\alpha} = 0$ take the form

$$[(Hu_\alpha)_{,\beta} - (Hu_\beta)_{,\alpha}]u^\beta = 0. \tag{1.12}$$

Now we recall two theorems which will be useful later. We give both of them in the special case of a four-dimensional manifold. Their general forms can be found in Refs. 10–12.

Theorem 1 (Darboux): Let ω be a differential form of the 1st order, then

(1) $(d\omega \wedge d\omega \neq 0) \Longleftrightarrow$ (there exists the set of functions σ, τ, ξ, η such that $\omega = \sigma d\tau + \eta d\xi$);

(2) $(d\omega \wedge d\omega = 0$ but $\omega \wedge d\omega \neq 0) \Longleftrightarrow (\sigma = 1$ above);

(3) $(\omega \wedge d\omega = 0$ but $d\omega \neq 0) \Longleftrightarrow (\xi = 1$ in (1));

(4) $(d\omega = 0) \Longleftrightarrow (\sigma = \xi = 1$ in (1)).

Its proof is given in Ref. 10.

For an antisymmetric tensor $F_{\alpha\beta}$ the following form can be defined:

$$Pf(F_{\alpha\beta}) = \tfrac{1}{8}\,\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} \tag{1.13}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol. We have

Theorem 2:

$$[Pf(F_{\alpha\beta})]^2 = \det(F_{\alpha\beta}).$$

The proof can be found in Refs. 11 and 12.

Now let $F_{\alpha\beta} \overset{\text{def}}{=} (Hu_\alpha)_{,\beta} - (Hu_\beta)_{,\alpha}$. We see from (1.12) that $\det(F_{\alpha\beta}) = 0$ and so from Theorem 2 $Pf(F_{\alpha\beta}) = 0$ which means that $F_{[\alpha\beta}F_{\gamma\delta]} = 0$.

Let us define $\omega = Hu_\alpha dx^\alpha$. Then $F_{\alpha\beta}dx^\alpha \wedge dx^\beta = -2d\omega$, and so $d\omega \wedge d\omega = 0$.

Now Theorem 1 implies that there exist functions τ, ξ, η such that $\omega = d\tau + \eta d\xi$, i.e.,

$$Hu_\alpha = \tau_{,\alpha} + \eta\xi_{,\alpha}, \tag{1.14}$$

$$F_{\alpha\beta} = \xi_{,\alpha}\eta_{,\beta} - \xi_{,\beta}\eta_{,\alpha}. \tag{1.15}$$

This representation of Hu_α is introduced and discussed in more detail in Ref. 9.

When $F_{\alpha\beta} = 0$ we call the motion irrotational. When $F_{\alpha\beta} \neq 0$ we call it rotational. To distinguish rotational and irrotational motions we can use as well the vorticity vector w^α:

$$w^\alpha = -(-g)^{-1/2}\epsilon^{\alpha\beta\gamma\delta}u_\beta u_{\gamma,\delta}. \tag{1.16}$$

In the local inertial frame at a point p [where $u^\alpha = \delta^\alpha_0$, $g_{\alpha\beta}(p) = \operatorname{diag}(+1,-1,-1,-1)$] the vector w^α has the components $w^\alpha = (0, -(1/c)\mathbf{W})$ where $\mathbf{W} = \operatorname{rot}\mathbf{v}$, $\mathbf{v}$ — the Newtonian velocity vector. Thus the differentiation between rotational and irrotational motions based on w^α agrees with that in Newtonian physics. Moreover, we have

Theorem 3:

$$(F_{\alpha\beta} = 0) \Longleftrightarrow (w^\alpha = 0).$$

Therefore, this differentiation agrees with that based on $F_{\alpha\beta}$, too. Consequently, we can consider $F_{\alpha\beta}$ to be the angular velocity tensor. But there is a definition of the angular velocity tensor, given by J. Ehlers[13,14]

$$\Omega_{\alpha\beta} = u_{[\alpha;\beta]} - u_{[\alpha;|\rho|}u^\rho u_{\beta]} \tag{1.17}$$

With the help of the equations of motion (1.12) it is easy to show that

$$F_{\alpha\beta} = 2H\Omega_{\alpha\beta}, \tag{1.18}$$

so our definition of rotational motion agrees with that of Ehlers.

