Magnetohydrodynamics

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Program of lectures, 20h (10x2hrs), 2 ECTS points

- Short review of classical electrodynamics
- Physics of plasmas, kinetic theory
- Derivation of General Ohm’s law
- Alfven waves
- Toroidal and poloidal fields
- Helmholtz decomposition
- Conductivity tensor
- Reconnection
- Boussinesq approximation
- Dynamo mechanism
- Cylindric Taylor-Couette flow & dynamo
- Nonlinear (hydromagnetic) dynamo
- Magnetorotational instability (MRI)
- Classical hydrodynamic and hydromagnetic instabilities
- Hartmann flow,
- Numerical simulations of turbulence (hands-on)

Literature

R. Courant & K.O. Friedrichs, Supersonic Flow and Schock Waves, Interscience Publishers Inc. , New York, 1948

P.A. Davidson, An Introduction to Magnetohydrodynamics, Cambridge University Press, 2001

U. Frisch, Turbulence, Cambridge University Press, 1955

D. J. Griffiths, Introduction to Electrodynamics, 4th ed, Pearson Education, Inc., 2013

J.D. Jackson, Classical Electrodynamics, 3rd ed., John Wiley & Sons Inc., 1962

L.D. Landau & E.M. Lifshitz, Vol. 6, Fluid Mechanics, Pergamon Press, Oxford, UK, 1959

L. D. Landau and E. M. Lifschitz, Electrodynamics of Continuous Media, Pergamon Press, Oxford, UK, 1960

H.K. Moffatt, Magnetic field generation in electrically conducting fluids, Cambridge University Press, 1978

Dalton D. Schnack, Lectures in Magnetohydrodynamics: With an Appendix on Extended MHD, Lect. Notes Phys. 780 (Springer, Berlin Heidelberg 2009), DOI 10.1007/ 978-3-642-00688-3

M. Vietri, Foundations of High-Energy Astrophysics, The University of Chicago Press, 2008

MHD, lecture 1

- Short review of classical electrodynamics
- Perturbation theory
- Frozen flux condition

Top secret physics

Not so long time ago (mid-20 ct.) this topic was under military "Confidential" notice: R. Courant 1944 issued a report for USA military:

AMG-NYU-62

SUPERSONIC FLOW AND SHOCK WAVES

A MANUAL ON THE MATHEMATICAL THEORY OF NON-LINEAR WAVE MOTION

APPLIED MATHEMATICS PANEL
NATIONAL DEFENSE RESEARCH COMMITTEE

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I. INTRODUCTION

Violent disturbances - such as result from detonation of explosives, from flow through nozzles of rockets, from supersonic flight of projectiles, or from impact on solids - differ greatly from the "linear" phenomena of sound, light, or electromagnetic signals. In contrast to the latter, their propagation is governed by non-linear differential equations, and as a consequence, the familiar laws of superposition, reflection, and refraction cease to be valid; but even more novel features appear, among which the occurrence of shock fronts is the most conspicu-

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[19] von Neumann, J., "Progress report on the theory of shock waves." NDRC, Division 8, OSRD No. 1140, 1943. Restricted. (Comprehensive report on the principles of compressible fluid flow, in particular on shocks. The notions of shock interaction, contact surface, shock reflection, and Mach effect are introduced. Further remarks on the process of detonation are included. Various reports of Division 8 of the NDRC, to which the authors had no access, are quoted here.)

SUPERSONIC FLOW AND SHOCK WAVES

R. COURANT and K. O. FRIEDRICHS
INSTITUTE FOR MATHEMATICS AND MECHANICS
NEW YORK UNIVERSITY, NEW YORK

1948

INTERSCIENCE PUBLISHERS, INC., NEW YORK
INTERSCIENCE PUBLISHERS LTD., LONDON

FIRST PRINTING, 1948
SECOND PRINTING, 1956

Top secret physics

This was released to public in 1948, but for some research it lasted longer: 10 years later, work on reconnection by Petcheck was still "Classified"! In 2013 I published an article on fast reconnection, but did not have the access to Petchek's work-only few years later there was a scan available online, for decades it was available only in the library somewhere in the USA.

Classical electrodynamics & MHD

•What we need to add to the CE knowledge for dealing with MHD?

•In MHD we will deal with ***Maxwell's equations*** and ***fluid equations***, plus ***gas laws***.

•A solid conductor will behave different from the fluid one (liquid or gas). Conductivity occurs with motion of electrons, but this is large, compared to atomic scales, even in solid bodies.

•In solid bodies, dynamic effects (e.g. conductivity, Hall effect) occur, but not the mass motion.

•In fluids or gasses, we observe the flow of mass, too; the magnetic field affects both electrons and ions.

•Physics of plasmas and MHD are not sharply separated, but they do deal with different regimes. Mostly we see the difference in the kinds of waves they are dealing with, and in time scales.

•*Plasma oscillations* and *MHD waves* are different. By the simple Drude model, the former are consequence of collisions, with effective, frictional drag forces, affecting the direction and speed of electrons which are accelerated by the applied fields. These forces are trying to restore the equilibrium after separation of charges, which occurs at the frequencies much larger than the collisional frequencies. The latter are those which are oscillating with lower frequencies, they include the movement of fluid but do not include the charge separation, and the displacement current in Ampere's law is neglected.

What is MHD?

Existence of Electromagnetic-Hydrodynamic Waves

2ove

IF a conducting liquid is placed in a constant magnetic field, every motion of the liquid gives rise to an E.M.F. which produces electric currents. Owing to the magnetic field, these currents give mechanical forces which change the state of motion of the liquid.

Thus a kind of combined electromagnetic-hydrodynamic wave is produced which, so far as I know, has as yet attracted no attention.

The phenomenon may be described by the electrodynamic equations

$$\text{rot } H = \frac{4\pi}{c} i$$

$$\text{rot } E = -\frac{1}{c}\frac{dB}{dt}$$

$$B = \mu H$$

$$i = \sigma(E + \frac{v}{c} \times B);$$

together with the hydrodynamic equation

$$\partial \frac{dv}{dt} = \frac{1}{c}(i \times B) - \text{grad } p,$$

where σ is the electric conductivity, μ the permeability, ∂ the mass density of the liquid, i the electric current, v the velocity of the liquid, and p the pressure.

Consider the simple case when $\sigma = \infty$, $\mu = 1$ and the imposed constant magnetic field H_0 is homogeneous and parallel to the z-axis. In order to study a plane wave we assume that all variables depend upon the time t and z only. If the velocity v is parallel to the x-axis, the current i is parallel to the y-axis and produces a variable magnetic field H' in the x-direction. By elementary calculation we obtain

$$\frac{d^2H'}{dz^2} = \frac{4\pi\partial}{H_0^2}\frac{d^2H'}{dt^2},$$

which means a wave in the direction of the z-axis with the velocity

$$V = \frac{H_0}{\sqrt{4\pi\partial}}.$$

Waves of this sort may be of importance in solar physics. As the sun has a general magnetic field, and as solar matter is a good conductor, the conditions for the existence of electromagnetic-hydrodynamic waves are satisfied. If in a region of the sun we have $H_0 = 15$ gauss and $\partial = 0{\cdot}005$ gm. cm.$^{-3}$, the velocity of the waves amounts to

$$V \sim 60 \text{ cm. sec.}^{-1}.$$

This is about the velocity with which the sunspot zone moves towards the equator during the sunspot cycle. The above values of H_0 and ∂ refer to a distance of about 10^{10} cm. below the solar surface where the original cause of the sunspots may be found. Thus it is possible that the sunspots are associated with a magnetic and mechanical disturbance proceeding as an electromagnetic-hydrodynamic wave.

The matter is further discussed in a paper which will appear in *Arkiv för matematik, astronomi och fysik*.

H. ALFVÉN.

Kgl. Tekniska Högskolan,
Stockholm.
Aug. 24.

•Encyclopedia Britannica: "Magnetohydrodynamics (MHD), the description of the behaviour of a plasma or, in general, any electrically conducting fluid in the presence of electric and magnetic fields.

A plasma can be defined in terms of its constituents, using equations to describe the behavior of the electrons, ions, neutral particles, etc. It is often more convenient, however, to treat it as a single fluid, even though it differs from fluids that are not ionized in that it is strongly influenced by electric and magnetic fields, both of which can be imposed on the plasma or generated by the plasma; the equations describing the behaviour of the plasma, therefore, must involve the close relationship between the plasma and the associated fields."

"Plasma, in physics, an electrically conducting medium in which there are roughly equal numbers of positively and negatively charged particles, produced when the atoms in a gas become ionized. It is sometimes referred to as the fourth state of matter, distinct from the solid, liquid, and gaseous states."

The MHD description of electrically conducting fluids was first developed by Hannes Alfvén in a 1942 paper published in Nature titled "Existence of Electromagnetic–Hydrodynamic Waves" which outlined his discovery of what are now referred to as Alfvén waves. His one-column article is indeed in a Scandinavian style (not valid for sagas!): short and novel. Learn from masters!

A fast-forward through the derivation of MHD equations: the mass conservation

The equations of *ideal MHD* are the **mass continuity equation**, the **momentum equation** with the inclusion of the Lorentz force, the **induction equation**, and the **divergence-free magnetic field** constraint div B=0. *Non-ideal MHD* includes the dissipative terms like viscosity and resistivity. Let's go through the preparatory material, so we'd have the equations handy. To describe the moving fluid, 5 equations are needed: density, pressure (or temperature or entropy, as through the eq. of state we can always compute the remaining quantities from any two of the thermodynamical quantities) and three velocity components. Approximation we introduce with such macroscopic eqs. is that "small volume" always includes *many* fluid particles.

$\Delta l = \vec{v}\, dt$

Mass of the fluid in volume V is $dm = \rho\, dV$ $\left(\rho \equiv \frac{dm}{dV}\right)$

Volume of the fluid passing through surface Δa is $\Delta l \cdot \Delta a = \vec{v}\,\Delta t\, \underset{\sim}{\Delta \vec{a}}$

$\Delta\vec{a} = \vec{n}\,\Delta a$

convention that (+) is to external $\vec{n}$

Decrease of the amount of fluid in the volume V is $-\frac{\partial}{\partial t}\int \rho\, dV$, and if we equal it to the outflow across a: $\oint_S \rho \vec{v}\, d\vec{a}$, we have

$$\frac{\partial}{\partial t}\int \rho\, dV = -\underbrace{\oint_S \rho\, \vec{v}\, d\vec{a}}_{\int \nabla(\rho\vec{v})\, dV} \Rightarrow \int\left[\frac{\partial \rho}{\partial t} + \nabla(\rho\vec{v})\right] dV = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla(\rho\vec{v}) = 0$$

The mass conservation eq.

A fast-forward through the derivation of MHD equations: the momentum conservation

Pressure $p \stackrel{\text{def}}{=} \frac{dF}{da}$ → force, with the total force acting from the outside on the fluid volume dV equal to

$$-\oint_S p\, d\vec{a} = -\int_V \nabla p\, dV$$

↳ direction opposite to $\vec{n}$

On each volume element dV, the surrounding fluid acts with the force $-\nabla p\, dV$ or, equivalently, on the unit of fluid volume acts force $-\nabla p$. Equaling this force with the mass of the unit volume of fluid, ρ and its acceleration $\frac{d\vec{v}}{dt}$, we obtain the equation of motion of the fluid:

$$\rho \frac{d\vec{v}}{dt} = -\nabla p$$

→ the change of velocity for the moving fluid in comoving coordinates, not in the specific point in space.

The convective derivative

We can write this acceleration in two parts:

1) $\frac{\partial \vec{v}}{\partial t} dt$, change of velocity in the point $\vec{r}_0$ during time dt

2) difference of velocities in the given t_0 in the two points $\vec{r}_0$, $\vec{r}_0 + \vec{v}\,dt$ $\quad d\vec{v} = \frac{\partial \vec{v}}{\partial t} dt + (d\vec{r} \cdot \nabla)\vec{v}$

In 1) $\vec{r}_0$ is fixed during time variation, in 2) the velocity in 2 points change, we have

$$\frac{d\vec{v}}{dt} = \lim_{dt \to 0} \frac{\vec{v}_1 - \vec{v}_0}{dt} =$$

$\vec{v}_0 = \vec{v}(\vec{r}_0, t_0)$

$\vec{v}_1 = \vec{v}(\vec{r}_0 + \vec{v}_0 dt, t_0 + dt)$

$$= \lim_{dt \to 0} \frac{\vec{v}(\vec{r}_0 + \vec{v}_0 dt, t_0 + dt) - \vec{v}(r_0, t_0)}{dt}$$

$$= \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}.$$

Think Cartesian:

$$dx \frac{\partial \vec{v}}{\partial x} + dy \frac{\partial \vec{v}}{\partial y} + dz \frac{\partial \vec{v}}{\partial z} = (d\vec{r} \cdot \nabla)\vec{v}$$

$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$, $\quad d\vec{r} = \vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz$

$$\frac{dX}{dt} = \frac{\partial X}{\partial t} + \vec{v} \cdot \nabla X$$

is called convective derivative, Lat. convehere = "to carry"

for a phys. quantity X not varying at a fixed position, but for a fixed mass element. Following such an element as if sitting on it, we would measure $\frac{dX}{dt}$.

Back to the momentum conservation: Euler equation

Now we can write $\rho \frac{d\vec{v}}{dt} = -\nabla P$ as

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = -\nabla P \quad \text{or}$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \nabla) \vec{v} = -\nabla P + \vec{f}_{ext} \quad \text{Euler equation}$$

This is the general equation for ideal fluid motion. We added the external forces $\vec{f}$, for example gravity will add $\rho \vec{g}$ or $-\rho \nabla \phi$ if expressed through the gravity potential. $\vec{f}$ can also include friction, which is important in astroph. fluids (viscosity).

Adiabatic flow

•If we ignore the exchange of heat between the fluid elements, we have adiabatic flow. Entropy of the fluid elements does not change

$$\frac{\partial S}{\partial t} + \vec{v} \cdot \nabla S = 0$$ adiabatic eq. of fluid.

often it is simply $S = \text{const}$,

such motion is called isentropic

We can use it for different form of Euler eq. Heat Q is $dQ = T\,dS + V\,dP$ for unit fluid mass ($V = \frac{1}{\rho}$, T = temperature)

With $S = \text{const} \Rightarrow dQ = V\,dP = \frac{1}{\rho}\,dP \Rightarrow \frac{1}{\rho}\nabla P = \nabla Q$ and

we can write Euler's eq. as $\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\nabla Q$

If we use $\frac{1}{2}\nabla v^2 = \vec{v}\times(\nabla\times\vec{v}) + (\vec{v}\cdot\nabla)\vec{v}$, we can write it as

$$\nabla\times \Big\backslash \frac{\partial \vec{v}}{\partial t} - \vec{v}\times(\nabla\times\vec{v}) = -\nabla\left(Q + \frac{v^2}{2}\right)$$, and after $\nabla\times$ at both sides, we have

$$\frac{\partial}{\partial t}\nabla\times\vec{v} = \nabla\times(\vec{v}\cdot\nabla\times\vec{v})$$, where we have only velocities.

Bernoulli equation

For a stationary flow $\frac{\partial \vec{v}}{\partial t}=0$ and we can write

$$\frac{1}{2}\nabla v^2-\vec{v}\times(\nabla\times\vec{v})=-\nabla Q$$

If we introduce the flow streamlines, whose tangents show the direction of the velocity vectors at a given instant of time, such lines for a stationary flow follow the paths of the fluid particles. If we multiply with the unit vector of the tangent in each point of the trajectory $\vec{l}$. We know that the projection of the gradient in some direction is obtained by performing a derivation in that direction, so that the projection of ∇Q becomes $\frac{\partial Q}{\partial l}$. Since $\vec{v}\times(\nabla\times\vec{v})\perp\vec{v}$ we know that its projection $=0$, so that $\frac{\partial}{\partial l}\left(\frac{v^2}{2}+Q\right)=0$, hence

$\frac{v^2}{2}+Q=\text{const}$. Bernoulli's eq. This means that $\frac{v^2}{2}+Q$ is constant along the flow line.

In the gravitation field, we need to add the grav. force, if we direct it in the $(-\hat{z})$ direction $\frac{\partial}{\partial l}\left(\frac{v^2}{2}+Q+gz\right)=0$ and $\frac{v^2}{2}+Q+gz=\text{const}$.

Not the part of the ideal MHD eqs. but let's derive the energy conservation eq.

Energy of a fluid element consists of kinetic energy per unit volume, $\frac{1}{2}\rho v^2$ and internal (thermal) energy $\rho\epsilon$ per unit volume (ϵ is the internal energy per unit mass; it depends on the gas temperature T). By equipartition theorem every degree of freedom has $\frac{1}{2}kT$, so $\epsilon = \frac{3}{2}kT\frac{1}{\mu m_H}$ in all astroph. cases except WDs and NSs and near the central part of stars, by the EOS for the ideal gas $p = \rho\frac{kT}{\mu m_H}$

$\mu = 1$ neutral H, m_H = m_p, actually Hydrogen atom mass

0.5 fully ionized H

and $\mu \in [0.5, 1]$ for the gas mixture, depending on the ionization state.

→ Last 30 yrs I never used anything than 0.5 or 1.

Energy eq. we can write in the form of continuity eq. simply adding the terms for the included processes.

The energy conservation eq.

In the continuity eq. $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$ we redefine

$\rho \to \frac{1}{2}\rho v^2 + \rho \epsilon$ and $\rho \vec{v} \to \frac{1}{2}\rho v^2 + \rho\epsilon + P$ → "work by pressure", adding the work of external forces, $\vec{f}_{ext} \cdot \vec{v}$

and we can write the energy conservation eq: if we heat the gas, it radiates, energy leaves the system, so $(-)$.