From now on we shall deal with rotating matter only, so we assume

$$F_{\alpha\beta} \neq 0. \tag{1.19}$$

It means that all the three functions in (1.14) have linearly independent gradients. Equation (1.12) implies that $u^\alpha\xi_{,\alpha} = u^\alpha\eta_{,\alpha} = 0$. This, together with the equation of continuity $[(-g)^{1/2}\rho u^\alpha]_{,\alpha} = 0$, allows us to define the fourth function ζ in the following way:

$$(-g)^{1/2}\rho u^\alpha = \epsilon^{\alpha\beta\gamma\delta}\xi_{,\beta}\eta_{,\gamma}\zeta_{,\delta}. \tag{1.20}$$

(For the details see again.[9]) By contraction of (1.14) and (1.20) we get

$$g = -g^{-2}H^{-2}\left(\frac{\partial(\tau,\xi,\eta,\zeta)}{\partial(x^0,x^1,x^2,x^3)}\right)^2. \tag{1.21}$$

If (1.14) and (1.20) are assumed, then the equations of motion and continuity are just identities.

Of course we can use the functions (τ, ξ, η, ζ) as new coordinates. If we do, then (1.14), (1.20), and (1.21) reduce to

$$u^\alpha = H\delta^\alpha_0, \tag{1.22}$$

$$u_\alpha = H^{-1}\delta^0_\alpha + x^2H^{-1}\delta^1_\alpha, \tag{1.23}$$

$$g = -\rho^{-2}H^{-2}. \tag{1.24}$$

We also have

$$w^\alpha = \rho H^{-1}\delta^\alpha_3 \tag{1.25}$$

and, since $u_\alpha = g_{\alpha\rho}u^\rho$,

$$\begin{aligned} g_{00} &= H^{-2},\\ g_{01} &= x^2H^{-2},\\ g_{02} &= g_{03} = 0. \end{aligned} \tag{1.26}$$

The functions (τ, ξ, η, ζ) are not unique. The coordinate transformations preserving the properties (1.22)–(1.26) are of the form:

$$x^0 = x^{0'} - S(x^{1'}, x^{2'}),$$
$$x^1 = F(x^{1'}, x^{2'}),$$
$$x^2 = G(x^{1'}, x^{2'}), \tag{1.27}$$
$$x^3 = x^{3'} + T(x^{1'}, x^{2'}),$$

where T is completely arbitrary, while F and G must obey the equation

$$F_{,1'} G_{,2'} - F_{,2'} G_{,1'} = 1. \tag{1.28}$$

S is fixed by the equations

$$GF_{,1'} - x^{2'} = S_{,1'}, \tag{1.29}$$
$$GF_{,2'} = S_{,2'}.$$

We see that one of the functions F and G is arbitrary and once it is fixed, the other is given by (1.28). Therefore, together with T we have two arbitrary functions in (1.27). Notice that all functions in (1.27) depend only on two variables x^1 and x^2.

Now the idea arises: If the whole metric tensor also depends only on x^1 and x^2, then the transformations (1.27) may allow us to simplify the metric further. So we assume

$$\left(\frac{\partial}{\partial x^0}\right) g_{\alpha\beta} = \left(\frac{\partial}{\partial x^3}\right) g_{\alpha\beta} = 0. \tag{1.30}$$

This condition is covariant with the transformations (1.27). As a consequence of (1.22) and (1.25) it can be written as

$$\partial_u g_{\alpha\beta} = \partial_w g_{\alpha\beta} = 0, \tag{1.31}$$

where $\partial_u = u^\alpha(\partial/\partial x^\alpha)$, $\partial_w = w^\alpha(\partial/\partial x^\alpha)$.

These two assumptions are sufficient to integrate the Einstein field equations for the metric fulfilling (1.24) and (1.26) to the very end. No additional simplifying assumptions are made here. We shall explain the geometrical meaning of the assumptions (1.31) later. Notice that the first of (1.31) means that u^α is colinear with a timelike Killing vector, so the expansion and shear of the velocity field vanish.

2. FIRST INTEGRALS OF THE FIELD EQUATIONS AND CLASSIFICATION OF THE SOLUTIONS

Since there are two arbitrary functions in (1.27), we can expect that it will be possible to make two more components of the metric tensor equal to 0. It is really the case. If we choose F, G, and T so that the equations

$$g^{22} F_{,1'} F_{,2'} - g^{12}(F_{,1'} G_{,2'} + F_{,2'} G_{,1'}) + g^{11} G_{,1'} G_{,2'} = 0 \tag{2.1}$$

and

$$T_{,1'} = -(g_{13}/g_{33}) F_{,1'} - (g_{23}/g_{33}) G_{,1'} \tag{2.2}$$

hold, then in the new coordinates $(x^{0'}, x^{1'}, x^{2'}, x^{3'})$ we have, in addition to (1.24) and (1.26),

$$g_{12} = g_{13} = 0 \tag{2.3}$$

The set of Eqs. (1.28)–(2.1) makes sense no matter what $g_{\alpha\beta}$ is. Equation (2.2) makes sense because. Theorem 3, (1.19), and (1.25) imply that $g_{33} \neq 0$.

Substituting (2.3) in (2.1) and (2.2), we get a new set of equations which determines the transformations (1.27) preserving all the properties (1.24), (1.26), and (2.3).

From now on there is no arbitrary function in (1.27).

It is time to use the field equations. If the right-hand side of the equations

$$R^\alpha{}_\beta = (\kappa/c^2)(T^\alpha{}_\beta - \tfrac{1}{2}\delta^\alpha{}_\beta T) + \Lambda\delta^\alpha{}_\beta, \quad \kappa = 8\pi k/c^2, \tag{2.4}$$

is given by (1.2), (1.11), and (1.10), then it must be $R^0{}_3 = R^1{}_3 = 0$. These two equations when integrated yield the result

$$g_{23} = K(x^2) g_{33}, \tag{2.5}$$

where K is an arbitrary function of one variable. Now we can verify that the coordinate transformation

$$x^0 = x^{0'} + x^{1'} x^{2'},$$
$$x^1 = x^{2'}, \quad x^2 = -x^{1'}, \tag{2.6}$$
$$x^3 = x^{3'} - \int K(x^2)\, dx^2$$

fulfills all Eqs. (1.28), (1.29), (2.1), (2.2), and yields, in addition,

$$g_{23} = 0. \tag{2.7}$$

In the new coordinates it is easier to compute the Ricci tensor. From the equations $R^1{}_0 = R^2{}_0 = 0$, we easily find that

$$g_{33} = G\rho^{-1} H^3, \quad G = \text{const} < 0. \tag{2.8}$$

We classify the solutions into three families:

Family I in which

$$\rho_{,\alpha} \neq 0, \quad p \neq 0. \tag{2.9}$$

Family II in which $\rho_{,\alpha} = 0$ and consequently

$$H_{,\alpha} = p_{,\alpha} = 0. \tag{2.10}$$

Family III in which

$$p = 0. \tag{2.11}$$

This classification is invariant. We are going to discuss each family separately.

3. THE FIRST FAMILY OF SOLUTIONS

Using the complete set of the field equations one can prove that by a suitable choice of coordinate system we obtain

$$\rho = \rho(x^2), \quad \text{and so} \quad H = H(x^2), \quad p = p(x^2). \tag{3.1}$$

Then the field equations reduce to the set of ordinary differential equations, and after integration they yield

$$ds^2 = H^{-2}(dx^0)^2 + 2x^2 H^{-2} dx^0 dx^1 + [(x^2)^2 - W/G] H^{-2}(dx^1)^2 + (W\rho H)^{-1}(dx^2)^2 + G\rho^{-1}H^3(dx^3)^2, \tag{3.2}$$

where

$$W = (G + \kappa(x^2)^2 + Bx^2 + E, \quad B, E = \text{const}, \tag{3.3}$$

$$\rho = D\frac{H^5}{W} \exp\left(\int \frac{Gx^2}{W}\, dx^2\right), \quad D = \text{const} < 0, \tag{3.4}$$

$$H = |Mu_1 + Nu_2|^{1/3}, \quad M, N = \text{const} \tag{3.5}$$

u_1 and u_2 are the linearly independent solutions of the equation

$$u_{,22} - \frac{W_{,2} - Gx^2}{W} u_{,2} + \frac{3}{4}\left(- \frac{W_{,22}}{W} + \frac{W_{,2}^2}{W^2} - \frac{Gx^2 W_{,2}}{W^2} + \frac{G}{W}\right) u = 0. \quad (3.6)$$

The pressure p is given by the formula resulting from (1.11):

$$p = c^2 \int \rho H_{,2} dx^2 + p_0. \quad (3.7)$$

Whenever an inequality for a constant appears above or below, it results from two conditions:

(1) $\rho, p, H > 0$.

(2) The signature of the metric is $(+ - - -)$.

The absolute value in (3.5) is needed to assure that $H > 0$.

The solutions of the first family divide into six types according as to whether W has two complex roots, two real roots, one real root or degenerates to a polynomial of a lower degree.

It is clear from (3.4) that when the sign of W is not the same for all values of x^2, then ρ may be positive only in some range of values of x^2. The boundaries of this range (i.e., the roots of W) are singular points of ρ, and outside of this range ρ would be negative. In such a situation we have to find some exterior metric and match it to (3.2) so that the complete space–time has no singularities. This is done in Sec. 7. In the formulas given below an auxiliary constant $a \overset{\text{def}}{=} G/(G + \kappa)$ is occasionally used.