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\epsilon\right) + \nabla \cdot \left[\left(\frac{1}{2}\rho v^2 + \rho\epsilon + P\right)\vec{v}\right] - \vec{f}_{ext} \cdot \vec{v} + \nabla \vec{\Phi}_{rad} + \nabla \vec{\phi}_{heat} = 0$$

In the stationary state $\frac{\partial}{\partial t} = 0$ and neglecting the radiation and heat conducting:

$$\nabla\left[\left(\frac{1}{2}\rho v^2 + \rho\epsilon + P\right)\vec{v}\right] = \vec{f}_{ext} \cdot \vec{v}$$

A fast-forward through the derivation of MHD equations: adding the electromagnetic part to the momentum eq.

Now we will add the electromagnetic part to the HD equations.

The mass continuity eq. remains the same.

In the momentum (i.e. force) and energy equations we need to add the electromagnetic part. We use the $\vec{I}$ text for this, and we readily write:

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla P + \vec{f}_{gr} + \vec{f}_{Lorentz}$$

Since the typical astrophysical fluids have zero charge density, the electric force is zero, only the magnetic part of the Lorentz force remains,

$\vec{f}_{Lorentz} = \rho\vec{E} + \frac{q}{c}\vec{v}\times\vec{B} \rightarrow \frac{\vec{J}\times\vec{B}}{c}$ and with $\vec{f}_{gr} = -\nabla\phi_g$ we have

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla P - \nabla\phi_g + \frac{\vec{J}\times\vec{B}}{c}.$$

Induction eq.

Faraday's Law (cgs): $\nabla\times\vec{\mathcal{E}} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$

Absence of magnetic monopoles: $\nabla\cdot\vec{B}=0$

For the completeness of dynamical eqs., we need the relation between the current density $\vec{J}$ and the fields $\vec{\mathcal{E}}$ and $\vec{B}$.

For the conducting medium with conductivity σ, Ohm's law applies $\vec{J}=\sigma\vec{\mathcal{E}}'$ for a medium at rest. If the fluid is moving with respect to the observer, and $\vec{\mathcal{E}}'$ and $\vec{B}'$ are measured in the fluid's comoving frame. The Lorentz transf. for the electric field is $\vec{\mathcal{E}}'=\gamma\left(\vec{\mathcal{E}}+\frac{\vec{v}\times\vec{B}}{c}\right)-\frac{\gamma^2}{\gamma+1}\frac{\vec{v}(\vec{v}\cdot\vec{B})}{c^2}$, $\vec{v}$ is measured in the observer's frame, $\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, Lorentz factor.

For a Newtonian fluid $v\ll c$, so

$\vec{\mathcal{E}}'\doteq\vec{\mathcal{E}}+\frac{\vec{v}\times\vec{B}}{c}$ in the fluid's frame, so the Ohm's law is $\vec{J}=\sigma\left(\vec{\mathcal{E}}+\frac{\vec{v}\times\vec{B}}{c}\right)$. The $\vec{J}'=\gamma^2\vec{J}\simeq\vec{J}$ follows from $v\ll c$.

Simpler: $\vec{\mathcal{E}}'=\frac{\vec{\mathcal{E}}+\vec{v}\times\vec{B}}{\sqrt{1-\frac{v^2}{c^2}}}=$

$\vec{\mathcal{E}}$ is in stationary frame

$=(\vec{\mathcal{E}}+\vec{v}\times\vec{B})\left(1-\frac{1}{2}\frac{v^2}{c^2}+\dots\right)$

$\simeq\vec{\mathcal{E}}+\vec{v}\times\vec{B}$

We assumed the zero density of local charge, so in Gauss law $\nabla\cdot\vec{\mathcal{E}}=\frac{4\pi\rho_e}{c}$ $\rho_e=0$ and $\nabla\cdot\vec{\mathcal{E}}=0$ and the $\vec{\mathcal{E}}$ exists only because of induction.

For $\vec{B}$ we know $\nabla\cdot\vec{B}=0$, $\nabla\times\vec{B}=\frac{4\pi}{c}\vec{J}+\cancel{\frac{1}{c}\frac{\partial\vec{\mathcal{E}}}{\partial t}}=\frac{4\pi}{c}\vec{J}$

displacement current neglected

We can eliminate $\vec{J}$ using in Ohm:

$\nabla\times\Big|\ \frac{c}{4\pi}\nabla\times\vec{B}=\sigma\left(\vec{\mathcal{E}}+\frac{\vec{v}\times\vec{B}}{c}\right)$ and taking the curl we have

$$\frac{c}{4\pi}\underbrace{\nabla\times(\nabla\times\vec{B})}_{\nabla(\nabla\cdot\vec{B})-\nabla^2\vec{B},\ \nabla\cdot\vec{B}=0}=\sigma\underbrace{\nabla\times\vec{\mathcal{E}}}_{-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}}+\frac{\sigma}{c}\nabla\times(\vec{v}\times\vec{B})$$

If $\sigma=$const

$$\Rightarrow\ \frac{\partial\vec{B}}{\partial t}=\nabla\times(\vec{v}\times\vec{B})+\frac{c^2}{4\pi\sigma}\nabla^2\vec{B}$$

// The induction eq.
we get rid of $\vec{\mathcal{E}}$ and $\vec{J}$.

Induction eq.

In $\frac{\partial \vec{B}}{\partial t} = \underbrace{\nabla \times (\vec{v} \times \vec{B})}_{\text{convective term}} + \underbrace{\frac{c^2}{4\pi\sigma} \nabla^2 \vec{B}}_{\text{diffusive term}}$ the 1st or 2nd term can prevail; for describing their importance we introduce the magnetic Reynolds number:

$$R_M = \frac{\text{convective term}}{\text{diffusive term}} = \frac{\frac{1}{L} v B}{\frac{\left(\frac{c^2}{4\pi\sigma}\right) B}{L^2}}$$

L is here the characteristic length of the variation of B. Diffusion or convection become dominant with $R_m \ll 1$ or $R_m \gg 1$

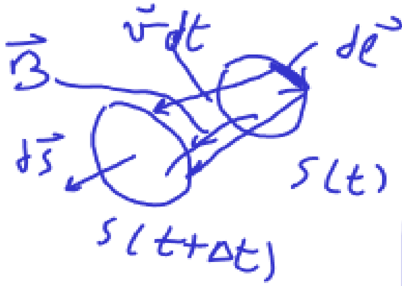
In lab experiments $R_m \ll 1$ usually (think Hg, Na), in astrophysics $R_m \gg 1$, usually $R_m \geq 10^3$.

Eg. in the active regions of solar corona $R_m \sim 10^6$!, diffusion is negligible there, ideal MHD should be enough. Is it? Reconnection? Current sheets complicate this picture because of changes in the field geometry.

Frozen flux condition

$\underbrace{\qquad}_{R_m \to \infty}$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B})$$



Consider a loop in varying $\vec{B}(\vec{r},t)$ co-moving with the fluid. The flux through the loop surface is $\Phi(S,t) = \int_{S(t)} ds\,\vec{n}\cdot\vec{B}(\vec{r},t)$

($\vec{n}$ is ort to S) $\quad \Phi(S(t+\Delta t), t+\Delta t) = \int_{S(t+\Delta t)} ds\,\vec{n}\,\vec{B}(\vec{r},t+\Delta t)$

We can write, by Taylor, to the 1st order in Δt:

$$\Phi(S(t+\Delta t), t+\Delta t) = \int_{S(t+\Delta t)} ds\,\vec{n}\,\vec{B}(\vec{r},t) + \int_{S(t)} ds\,\vec{n}\,\frac{\partial \vec{B}}{\partial t}\Delta t \quad (\star)$$

Contribution by the $S(t+\Delta t)$ is of the 2nd order and we can neglect it.

The mag. flux through the sides of the flux tube described by the moving loop is $\Phi(S_{side},t) = \oint_S (d\vec{s}\times\vec{v}\Delta t)\cdot\vec{B}(\vec{r},t)$ (to the 1st order in Δt)

For non-divergent $\vec{B}$, without sources and drains, $\Phi(S,t) = \Phi(S(t+\Delta t),t) + \Phi(S_{side},t)$

$$\Phi(S(t+\Delta t),t+\Delta t) - \Phi(S,t) = \int_S ds\,\vec{n}\,\frac{\partial\vec{B}}{\partial t}\Delta t - \oint_S \underbrace{(d\vec{s}\times\vec{v}\Delta t)\cdot\vec{B}(\vec{r},t)}_{d\vec{s}\cdot(\vec{v}\times\vec{B})}, \text{ in limit } \Delta t \to 0$$

$$\frac{d\Phi}{dt} = \int_S ds\,\vec{n}\,\frac{\partial\vec{B}}{\partial t} - \underbrace{\oint_S d\vec{s}\cdot(\vec{v}\times\vec{B})} \Rightarrow \frac{d\Phi}{dt} = \int_S ds\,\vec{n}\cdot\Big[\frac{\partial\vec{B}}{\partial t} - \nabla\times(\vec{v}\times\vec{B})\Big] = 0$$

Stokes: $= \int_S ds\,\vec{n}\,\nabla\times(\vec{v}\times\vec{B})$

$$\frac{\partial\vec{B}}{\partial t} \;\Big\|\; \nabla\times(\vec{v}\times\vec{B})$$

Mag. flux through the surface S is constant when S co-moves with the fluid. = "Frozen flux condition".