Type I

$$W = (G + \kappa)(x^2 - b)(x^2 - c'), \qquad c' = b^*, \qquad a > 1. \quad (3.8)$$

$$u_1 = u + u^*, \qquad u_2 = -i(u - u^*). \quad (3.9)$$

$$u = \left(\frac{x^2 - b}{c' - K}\right)^{\beta} \left(\frac{x^2 - c'}{b - K}\right)^{\gamma} \times F\left(\alpha + \beta + \gamma, \alpha' + \beta + \gamma, 1 + \beta - \beta', \frac{x^2 - b}{c' - b}\right), \quad (3.10)$$

$K = \text{const} = K^*$, $F(.,.,.,.)$—the hypergeometric function.

$$\left.\begin{matrix}\alpha \\ \alpha'\end{matrix}\right\} = \tfrac{1}{2}[a - 3 \pm (a^2 - 3a + 3)^{1/2}]. \quad (3.11)$$

$$\left.\begin{matrix}\beta \\ \beta'\end{matrix}\right\} = \frac{1}{2(b - c')}\{-(a - 2)b - 2c' \mp [a^2 b^2 + (b - c')(b - c' - ab)]^{1/2}\}. \quad (3.12)$$

$$\left.\begin{matrix}\gamma \\ \gamma'\end{matrix}\right\} = \frac{1}{2(b - c')}\{2b + (a - 2)c' \pm [a^2 c'^2 + (b - c')(b - c' + ac')]^{1/2}\} = \left\{\begin{matrix}\beta^* \\ \beta'^*\end{matrix}\right. . \quad (3.13)$$

Type II

$$W = (G + \kappa)(x^2 - b)(x^2 - c'), \qquad b \text{ and } c' \text{ real}, \qquad b < c'. \quad (3.14)$$

u_1 is given by (3.10), u_2 is the standard second linearly independent solution.[15-17] The formulas for $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ are identical with (3.11)–(3.13). This time no analogue of the equations $\gamma = \beta^*$ and $\gamma' = \beta'^*$ holds.

The sign of W is not constant. For $x^2 = b$ and $x^2 = c'$ the solution has singularities. When $a < 0$, the density of matter is positive in the region $b < x^2 < c'$; when $a > 1$ it is positive in the nonconnected region $x^2 < b$ and $x^2 > c'$.

Type III

$$W = (G + \kappa)(x^2 - b)^2, \qquad b \neq 0, \qquad a > 1. \quad (3.15)$$

$$u_i = (x^2 - b)^{q_i} F\left(\tfrac{3}{2} - q_i, 4 - a - 2q_i, \frac{ab}{x^2 - b}\right), \qquad i = 1, 2, \quad (3.16)$$

$F(.,.,.)$—the confluent hypergeometric function.

$$\left.\begin{matrix}q_1 \\ q_2\end{matrix}\right\} = \tfrac{1}{2}[3 - a \pm (a^2 - 3a + 3)^{1/2}]. \quad (3.17)$$

W has a constant sign, but $x^2 = b$ is a singular point of the solution.

Type IV

$$W = (G + \kappa)(x^2)^2, \qquad a > 1. \quad (3.18)$$

$$u_i = (x^2)^{q_i}, \qquad i = 1, 2, \qquad q_i \text{ given by (3.17).} \quad (3.19)$$

Again W has a constant sign, but $x^2 = 0$ is a singular point.

Type V

$$W = Bx^2 + E. \quad (3.20)$$

Here the coordinates can be chosen so that $B = \kappa$. We denote $E = \kappa E_0$ and we get

$$u_i = [\exp(x^2 + E_0)](-x^2 - E_0)^{q_i} F(q_i + E_0 - 1, 2q_i + E_0 - 1, -x^2 - E_0), \qquad i = 1, 2, \quad (3.21)$$

$$\left.\begin{matrix}q_1 \\ q_2\end{matrix}\right\} = \tfrac{1}{2}[2 - E_0 \pm (E_0^2 - E_0 + 1)^{1/2}]. \quad (3.22)$$

The density of matter is positive in the region $x^2 < -E_0$.

Type VI

$$W = E = \text{const} < 0, \quad (3.21)$$

$$u_1 = F(\tfrac{3}{8}, \tfrac{1}{2}, (\kappa/2E)(x^2)^2), \qquad u_2 = x^2 F(\tfrac{7}{8}, \tfrac{3}{2}, (\kappa/2E)(x^2)^2). \quad (3.22)$$

4. THE SECOND FAMILY OF SOLUTIONS

Here the field equations reduce to one partial differential equation. Again it can be shown that by a suitable choice of coordinates the metric can be made dependent only on one variable x^2. Then the solution appears to be unique (exact to coordinate transformations):

$$ds^2 = H^{-2}(dx^0 + x^2 dx^1)^2 - \tfrac{1}{2} H^{-2}(x^2)^2(dx^1)^2 - [\kappa\rho H(x^2)^2]^{-1}(dx^2)^2 - 2\kappa\rho^{-1}H^3(dx^3)^2, \quad (4.1)$$

where

$$\rho, p > 0 \quad \text{are arbitrary constants,} \quad (4.2)$$

$$H = 1 + (p/c^2\rho) = \text{const}, \quad (4.3)$$

$$\Lambda = \tfrac{1}{2}\kappa[\rho - (p/c^2)] \text{ is the cosmological constant.} \quad (4.4)$$

The metric (4. 1) was found for the first time by H. M. Raval and P. C. Vaidya[18] and is a generalization to the case of constant but nonzero pressure of the well-known solution of Gödel[19] [if $p = 0$ then (4. 1) is precisely the Gödel's metric]. It is the limiting case $a = 2$, $M = 0$ of the Type IV solution from the first family.

5. THE THIRD FAMILY OF SOLUTIONS

One sees that when $p = 0$ (and consequently $\epsilon = \rho c^2$), then the equations of motion (1. 1) can be written in the form $(u_{\alpha,\beta} - u_{\beta,\alpha}) u^\beta = 0$, and thus are the special case $H = 1$ of Eq. (1. 12). Therefore all the formulas up to (2. 8) hold for dust if $p = 0$ and $H = 1$ is substituted there. This time again one verifies that such coordinates exist, in which $\rho = \rho(x^2)$. The solution is unique:

$$ds^2 = (dx^0)^2 + 2x^2 dx^0 dx^1 + x^2(x^2+1)(dx^1)^2 + \frac{e^{x^2}}{\kappa a x^2}(dx^2)^2 - \frac{\kappa}{a} e^{x^2}(dx^3)^2, \quad (5.\ 1)$$

where

$$\rho = a e^{-x^2}, \quad a = \text{const} > 0. \quad (5.\ 2)$$

The metric has the proper signature in the region $x^2 < 0$. Here necessarily $\Lambda = 0$. This metric was found by K. Lanczos[3] in 1924 and was the first exact solution with rotating matter in the history of relativity. It was rediscovered next by W. J. van Stockum[20] in 1937 and J. P. Wright[21] in 1965. In fact, Lanczos and Wright also found the generalization of (5. 1) to the case $\Lambda \neq 0$, but this generalization does not fulfill the second of (1. 30).

Equation (5. 1) is the limiting case $E_0 = N = 1$, $M = \frac{3}{2}$ of the Type V solution from the first family (represented in slightly different coordinates, related to those of Type V by the transformation $x^0 = x^{0\prime} + x^1$, $x^2 = x^{2\prime} - 1$).

6. SYMMETRIES OF THE SOLUTIONS

I will not investigate the second and third family of solutions as they have been considered by many other authors.[3,6,18-22] The symmetry group for all the types of the first family solutions consists of the following transformations:

$$\begin{aligned} x^0 &= x^{0\prime} + t_0, \\ x^1 &= x^{1\prime} + t_1, \\ x^2 &= x^{2\prime}, \\ x^3 &= x^{3\prime} + t_3 \end{aligned} \quad (6.\ 1)$$

with $t_0, t_1, t_3 = \text{const}$.

Thus it is 3-parametric Abelian group with the Killing vectors $k^\mu_{(i)} = \delta^\mu_i$, $i = 0, 1, 3$. It acts simply transitively on the timelike hypersurfaces $x^2 = \text{const}$. Such groups were classified by Bianchi into nine types.[23] (In fact, Bianchi classification is usually applied to groups acting on spacelike hypersurfaces, but no specific signature of the metric on the hypersurface is assumed and therefore such a classification is true for timelike homogeneous hypersurfaces, too). Since the group of transformations (6. 1) is Abelian, it is of Bianchi Type I, and the hypersurfaces $x^2 = \text{const}$ are flat. Notice that the group (6. 1) is completely characterized by four statements:

(1) There exist three commuting Killing vectors $\underset{(0)}{k^\mu}, \underset{(1)}{k^\mu}, \underset{(3)}{k^\mu}$ whose integral lines are the coordinate lines (x^0, x^1, x^3). The x^0 line is timelike.

(2) The x^2 line is orthogonal to all the three (x^0, x^1, x^3) lines.

(3) $g_{\mu\nu} \underset{(0)}{k^\mu} \underset{(3)}{k^\nu} = g_{\mu\nu} \underset{(1)}{k^\mu} \underset{(3)}{k^\nu} = 0.$

(4) $g_{\mu\nu} \underset{(0)}{k^\mu} \underset{(1)}{k^\nu} \neq 0.$

7. EXTERIOR SOLUTIONS

It is reasonable to look for exterior solutions having the same symmetry group as the interior ones to which they are to be matched. Taking Statements (1)–(4) above as axioms, we arrive at the metric form

$$ds^2 = (\alpha dx^0 + \beta dx^1)^2 - (\gamma dx^1)^2 - (\delta dx^2)^2 - (\epsilon dx^3)^2, \quad (7.\ 1)$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ are functions of one variable x^2. Two cases must be considered separately: $(\beta/\alpha)_{,2} = 0$ and $(\beta/\alpha)_{,2} \neq 0$.