The field lines can be assumed as attached to the fluid velocity streamlines (and v. versa!) $\Rightarrow$ the fluid can not move across the magnetic field, but can slide along $\vec{B}$.

Perturbation theory

For stationary flow and without energy losses, we can write, in the nonmagnetic case: $\nabla(\rho\cdot\vec{v})=0$, $\rho(\vec{v}\cdot\nabla)\vec{v}=-\nabla P+\vec{f}$

$$\nabla[(\tfrac{1}{2}\rho v^2+\rho\epsilon+P)\cdot\vec{v}]=\vec{f}\cdot\vec{v}$$

For hydrostatic balance $\vec{v}=0 \Rightarrow$ only $\nabla P=\vec{f}$

The hydrostatic balance is the base state with respect to which we compute the perturbations. If the initial values are (P_0,ρ_0) and perturbations (P',ρ'), $P=P_0+P'$, $\rho=\rho_0+\rho'$, $\vec{v}=\vec{v}'$. Small perturbations can be, depending on the process, adiabatic or isothermal. for adiabatic processes

Adiabatic: entropy $S=P\rho^{-\gamma}=\text{const}$ $\rightarrow$ for monoatomic gases $\gamma=5/3$

Isothermal: $P=\frac{\rho kT}{\mu m_H}$. Both can be written as $P=k\rho^{\gamma}$, with $\gamma=5/3$ for adiabatic and $\gamma=1$ for isothermal cases.

If we linearize the continuity eq. ($\nabla\rho'\cdot\vec{v}=0$ in the 1st approx)

$$\frac{\partial\rho}{\partial t}+\nabla(\rho\vec{v})=\frac{\partial}{\partial t}(\rho_0+\rho')+\nabla\cdot[(\rho_0+\rho')\cdot\vec{v}']=\frac{\partial\rho'}{\partial t}+\rho_0\nabla\cdot\vec{v}'=0$$

The same with Euler's eq:

$$(\rho_0+\rho')\frac{\partial\vec{v}'}{\partial t}+(\rho_0+\rho')\vec{v}'\nabla\cdot\vec{v}'=-\nabla(P_0+P')+\vec{f}$$

$$\rho_0\frac{\partial\vec{v}'}{\partial t}+\rho_0\vec{v}'\nabla\cdot\vec{v}'=-\underbrace{\nabla P_0}_{=\vec{f}}-\nabla P'+\vec{f}\,,\quad \rho_0\frac{\partial\vec{v}'}{\partial t}=-\nabla P' \text{ in the 1st order}$$

We obtained 2 eqs:

$$\frac{\partial\rho'}{\partial t}+\rho_0\nabla\cdot\vec{v}'=0 \quad (1)$$

$$\rho_0\frac{\partial\vec{v}'}{\partial t}=-\nabla P' \rightarrow \frac{\partial\vec{v}'}{\partial t}=-\frac{1}{\rho_0}\nabla P' \quad (2)$$

From $P_0+P'=k(\rho_0+\rho')^{\gamma}$ we see $P=f(\rho)\Rightarrow$ $\nabla P'=\left(\frac{\partial P}{\partial\rho}\right)_0\nabla\rho'$ to the 1st order (equilibrium solution)

So we have from (2)

$$\nabla\cdot\left\backslash\frac{\partial\vec{v}'}{\partial t}+\frac{1}{\rho_0}\left(\frac{\partial P}{\partial\rho}\right)_0\nabla\rho'=0\right.$$

$$\nabla\cdot\frac{\partial\vec{v}'}{\partial t}+\left[\frac{1}{\rho_0}\left(\frac{\partial P}{\partial\rho}\right)_0\nabla^2\rho'\right]=0\,/\rho_0$$

$$\rho_0\nabla\cdot\frac{\partial\vec{v}'}{\partial t}+\left(\frac{\partial P}{\partial\rho}\right)_0\nabla^2\rho'=0 \quad (A)$$

From $\frac{\partial}{\partial t}$(1): $\frac{\partial^2\rho'}{\partial t^2}+(\rho_0\nabla\cdot\frac{\partial\vec{v}'}{\partial t})=0$ (B)

(A)-(B): $\frac{\partial^2\rho'}{\partial t^2}-\left(\frac{\partial P}{\partial\rho}\right)_0\nabla^2\rho'=0$ Wave eq!

$$\boxed{\frac{\partial^2\rho'}{\partial t^2}-c_s^2\nabla^2\rho'=0}$$

sound speed, small perturbations around hydrostatic equilibrium propagate with speed of sound, as sound waves.

2 sound speeds:

adiabatic: $c_s^{ad}=\sqrt{\frac{5}{3}\frac{P}{\rho}}=\sqrt{\frac{5}{3}\frac{kT}{\mu m_H}}\propto\rho^{1/3}$

isothermal: $c_s^{isot}=\sqrt{\frac{P}{\rho}}=\sqrt{\frac{kT}{\mu m_H}}$

// $c_s^2=\gamma\frac{kT}{\mu m_H}=\gamma\frac{\mathcal{R}T}{m}$ // (ideal gas const, 8.31 erg K^{-1}mol^{-1})

These two expressions, for isothermal and adiabatic sound speeds, are most often used. Keep in mind that they are different!

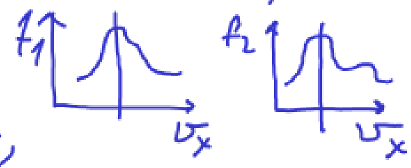
MHD, lecture 2

- Physics of plasmas, kinetic theory
- Derivation of Fokker-Planck eq.

Kinetic theory

We use the kinetic theory when distribution of velocities for each kind of the particles is important. In the 'fluid' approach we assumed Maxwellian distribution of velocities for each kind of particles, so <u>we could describe them by one number</u>, temperature T.

High temperature plasmas could render collisions of particles rare, so that departures from thermal equilibrium could last long - in the fluid description we can not distinguish the two distributions $f_1(v_x)$ and $f_2(v_x)$ if the area below f is the same, but their behavior will be completely different:



The number density $n = n(\vec{r}, t)$ is the function of 4 scalar variables, components of the position and the time, to describe the velocity distribution we need 7 independent variables: $f = f(\vec{r}, \vec{v}, t)$.

If we assume continuity of the flow, we can describe the number of particles with $f(x,y,z,v_x,v_y,v_z,t)\,dv_x\,dv_y\,dv_z$ particles in cm^3 at position $\vec{r}$ at a moment t, having the velocities between v_x and $v_x + dv_x$ (y, z), the density of particles is an integral of f, which we can write different ways:

$$n(\vec{r},t) = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z\, f(\vec{r},\vec{v},t) = \int_{-\infty}^{\infty} f(\vec{r},\vec{v},t)\,\overbrace{d^3v}^{\text{3D volume element}} = \int_{-\infty}^{\infty} f(\vec{r},\vec{v},t)\,dV$$

If we normalize f so that $\int_{-\infty}^{\infty} \hat{f}(\vec{r},\vec{v},t)\,d\vec{v} = 1$, then $\hat{f}$ is the probability, with $f(\vec{r},\vec{v},t) = n(\vec{r},t)\,\hat{f}(\vec{r},\vec{v},t)$. $\hat{f}$ is still the function of 7 variables, since the shape of the distribution can change in $\vec{r}$ and t, the same as for density.

Some would say the only real plasma theory is kinetic theory, but fluid approach is surprisingly successful when the approximations it uses apply.

One important distribution function is Maxwell's:

$$\hat{f} = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-v^2/v_{th}^2}$$

with $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$,

$v_{th} = \sqrt{2kT/m}$, $k = k_{Boltz} = 1.38\cdot10^{-16}\ \mathrm{erg/K} = 1.38\cdot10^{-23}\ \mathrm{J/K}$

We use the $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$, in

$$\int_{-\infty}^{\infty} \hat{f}_m\, dv_x\, dv_y\, dv_z = \left(\frac{2kT}{m} = A\right) =$$

$$= \left(\frac{1}{\pi A}\right)^{3/2} \int_{-\infty}^{\infty} e^{-v_x^2/A} dv_x \int_{-\infty}^{\infty} e^{-v_y^2/A} dv_y \int_{-\infty}^{\infty} e^{-v_z^2/A} dv_z$$

$$= \left(\int e^{-x^2} dx = \sqrt{\pi},\quad u = x\sqrt{A},\quad dx = \frac{du}{\sqrt{A}} \right.$$

$$\left. \int_{-\infty}^{\infty} e^{-u^2/A} \frac{du}{\sqrt{A}} = \frac{1}{\sqrt{A}} \int_{-\infty}^{\infty} e^{-u^2/A} du = \sqrt{\pi} \Rightarrow \int_{-\infty}^{\infty} e^{-u^2/A} du = \sqrt{A\pi} \right)$$

$$= \left(\frac{1}{\pi A}\right)^{3/2} \sqrt{A^3 \pi^3} = 1.$$

Equations of the kinetic theory

The total derivation in time for f is given with

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underbrace{\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}}_{\vec v \cdot \nabla f} + \underbrace{\frac{\partial f}{\partial v_x}\frac{dv_x}{dt} + \frac{\partial f}{\partial v_y}\frac{dv_y}{dt} + \frac{\partial f}{\partial v_z}\frac{dv_z}{dt}}_{\frac{\partial f}{\partial \vec v}\frac{d\vec v}{dt} = \nabla_v \frac{d\vec v}{dt}}$$

〃 rate of change, as seen from the frame comoving with particles. We are in 6D space now $(\vec r, \vec v)$, $\frac{df}{dt}$ is a convective derivation in phase space.