In the first case the metric (7. 1) is static. The only nonflat solution of the empty space field equations (with $\Lambda = 0$) is then

$$ds^2 = A_0^2 (x^2)^{2a}(dx^0 + S dx^1)^2 - A_1^2 (x^2)^{-2(a-1)}(dx^1)^2 - A_2^2 (x^2)^{2a(a-1)}(dx^2)^2 - A_3^2 (x^2)^{2a(a-1)}(dx^3)^2, \quad (7.\ 2)$$

where $A_0, \ldots, A_3, S, a = \text{const}$. If $a = 0$ or $a = 1$, then (7. 2) is flat.

In the second case (7. 1) is stationary, nonstatic, and (β/α) can be taken as a new coordinate x^2. Then (7. 1) becomes closely analogous to (3. 2). The solutions of the empty space Einstein equations with the Λ term divide into four types and are given by the formulas

$$ds^2 = f^{-2}(dx^0)^2 + 2x^2 f^{-2} dx^0 dx^1 + [(x^2)^2 - V] f^{-2}(dx^1)^2 - \frac{J^2}{s f^6} \exp\left(-\int \frac{x^2}{V} dx^2\right)(dx^2)^2 - \frac{V}{s f^2} \exp\left(-\int \frac{x^2}{V} dx^2\right)(dx^3)^2, \quad (7.\ 3)$$

where $J^2 \neq 0 < s$ are constants and

$$V = (x^2)^2 + p x^2 + q, \quad p, q = \text{const}, \quad (7.\ 4)$$

$$f = (P v_1 + Q v_2)^{1/3}, \quad P, Q = \text{const}. \quad (7.\ 5)$$

v_1 and v_2 are two linearly independent solutions of the equation

$$v_{,22} - V^{-1}(V_{,2} - x^2) v_{,2} + \frac{3}{4}\left(-\frac{V_{,22}}{V} + \frac{V_{,2}^2}{V^2} - \frac{x^2 V_{,2}}{V^2} + \frac{1}{V}\right) v = 0. \quad (7.\ 6)$$

Now compare (7. 3)–(7. 6) with (3. 2)–(3. 6) and note the similarity.

Type A

$$V = (x^2 - p_0)(x^2 - q_0), \quad q_0 = p_0^*. \quad (7.\ 7)$$

$$\begin{aligned} v_1 &= \left(\frac{x^2 - p_0}{q_0 - L}\right)^\mu \left(\frac{x^2 - q_0}{p_0 - L}\right)^\nu, \\ v_2 &= \left(\frac{x^2 - p_0}{q_0 - L}\right)^{\mu'} \left(\frac{x^2 - q_0}{p_0 - L}\right)^{\nu'}, \quad L = L^* = \text{const}. \end{aligned} \quad (7.\ 8)$$

$$\left.\begin{matrix}\mu \\ \mu'\end{matrix}\right\} = \frac{1}{2(p_0 - q_0)}\,[p_0 - 2q_0 \pm (p_0^2 - p_0 q_0 + q_0^2)^{1/2}]. \quad (7.\ 9)$$

$$\left.\begin{matrix}\nu \\ \nu'\end{matrix}\right\} = \frac{1}{2(p_0 - q_0)}\,[2p_0 - q_0 \mp (p_0^2 - p_0 q_0 + q_0^2)^{1/2}] = \left\{\begin{matrix}\mu^* \\ \mu'^*.\end{matrix}\right. \quad (7.\ 10)$$

$$\Lambda = (s/3J^2)PQ(p_0^2 - p_0 q_0 + q_0^2)(p_0 - L)^k (q_0 - L)^l, \tag{7.11}$$

where $k = -q_0/(p_0 - q_0)$, $l = p_0/(p_0 - q_0)$.

Type B

$$V = (x^2 - p_0)(x^2 - q_0), \quad p_0 \text{ and } q_0 \text{ real}, \quad p_0 < q_0. \tag{7.12}$$

The formulas for $v_1, v_2, \mu, \mu', \nu, \nu'$ are identical with (7.8)–(7.10), but now $\mu^* = \mu$, $\mu'^* = \mu'$. For $x^2 = p_0$ and $x^2 = q_0$ the metric has singularities, and it has the proper signature in the nonconnected region $x^2 < p_0$ and $x^2 > q_0$. The cosmological constant is given by (7.11).