$\vec F = m\frac{d\vec v}{dt}$; force acting on particles

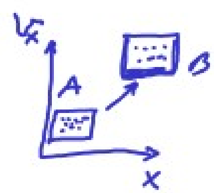
$\frac{\vec F}{m}$ $\frac{d\vec v}{dt}$

↳ is a gradient in velocity space

The Boltzmann eq. states that $\frac{df}{dt} = 0$, except if collisions of particles occur:

$$\frac{\partial f}{\partial t} + \vec v \cdot \nabla f + \frac{\vec F}{m}\frac{\partial f}{\partial \vec v} = \left(\frac{\partial f}{\partial t}\right)_c$$

1) Illustration of $\frac{df}{dt} = 0$ for non-collisional cases:



a group of particles A moves with v_x in direction x, described by $f(x, v_x)$.
If the particles move to B in time t, all with the v_x, in B they will still have v_x. If there are collisions, $\left(\frac{\partial f}{\partial t}\right)_c \neq 0$.

In the hot enough plasma, we can neglect the collisions. Under the electromagnetic force $\vec F$, we can write

$$\frac{\partial f}{\partial t} + \vec v \cdot \nabla f + \frac{q}{m}(\vec E + \vec v \times \vec B)\frac{\partial f}{\partial \vec v} = 0$$ Vlasov's eq.

This is the simplest case in kinetic theory.

For Coulomb's collisions, Boltzmann's eq. becomes

$$\frac{df}{dt} = -\frac{\partial}{\partial \vec v}(f\langle \Delta \vec v\rangle) + \frac{1}{2}\frac{\partial^2}{\partial \vec v \partial \vec v}(f\langle \Delta \vec v \Delta \vec v\rangle)$$

Fokker-Planck eq., short cut writing of a more complicated general version.

Equations of the kinetic theory-derivation of fluid eqs.

Fluid eqs. can be understood as moments eqs. of Boltzmann eq.

The lowest moment we obtain by integration of B's eq:

$$\underbrace{\int \frac{\partial f}{\partial t} d\vec{v}} + \underbrace{\int \vec{v}\cdot\nabla f \cdot d\vec{v}} + \frac{q}{m} \underbrace{\int (\vec{E} + \vec{v}\times\vec{B})} \cdot \frac{\partial f}{\partial \vec{v}} d\vec{v} = \underbrace{\int \left(\frac{\partial f}{\partial t}\right)_c d\vec{v}}$$

$\frac{\partial}{\partial t}\int f\, d\vec{v} = \frac{\partial n}{\partial t}$

from the def. on n 2 pages earlier

$\vec{v}$ is independent of ∇

$= \nabla \cdot \underbrace{\int \vec{v} f\, d\vec{v}} = \nabla \cdot n\vec{u}$

$n\bar{\vec{v}} = n\vec{u}$

↓

average fluid velocity

$= 0$ since the collisions do not change the number of particles [reconnection is ignored here]

$\int \vec{E} \frac{\partial f}{\partial \vec{v}} d\vec{v} = \nabla_v (f\vec{E})\, d\vec{v} =$

divergence is integrated on the surface with $v = \infty$, $f\vec{E}$ vanishes for $f \to 0$ faster than v^{-2} when $v \to \infty$, which is a necessary condition for any distribution with finite energy. The same reasoning here gives $\underline{\underline{= 0}}$

$\int (\vec{v}\times\vec{B}) \frac{\partial f}{\partial \vec{v}} d\vec{v} =$

$\int \frac{\partial}{\partial \vec{v}} (f \vec{v}\times\vec{B})\, d\vec{v} -$

$- \underbrace{\int f \frac{\partial (\vec{v}\times\vec{B})}{\partial \vec{v}} d\vec{v}}$

$= 0$ since $\vec{v}\times\vec{B} \perp \nabla_v$

We stay with $\frac{\partial n}{\partial t} + \nabla \cdot n\vec{u} = 0$,

the continuity eq.

Equations of the kinetic theory-derivation of fluid eqs.

The next moment eq. of Boltzmann eq. we obtain multiplying it with $m\vec{v}$ and integrating by $d\vec{v}$:

$$\underbrace{m\int \vec{v}\frac{\partial f}{\partial t}d\vec{v}}+m\int \underbrace{\vec{v}\cdot(\vec{v}\cdot\nabla)}f\,d\vec{v}+q\int \underbrace{\vec{v}(\vec{E}+\vec{v}\times\vec{B})\frac{\partial f}{\partial \vec{v}}}d\vec{v}=\underbrace{\int m\vec{v}\left(\frac{\partial f}{\partial t}\right)_c}d\vec{v}$$

$$m\frac{\partial}{\partial t}\underbrace{\int \vec{v}f\,d\vec{v}}_{n\vec{u}}=m\frac{\partial}{\partial t}(n\vec{u})$$

since ∇ does not act on $\vec{v}$, $\int \nabla\cdot(f\vec{v}\cdot\vec{v})d\vec{v}=$
$=\nabla\cdot\int f\vec{v}\vec{v}\,d\vec{v}$

by the def. of "average"
$\frac{1}{n}\cdot$(integral on $\vec{v}$), "weight" $\Rightarrow \nabla\cdot\int f\vec{v}\cdot\vec{v}\,d\vec{v}=\nabla\cdot n\overline{\vec{v}\cdot\vec{v}}$

$$\underbrace{\int\frac{\partial}{\partial\vec{v}}[f\vec{v}(\vec{E}+\vec{v}\times\vec{B})]d\vec{v}}_{=0 \text{ from previous slide}}-\int f(\vec{E}+\vec{v}\times\vec{B})\underbrace{\frac{\partial}{\partial\vec{v}}\vec{v}}_{\tilde{I}\text{ unit tensor}}d\vec{v}=$$

$$=-\int(\vec{E}+\vec{v}\times\vec{B})f\,d\vec{v}=$$

$$=-n(\vec{E}+\vec{u}\times\vec{B})$$

change of momentum because of collision; $\tilde{P}_{ij}$ = tensor

We separate $\vec{v}$ into average kinetic velocity $\vec{u}$ and thermal velocity $\vec{w}$: $\vec{v}=\vec{u}+\vec{w}$. We already know $\vec{u}$ as the average velocity, so $\nabla\cdot(n\overline{\vec{v}\cdot\vec{v}})=\nabla\cdot n\vec{u}\vec{u}+\nabla\cdot n\overline{\vec{w}\vec{w}}+2\nabla\cdot n\underbrace{\vec{u}\overline{\vec{w}}}_{=0\text{, obviously.}}$

A quantity $mn\overline{\vec{w}\vec{w}}=\tilde{P}$ stress tensor, $\nabla\cdot n\vec{u}\vec{u}=\vec{u}\,\nabla\cdot(n\vec{u})+n(\vec{u}\cdot\nabla)\vec{u}$

We can write, finally: $\underbrace{m\frac{\partial}{\partial t}(n\vec{u})+m\vec{u}\,\nabla\cdot(n\vec{u})}+mn(\vec{u}\cdot\nabla)\vec{u}+\nabla\cdot\tilde{P}-qn(\vec{E}+\vec{u}\times\vec{B})=\tilde{P}_{ij}$

$$m\left(\frac{\partial n}{\partial t}\vec{u}+n\frac{\partial\vec{u}}{\partial t}\right)+m\vec{u}\cdot\nabla\underbrace{(n\vec{u})}_{\text{from contin. eq.}=-\frac{\partial n}{\partial t}}=m\left(\frac{\partial n}{\partial t}\vec{u}+n\frac{\partial\vec{u}}{\partial t}-\vec{u}\frac{\partial n}{\partial t}\right)=$$

$$=mn\frac{\partial\vec{u}}{\partial t}$$

So that the eq. of motion for fluid is:

$$mn\left[\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}\right]=qn(\vec{E}+\vec{u}\times\vec{B})-\nabla\cdot\tilde{P}+\tilde{P}_{ij}$$

, describing the flux of momentum in a fluid.

The third moment of the Boltzmann eq. we would obtain multiplying it with 0.5m**vv**. Instead of the stress tensor $\mathcal{P}$ we would then obtain the heat conducting coefficient-→setting it to zero, we would obtain the simplest eq of state, P= n^γ

Electrostatic oscillations in warm plasma

Two-fluid plasma shows growing instabilities with decreasing velocities of particles in fluids, which is counterintuitive when we know that a system stabilizes with the velocities of flows decreases to zero. To resolve the inconsistency, we need a theory taking into account the thermal movement of electrons and ions in the plasma. We will work one example in detail, to see how it goes.

We define the distribution function $f(\vec{r},\vec{v},t)$ as few slides back and the number of particle in the $d\vec{v}$ with velocities in the interval $\vec{v}$ to $\vec{v}+d\vec{v}$ at given with $d^6N = f(\vec{r},\vec{v},t)\,d\vec{r}\,d\vec{v}$.

We have to be careful with this def: if f is defined in a lim $\begin{matrix} d^3\vec{r} \\ d^3\vec{v} \end{matrix} \to 0 \Rightarrow f \to \infty$ for the volume elements centered on a particle, or $f \to 0$ if they are not centered on a particle. We avoid the problem by imagining not large values of the spatial and velocity gradients; $\Delta r, \Delta v$ in $f(\Delta r)^3(\Delta v)^3 \gg 1 \Rightarrow f$ is finite in the volumes small enough with respect to $(\Delta r)^3$ and speed $(\Delta v)^3$ for which d^3N is large with respect to unity. Another possibility for definition of d^6N is through the "ensemble" of the system and its average values, or through the probability of finding a particle in a given volume of $\vec{r}$ and $\vec{v}$.

Electrostatic oscillations in warm plasma

The flux of particles $\vec{\Psi} = n\langle\vec{v}\rangle = \int d^3v\, f(\vec{r},\vec{v},t)\,\vec{v}$

average velocity $\langle\vec{v}\rangle = \frac{1}{n}\int d^3v\, f(\vec{r},\vec{v},t)\,\vec{v}$

number density $n(\vec{r},t) = \int d^3v\, f(\vec{r},\vec{v},t)$

Analogous to this, $\langle v^2\rangle = \frac{1}{n}\int d^3v\, f(\vec{r},\vec{v},t)\, v^2$ is an average for higher moments of velocity.

For the plasma description, we further need the charge densities and currents expressed through the distribution function f.

If a plasma consists of $\sigma = e, 1, 2, 3, \ldots$ kinds of particles, charge density is $n_q = \sum_\sigma q_\sigma \int d^3v\, f_\sigma\, f(\vec{r},\vec{v},t)\,\vec{v}$, with the current density

$$\vec{j} = \sum_\sigma \frac{q_\sigma}{c}\int d^3v\, f_\sigma(\vec{r},\vec{v},t)\,\vec{v}.$$

To obtain the eq. of motion through f we define the vector field of accelerations $\vec{a}(\vec{r},\vec{v},t)$, and we compute for any kind of particles

$$\vec{a}(\vec{r},\vec{v},t) = \frac{d\vec{v}}{dt} = \frac{q}{m}\vec{E} + \frac{q}{mc}\,\vec{v}\times\vec{B} \qquad \left(\text{since } \vec{F} = m\vec{a} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}\right)$$

In the time interval dt, position and velocity change as

$\vec{r} \to \vec{r}\,' = \vec{r} + \vec{v}\,dt$, $\quad \vec{v} \to \vec{v}\,' = \vec{v} + \vec{a}\,dt$, with the change in distribution function $f \to f' = f + \frac{\partial f}{\partial t}dt$

[Eulerian way, we follow a change in the number of particles at a fixed position and velocity.]

Without collisions, the number of particles in a volume $d^3r\, d^3v$ remain the same in a later volume $d^3r'\, d^3v' \Rightarrow f'(\vec{r}\,',\vec{v}\,',t+dt)\,d^3r'\,d^3v' =$

$= f(\vec{r},\vec{v},t)\,d^3r\,d^3v$ <u>Liouville's theorem</u>

Electrostatic oscillations in warm plasma

with a bit of fiddling with Jacobians, we can write the Liouville's eq. to the 1st order as "collision term" we simply added here

$$\left(f+\frac{\partial f}{\partial \vec r}d\vec r+\frac{\partial f}{\partial \vec v}d\vec v+\frac{\partial f}{\partial t}dt\right)d^3r'\,d^3v'=f+\left(\frac{\partial f}{\partial t}\right)_c dt\,,\quad \vec v=\frac{d\vec r}{dt},\ \vec a=\frac{d\vec v}{dt},$$

$$\left(f+\frac{\partial f}{\partial \vec r}\vec v dt+\frac{\partial f}{\partial \vec v}\vec a dt+\frac{\partial f}{\partial t}dt\right)\left(1+\frac{d\vec a}{d\vec v}dt\right)=f+\left(\frac{\partial f}{\partial t}\right)_c dt\,,$$ we group the terms with linear dependence on dt:

$$f\frac{\partial \vec a}{\partial \vec v}dt+\frac{\partial f}{\partial \vec r}\vec v dt+\frac{\partial f}{\partial \vec v}\vec a dt+\frac{\partial f}{\partial t}dt=\left(\frac{\partial f}{\partial t}\right)_c dt \quad \Big/ \frac{1}{dt}$$

$$\frac{\partial f}{\partial t}+\vec v\frac{\partial f}{\partial \vec r}+\underbrace{f\frac{\partial \vec a}{\partial \vec v}+\frac{\partial f}{\partial \vec v}\vec a}_{=\frac{\partial}{\partial \vec v}(\vec a f)}=\left(\frac{\partial f}{\partial t}\right)_c \Rightarrow \frac{\partial f}{\partial t}+v_i\frac{\partial f}{\partial r_i}+\frac{\partial}{\partial v_i}(a_i f)=\left(\frac{\partial f}{\partial t}\right)_c$$

$(i=1,2,3)$

Since $\vec a=\frac{d\vec v}{dt} \Rightarrow \frac{\partial a_i}{\partial v_i}=0$, we remain with $\frac{\partial f}{\partial t}+v_i\frac{\partial f}{\partial r_i}+a_i\frac{\partial f}{\partial v_i}=\left(\frac{\partial f}{\partial t}\right)_c$,

which is the Boltzmann's eq. from few slides earlier (24), where now $\left(\frac{\partial f}{\partial t}\right)_c$ represents the case of collisions under the large angles.

For small angles, it will be equal to Fokker-Planck eq.

For large angles $\left(\frac{\partial f}{\partial t}\right)_c=0 \Rightarrow$ Vlasov's eq.

["Boltzmann's eq. without collisions",

$$\frac{\partial f}{\partial t}+v_i\frac{\partial f}{\partial r_i}+\left(\frac{q}{m}E_i+\frac{q}{mc}\epsilon_{ist}v_s B_t\right)\frac{\partial f}{\partial v_i}=0$$

"Fiddling": volume elements are connected through Jacobians:

$$d^3r'\,d^3v'=\frac{\partial(r_1',r_2',r_3',v_1',v_2',v_3')}{\partial(r_1,r_2,r_3,v_1,v_2,v_3)}d^3r\,d^3v,$$

for $\vec r\to\vec r'=\vec r+\vec v dt$, $\vec v\to\vec v'=\vec v+\vec a dt$, to the 1st order in dt

as $d^3r'\,d^3v'=\left(1+\frac{\partial a_i}{\partial v_i}dt\right)d^3r\,d^3v$,

Jacobian is a determinant $\begin{vmatrix}\frac{\partial \vec r'}{\partial \vec r} & \frac{\partial \vec r'}{\partial \vec v}\\ \frac{\partial \vec v'}{\partial \vec r} & \frac{\partial \vec v'}{\partial \vec v}\end{vmatrix}=$

$=\begin{vmatrix}1 & dt\\ 0 & 1+\frac{d\vec a}{d\vec v}dt\end{vmatrix}=1+\frac{\partial \vec a}{\partial \vec v}dt$, written symbolically with vectors, with indices r in $\frac{\partial a_r}{\partial v_r}$ is with $r=1,2,3$

Electrostatic oscillations in warm plasma-Fokker-Planck eq.

When we can not neglect collisions in plasma, 2 situations are considered:

a) plasma contains neutral particles = large collision angles → Boltzmann eq.

b) if neutral particles can be neglected, collisions are most affected by Coulomb forces [remember, we are doing electrostatic case!] = small angles of collisions, for this we will use Fokker-Planck eq. We wrote it few slides back (24), let's derive it. We consider the distribution of particles with the number density $n(\vec{r})$. Motion of a particle from $\vec{r}$ to $\vec{r}\,'$ we can describe with $\vec{r}\,'(\vec{r},\lambda)=\vec{r}+\lambda\xi(\vec{r})$. Particle moved for amount $\lambda\xi(\vec{r})$, the initial density distribution $n_0(\vec{r})$ changes to $n'(\vec{r})$, which we want to compute using $n(\vec{r})$, $\xi(\vec{r})$ i λ (scalar). We introduce Fourier transforms $n(\vec{r})=\int d^3k\, e^{i\vec{k}\cdot\vec{r}}\,\tilde{n}(\vec{k})$

and $\tilde{n}(\vec{k})=\frac{1}{(2\pi)^3}\int d^3r\, e^{-i\vec{k}\cdot\vec{r}}\, n(\vec{r})$

Assuming the particle number conservation (they do not decay or glue together)

$$n'(\vec{r})\,d^3\vec{r}\,'=n(\vec{r})\,d^3\vec{r},\ \text{so}$$

$$n'(\vec{k})=\frac{1}{(2\pi)^3}\int d^3r\, n(\vec{r})\, e^{-i\vec{k}[\vec{r}+\lambda\xi(\vec{r})]}$$

$$n(\vec{r})=\frac{1}{(2\pi)^3}\int d^3k\, e^{ik\cdot\vec{r}}\int d^3r'\, n(\vec{r}\,')\, e^{-i\vec{k}\cdot[\vec{r}\,'+\lambda\xi(\vec{r}\,')]}$$