Type C

$$V = (x^2 - p_0)^2, \quad p_0 \neq 0, \tag{7.13}$$

$$v_1 = |x^2 - p_0|^{3/2}, \quad v_2 = v_1 \exp[p_0/(x^2 - p_0)]. \tag{7.14}$$

$$\Lambda = (s/3J^2)PQp_0^2. \tag{7.15}$$

The signature is proper for all values of x^2, but $x^2 = p_0$ is a singular point.

Type D

$$V = (x^2)^2, \tag{7.16}$$

$$v_1 = |x^2|^{3/2}, \quad v_2 = |x^2|^{1/2}, \tag{7.17}$$

$$\Lambda = -(s/3J^2)Q^2. \tag{7.18}$$

Again the signature is proper everywhere, but $x^2 = 0$ is a singular point.

Now we have to say how these solutions are matched to the interior ones. The solutions of Types I, III, IV, and VI can have the exterior metric of Type A only. For the solutions of Types II and V the exterior metric is of Type A if the joining point $x^2 = r_0$ is at a distance from the singular point greater than some critical value. Otherwise the exterior metric is of Type B (or C), but both singularities of Type B metric (or the singular point of Type C metric) appear outside of matter.

It is interesting that all the four types of stationary exterior solutions can be obtained from the first family solutions by a formal substitution $\kappa = 8\pi k/c^2 \to 0$. Then Type I reduces to Type A, II reduces to B, III to C and IV to D. For obvious reasons the Types V and VI have no such analogs. The static metric (7.2) was discovered by E. Kasner[24] in 1925. Some cylindrically symmetric empty space solutions were considered by T. Lewis[25] in 1932. Kasner's solution (7.2) was one of them, but also there appeared Type A metric in the case $Q = 0$. The Type C metric in the case $P = 0$ is contained in Lewis' class,[25] but it is not given explicitly there. Finally, in the case $\Lambda = 0$ all the metrics from the present section are of the form given by Dautcourt, Papapetrou, and Treder.[26,27]

It should be emphasized that these references are rather accidental. The empty space metrics play only an auxiliary role in my paper, so I did not carry out any systematic search in the literature. In particular, I do not guarantee that the generalization of the stationary metrics to the case $\Lambda \neq 0$ is a new result. The generalization of (7.2) to the case $\Lambda \neq 0$ is unexpectedly very involved, so I do not present it here.

8. THE TYPE OF CONFORMAL CURVATURE

A special solution of Type IV from the first family is of Petrov type II. It is the metric

$$\begin{aligned} ds^2 = {} & N^{-2/3}(x^2)^{1-\sqrt{2}}[(dx^0)^2 + 2x^2 dx^0 dx^1 \\ & + 2(\sqrt{2}-1)(x^2)^2(dx^1)^2] + DN^{-2}(x^2)^{-5\sqrt{2}}(dx^2)^2 \\ & + (4DN^{2/3})^{-1}\kappa^2(x^2)^{-3\sqrt{2}}(dx^3)^2. \end{aligned} \tag{8.1}$$

All the other first-family solutions are of Petrov type I (general).

9. GEOMETRY OF THE SPACE-TIME

We have noticed in Sec. 6 that the hypersurfaces $x^2 =$ const are flat. Therefore, they can be embedded into the Minkowski space, i.e., they can be realized as some hypersurfaces $x^2 =$ const in the Minkowski space. It appears that this may be done only in four ways. The surfaces $x^0 =$ const, $x^2 =$ const can have the following geometry:

(1) Euclidean plane,

(2) surface of a cylinder with x^3 as the azimuthal angle and the x^1 line as the generator,

(3) surface of a cylinder with x^1 as the azimuthal angle and the x^3 line as the generator, parametrized by an observer at rest,

(4) the same surface as in (3), parametrized by an observer rotating about the axis of symmetry.

Since the velocity field is given by (1.22) and (1.23), we see that the particles of the fluid move inside the $x^2 = C$ hypersurface and follow the x^1 lines. Moreover, we know from (1.25) that the vorticity vector is tangent to x^3 lines, and we can compute quite easily the acceleration vector $\dot{u}_\alpha = H^{-1}H_{,2}\delta^2_\alpha$, which is tangent to x^2 lines.

This is enough to decide which case listed above is realized in our space–times of the first family. In cases (1) and (2) the acceleration, if present, is tangent to x^1 lines because the streamlines are straight. In cases (3) and (4) the acceleration has the direction of the radial line x^2, just as in our metrics. We decide that case (4) is a better model of our space–time since we do not expect that in the presence of rotating matter an observer at absolute rest would exist. It means that our space–time, when realized nonrelativistically, consists of co-axial cylinders rotating around an axis of symmetry with different angular velocities. All the physical quantities are constant on the surface of a fixed cylinder, but they vary from one cylinder to the other. The x^2-lines are geodesics orthogonal to the cylinders, x^1 lines are azimuthal circles, and x^3 lines are generators. Now we see that the second of the assumptions (1.30) meant just homogeneity in the direction of generators. We did not assume axial symmetry, but it resulted from the field equations.