Changing the order of integration and writing in powers of λ:

$$n'(\vec{r})=\frac{1}{(2\pi)^3}\int d^3r'\, n(\vec{r}\,')\int d^3k\, e^{i\vec{k}(\vec{r}-\vec{r}\,')}\,\underbrace{e^{-i\vec{k}\cdot\lambda\xi(\vec{r}\,')}}_{=1-i\lambda\vec{k}\cdot\xi(\vec{r}\,')-\frac{1}{2}\lambda^2[\vec{k}\cdot\xi(\vec{r}\,')]^2+\dots}$$

$$=\frac{1}{(2\pi)^3}\int d^3r'\, n(\vec{r}\,')\int d^3k\, e^{ik(\vec{r}-\vec{r}\,')}\left\{1-i\lambda\vec{k}\,\xi(\vec{r}\,')-\frac{1}{2}\lambda^2[\vec{k}\cdot\xi(\vec{r}\,')]^2+\dots\right\}$$

For Dirac's δ-function in 1D we know $\int\limits_{-\infty}^{\infty}e^{-i(k-l)x}dx=2\pi\delta(k-l)$, in 3D $\int\limits_{-\infty}^{\infty}e^{i(\vec{k}-\vec{l})\vec{r}}d^3r=(2\pi)^3\delta^3(\vec{k}-\vec{l})$

Integrating by $\vec{k}$, we obtain

$$n'(\vec{r})=\int d^3r'\, n(\vec{r}\,')\Big\{\delta^3(\vec{r}-\vec{r}\,')-\lambda\xi_i(\vec{r}\,')\frac{\partial}{\partial r_i}\delta^3(\vec{r}-\vec{r}\,')+$$

$$+\frac{1}{2}\lambda^2\xi_i(\vec{r}\,')\xi_j(\vec{r}\,')\frac{\partial^2}{\partial r_i\partial r_j}\delta^3(\vec{r}-\vec{r}\,')+\dots\Big\}\quad (i,j=1,2,3)$$

Electrostatic oscillations in warm plasma-Fokker-Planck eq.

Since $\frac{\partial}{\partial r_i}\delta^3(\vec{r}-\vec{r}\,') = -\frac{\partial}{\partial r_i'}\delta^3(\vec{r}-\vec{r}\,')$, [from $\delta(r)=\delta(-r)$]

with partial integration, we have

$$n'(\vec{r}) = \int d^3r'\,\delta^3(\vec{r}-\vec{r}\,')\left\{n(\vec{r}\,') - \lambda\frac{\partial}{\partial r_i'}\left[n(\vec{r}\,')\,\xi_i(\vec{r}\,')\right] + \frac{1}{2}\lambda^2\frac{\partial^2}{\partial r_i'\partial r_j'}\left[n(\vec{r}\,')\,\xi_i(\vec{r}\,')\,\xi_j(\vec{r}\,')\right]+\dots\right\}$$

$$\int d^3r'\,n(\vec{r}\,')\lambda\xi_i(\vec{r}\,')\frac{\partial}{\partial r_i}\delta^3(\vec{r}-\vec{r}\,') = \left[\begin{array}{l}\int u\,dv = uv - \int v\,du \\ u = n(\vec{r}\,')\lambda\xi_i(\vec{r}\,'),\ du = \lambda\frac{\partial}{\partial r_i'}\left[n(\vec{r}\,')\,\xi_i(\vec{r}\,')\right] \\ dv = d^3r'\,\partial^3(\vec{r}-\vec{r}\,'),\ v = \delta^3(\vec{r}-\vec{r}\,') \\ uv = 0 \text{ for } \vec{r}\neq\vec{r}\,'\end{array}\right] =$$

$$= \int\delta^3(\vec{r}-\vec{r}\,')\lambda\frac{\partial}{\partial r_i}\left[n(\vec{r}\,')\,\xi_i(\vec{r}\,')\right]$$

The same way for the 2nd term, at the end we have

$$n'(\vec{r}) = n(\vec{r}) - \lambda\frac{\partial}{\partial r_i}\left[n(\vec{r})\,\xi_i(\vec{r})\right] + \frac{1}{2}\lambda^2\frac{\partial^2}{\partial r_i\partial r_j}\left[n(\vec{r})\,\xi_i(\vec{r})\,\xi_j(\vec{r})\right]+\dots$$

This is the generalized Lagrange's series. We will use it for evolution of the distribution function for different kinds of particles. State of the particle is described with a 6D vector $(r_1, r_2, r_3, v_1, v_2, v_3)$ we need a 6D Lagrange series. We write the changes $t\to t+\Delta t$, $\vec{r}\to\vec{r}+\Delta\vec{r}$, $\vec{v}\to\vec{v}+\Delta\vec{v}$ writing for the distribution function f (and not for the particle number density)

$$f(\vec{r},\vec{v},t+\Delta t) = f(\vec{r},\vec{v},t) - \frac{\partial}{\partial r_i}(f\Delta r_i) - \frac{\partial}{\partial v_i}(f\Delta v_i) + \frac{1}{2}\frac{\partial^2}{\partial r_i\partial r_j}(f\Delta r_i\Delta r_j) + \frac{\partial^2}{\partial r_i\partial v_j}(f\Delta r_i\Delta v_j) + \frac{1}{2}\frac{\partial^2}{\partial v_i\partial v_j}(f\Delta v_i\Delta v_j) - \dots$$

(r.h.s. terms are evaluated at t)

Term $\Delta\vec{r}$ contains variations of the 1st order because of $v(t)$ and terms which are of the second order in Δt because of acceleration. There are no changes in position caused by collisions, so that to the 1st order in Δt we have $\Delta r_i = v_i\Delta t$

The Δv_i term contains 2 contributions: one because of mag. field, and other because of collisions: $\Delta\vec{v} = \vec{a}\Delta t + (\Delta\vec{v})_c$

Electrostatic oscillations in warm plasma-Fokker-Planck eq.

Collisions being random processes, their effect needs to be treated statistically. With statistic averages of $(\Delta v_i)_c (\Delta v_i)_c (\Delta v_j)_c$ etc, we find that the first and second moments $(\Delta v)_c$ contain contributions scaling like Δt, but higher moments scale with higher powers of Δt. This is consistent with random walk processes for which average quadratic shift increases linearly with time.

We can introduce the following moments:

$$\left\langle \frac{(\Delta v_i)_c}{\Delta t} \right\rangle = \left\langle \frac{\Delta v_i}{\Delta t} \right\rangle, \quad \left\langle \frac{(\Delta v_i)_c (\Delta v_j)_c}{\Delta t} \right\rangle \to \left\langle \frac{\Delta v_i \, \Delta v_j}{\Delta t} \right\rangle$$

Inserting this and $\Delta r_i = v_i \Delta t$, $\Delta \vec{v} = \vec{\Omega} \Delta t + (\Delta \vec{v})_c$ into $f(\vec{r}, \vec{v}, t+\Delta t)$ from the previous slide, we have

$$f + \frac{\partial f}{\partial t}\Delta t = f - \frac{\partial}{\partial r_i}(f v_i \Delta t) - \frac{\partial}{\partial v_i}\left[f a_i \Delta t + \left\langle \frac{\Delta v_i}{\Delta t}\right\rangle \Delta t\right] + \frac{1}{2}\frac{\partial^2}{\partial r_i \partial r_j}[f v_i \Delta t \underbrace{v_j \Delta t}_{\to 0}] +$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial r_i \partial r_j}[f v_i \underbrace{\Delta t v_j \Delta t}_{\to 0}] + \frac{\partial^2}{\partial r_i \partial v_j}\left[f v_i \Delta t \left(a_j \Delta t + \left\langle \frac{\Delta v_j}{\Delta t}\right\rangle \Delta t\right)\right] +$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left[f \left(a_i \Delta t + \left\langle \frac{\Delta v_i}{\Delta t}\right\rangle \Delta t\right)\left(a_j \Delta t + \left\langle \frac{\Delta v_j}{\Delta t}\right\rangle \Delta t\right)\right] - \dots =$$

$$= f - \frac{\partial}{\partial r_i}(f v_i \Delta t) - \frac{\partial}{\partial v_i}(f a_i \Delta t) - \frac{\partial}{\partial v_i}\left(f \left\langle \frac{\Delta v_i}{\Delta t}\right\rangle \Delta t\right) + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\Big(\underbrace{\left\langle \frac{\Delta v_i}{\Delta t}\right\rangle \Delta t \left\langle \frac{\Delta v_j}{\Delta t}\right\rangle \Delta t}_{\left\langle \frac{\Delta v_i \Delta v_j}{\Delta t}\right\rangle \Delta t}\Big) - \dots =$$

$$= f - \frac{\partial}{\partial r_i}(f v_i \Delta t) - \frac{\partial}{\partial v_i}(f a_i \Delta t) - \frac{\partial}{\partial v_i}\left(f \left\langle \frac{\Delta v_i}{\Delta t}\right\rangle \Delta t\right) + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left(f \left\langle \frac{\Delta v_i \Delta v_j}{\Delta t}\right\rangle \Delta t\right)$$

Since r_i and v_i are independent variables, $\frac{\partial v_i}{\partial r_i} = 0$

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial r_i} + f\frac{\partial v_i}{\partial r_i} + \frac{\partial f}{\partial v_i} + \underbrace{\frac{\partial f}{\partial v_i} a_i + f \frac{\partial a_i}{\partial v_i}}_{\frac{\partial}{\partial v_i}(a_i f)} = \frac{\partial}{\partial v_i}\left(f \left\langle \frac{\Delta v_i}{\Delta t}\right\rangle\right) + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left(f \left\langle \frac{\Delta v_i \Delta v_j}{\Delta t}\right\rangle\right) \Rightarrow$$

$$\Rightarrow \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial r_i} + \frac{\partial}{\partial v_i}(f a_i) = -\frac{\partial}{\partial v_i}\left(\left\langle \frac{\Delta v_i}{\Delta t}\right\rangle f\right) + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left(\left\langle \frac{\Delta v_i \Delta v_j}{\Delta t}\right\rangle f\right)$$

Fokker-Planck's eq.