10. PHYSICAL PROPERTIES OF THE SOLUTIONS

The velocity field has no shear or expansion, but it has rotation and acceleration. Rotation produces no red shift. According to Ehlers' formula[13] the red shift is equal to

$$(d\lambda)/\lambda = -\dot{u}_\alpha \delta_\perp x^\alpha, \tag{10.1}$$

where $\dot{u}_\alpha = H^{-1}H_{,2}\delta^2{}_\alpha$, and

$$\delta_\perp x^\alpha = (\delta^\alpha{}_\beta - u^\alpha u_\beta)\delta x^\beta, \tag{10.2}$$

δx^β being the infinitesimal vector pointing from the observer to the particle sending light signals to him. The

TABLE I. Models of rotating matter.

Lanczos[3] 1924 van Stockum[20] 1937 Wright[21] 1965 *Ellis[22] 1967 *Krasiński 1973	Gödel[19] 1949 *Wright[21] 1965 *Ozsváth[30] 1965 *Raval-Vaidya[18] 1966 *Ellis[22] 1967 *Wainwright[6] 1970 *Ozsváth[31] 1970 Bray[29] 1972 *Krasiński 1973	Ozsváth-Schücking[32] 1962 *Ozsváth[30] 1965	Ozsváth[30] 1965	
Maitra[33] 1966	Raval-Vaidya[18] 1966 Stewart-Ellis[5] 1968 Wainwright[6] 1970 *Krasiński 1973	Ellis[22] 1967 *Wainwright[6] 1970	Stewart-Ellis[5] 1968 *Wainwright[6] 1970	
Wahlquist[7] 1968	Wainwright[6] 1970	Ozsváth[31] 1970	Herlt[8] 1972	Krasiński 1973

red shift given by (10. 1) is strongly anisotropic and thus rather not realistic. However, it is not obvious that red shift computed with respect to distant sources of light would also have such a strong anisotropy.

It is interesting that $\rho = \rho(x^2)$ and $p = p(x^2)$, and so we have an equation of state $\rho = \rho(p)$ given in a parametric way, which resulted from the field equations. It might be unexpected as one usually considers an equation of state to be independent of the field equations. But if the metric tensor depends only on one variable, the equation of state is always determined by the field equations.

Now look at (3. 4), (3. 5), (3. 7), and (3. 8)–(3. 13). There are six independent arbitrary constants—D, M, N, G, b, c' entering the equation of state $\rho = \rho(p)$. In fact, this is a large class of equations of state.

One may expect simpler results when the parameters of the hypergeometric function in (3. 10) are such that $F(.\,, \,.\,, \,.\,, \,.\,)$ degenerates to a polynomial. J. Plebański[28] even suggested that then it would be possible to obtain the equation of state in the form of the van der Waals isotherms.

This question has not been investigated.

In Type IV solutions if $M = 0$ or $N = 0$ then ρ and p obey the polytrope type equation of state $p \cdot \rho^{-\gamma} = \text{const}$, with $[5a - 6 + \epsilon(a^2 - 3a + 3)^{1/2}]\gamma = 6(a-1)$, where $\epsilon = +1$ corresponds to $N = 0$ and $\epsilon = -1$ corresponds to $M = 0$. The condition $a > 1$ implies that $\gamma < 0$, $1 < \gamma < \frac{4}{3}$ or $\gamma > \frac{3}{2}$.

11. A SURVEY OF MODELS OF ROTATING PERFECT FLUID OR DUST

This survey is made in the form of a table. Each "cell" of the table represents one solution obtained by different authors. A star preceding author's name means that he knew his predecessors and did not expect to be the first inventor of the solution. There is no star at Bray's name in the "Gödel's cell" because his solutions, when the electromagnetic field is absent, reduce precisely to the metric of Gödel, but this fact was not indicated in Bray's paper.[29]

For each of the solutions the coordinates (τ, ξ, η, ζ) from (1. 14), (1. 20), and (1. 21) can be introduced, but this might be a subject of another paper.

No approximate solutions are taken into account.

A large expansion of this article is currently submitted to Acta Physica Polonica.

ACKNOWLEDGMENTS

I am indebted to Professor Jerzy Plebański for pointing out this subject to me as well as for constant encouragement and valuable advice while writing this paper. It owes much to his remarks and patient discussions with me. I deeply appreciate the kind help of Dr. Marek Demiański in tracing the references. My special thanks to Dr. Leonid Grishchuk from the Sternberg Astronomical Institute in Moscow for only one but very profitable discussion and to Dr. Bogdan Mielnik for some very useful remarks during the seminars.

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