In the case of nonrelativistic particles in el.mag. field $a_i = \frac{q}{m}\left(E_i + \frac{1}{c}\epsilon_{ijk} v_j B_k\right)$, $\frac{\partial a_i}{\partial v_i} = 0$ when the Coulomb collisions are included in the small angle approx.:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial r_i} + \frac{q}{m}\left(E_i + \frac{1}{c}\epsilon_{ijk} v_j B_k\right)\frac{\partial f}{\partial v_i} = -\frac{\partial}{\partial v_i}\left(\left\langle \frac{\Delta v_i}{\Delta t}\right\rangle f\right) + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left(\left\langle \frac{\Delta v_i \Delta v_j}{\Delta t}\right\rangle f\right)$$

MHD, lecture 3

- Derivation of general Ohm’s law
- Short, usually used version of Ohm’s law
- Further approximations, large conductivity limit

General Ohm's law

We consider a two-fluid (electric charge) neutral plasma consisting of electrons and protons, assuming the Maxwell distribution for both species, with taking into account the temperature difference between them.

Instead of number densities n_e, n_p we introduce mass and charge densities

$$\rho = n_p m_p + n_e m_e, \quad \xi = e(n_p - n_e)$$

The first moment renders average velocities $\langle v_{p,i}\rangle = u_{p,i}$, $\langle v_{e,i}\rangle = u_{e,i}$ $(i = 1, 2, 3)$

We introduce a "mass velocity" = the speed of the mass

$$U_i \stackrel{\text{def}}{=} \frac{n_p m_p u_{p,i} + n_e m_e u_{e,i}}{n_p m_p + n_e m_e}$$

and the electric current density $j_i = \frac{e}{c}(n_p u_{p,i} - n_e u_{e,i})$

For higher moments of distribution function, it is useful to connect particle velocity with the average mass velocity: $w_{p,i} = v_{p,i} - U_i$, $w_{e,i} = v_{e,i} - U_i$, which give the first moments $\langle w_{p,i}\rangle = u_{p,i} - U_i$, $\langle w_{e,i}\rangle = u_{e,i} - U_i$.

The second moment gives us the symmetric pressure tensors: $P_{p,ij} = m_p \int d^3v\, f_p w_i w_j = n_p m_p \langle w_{p,i} w_{p,j}\rangle$, $P_{e,ij} = m_e \int d^3v\, f_e w_i w_j = n_e m_e \langle w_{e,i} w_{e,j}\rangle$

The third moment gives the energy equation: $Q_i = \frac{1}{2} n_p m_p \langle w_p^2 w_{p,i}\rangle + \frac{1}{2} n_e m_e \langle w_e^2 w_{e,i}\rangle$, where the third moment of the velocity components, which represents the heat flux vector.

What about collisions?

Since we are considering fully ionized hydrogen plasma, we can neglect ionization and recombination and

$$\left(\frac{\partial n_p}{\partial t}\right)_c = \left(\frac{\partial n_e}{\partial t}\right)_c = 0$$

General form of the eq. of moments

On slide 29 we obtained the nonrelativistic distribution function

$$(\star)\quad \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial r_i} + \frac{q}{m}\left(\mathcal{E}_i + \frac{1}{c}\epsilon_{ijk} v_j B_k\right)\frac{\partial f}{\partial v_i} = \left(\frac{\partial f}{\partial t}\right)_c$$

Multiplying it with various functions of velocity $\psi(\vec{v})$, we obtained moment eqs., (after integration in velocity space).

Average of any function $\phi(\vec{r},\vec{v},t)$ in the velocity space: $\langle \phi(\vec{r},t)\rangle \overset{\text{def}}{=} \frac{1}{n(\vec{r},t)}\int d^3v\, \phi(\vec{r},\vec{v},t) f(\vec{r},\vec{v},t)$,

with the density $n(\vec{r},t) = \int d^3v f(\vec{r},\vec{v},t)$.

$\star / \cdot \psi(\vec{v}) \int d^3v$, the 1st term is $\int d^3v\, \psi \frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\int d^3v\, \psi f = \frac{\partial}{\partial t}\left(n\langle\psi\rangle\right)$

The 2nd term " $\int d^3v\, \psi v_i \frac{\partial f}{\partial r_i} = \frac{\partial}{\partial r_i}\int d^3v\, \psi v_i f = \frac{\partial}{\partial r_i}\left(n\langle \psi v_i\rangle\right)$

The 3rd term: $\int d^3v\, \psi \frac{q}{m}\left(\mathcal{E}_i + \frac{1}{c}\epsilon_{ijk} v_j B_k\right)\frac{\partial f}{\partial v_i} = \frac{q}{m}\mathcal{E}_i \int d^3v\, \psi \frac{\partial f}{\partial v_i} + \frac{q}{mc}\epsilon_{ijk} B_k \int d^3v\, \psi v_j \frac{\partial f}{\partial v_i} =$

$$= \frac{q}{m}\mathcal{E}_i \int d^3v \left[\frac{\partial}{\partial v_i}(\psi f) - f\left(\frac{\partial \psi}{\partial v_i}\right)\right] + \frac{q}{m}\epsilon_{ijk} B_k \int d^3v \left[\frac{\partial(\psi f)}{\partial v_i} v_j - \frac{\partial \psi}{\partial v_i} v_j f\right] =$$

$= \frac{\partial}{\partial v_i}\int d^3v (\psi f) = \frac{\partial}{\partial v_i} n\langle\psi\rangle$, and since $\langle\psi\rangle - \langle\psi(\vec{r},t)\rangle$ but not an average of the function of $v_i \Rightarrow \frac{\partial}{\partial v_i}\langle\psi\rangle = 0$

$$= -\frac{q}{m}\mathcal{E}_i \int d^3v \frac{\partial \psi}{\partial v_i} f - \frac{q}{mc}\epsilon_{ijk} B_k \int d^3v \frac{\partial \psi}{\partial v_i} v_j f = -\frac{q}{m}\mathcal{E}_i \left\langle \frac{\partial \psi}{\partial v_i}\right\rangle - \frac{q}{mc}\epsilon_{ijk} B_k n \left\langle \frac{\partial \psi}{\partial v_i} v_j\right\rangle$$

Finally, the general moment eq. is

$$\boxed{\frac{\partial}{\partial t}\left(n\langle\psi\rangle\right) + \frac{\partial}{\partial r_i}\left(n\langle\psi v_i\rangle\right) - \frac{qn}{m}\mathcal{E}_i\left\langle\frac{\partial \psi}{\partial v_i}\right\rangle - \frac{qn}{mc}\epsilon_{ijk} B_k \left\langle \frac{\partial \psi}{\partial v_i} v_j\right\rangle = \underbrace{\frac{\partial}{\partial t}\left(n\langle\psi\rangle\right)_c}_{\int d^3v\, \psi \left(\frac{\partial f}{\partial t}\right)_c}}$$

Sometimes it is also called 'transfer eq', as it represents the transfer of mass, momentum, energy etc, depending on the choice of ψ.

Eqs. of moments

The moments eq. for each species of particles: general eq. is

$$\frac{\partial}{\partial t}(nm\langle\psi\rangle)+\frac{\partial}{\partial r_i}(nm\langle\psi v_i\rangle)-qnE_i\langle\frac{\partial\psi}{\partial v_i}\rangle-\frac{qn}{c}\epsilon_{ijk}B_k\langle\frac{\partial\psi}{\partial v_i}v_j\rangle=\int d^3v\,\psi\left(\frac{\partial f}{\partial t}\right)_c$$

$\psi=1 \Rightarrow \frac{\partial}{\partial t}(nm)+\frac{\partial}{\partial r_i}(nm\langle v_i\rangle)=0 \,/\frac{1}{m}$, $\quad \frac{\partial n}{\partial t}+\frac{\partial(nu_i)}{\partial r_i}=0$

$\psi=v_i \Rightarrow \frac{\partial}{\partial t}(nmu_i)+\frac{\partial}{\partial r_j}(nm\langle v_iv_j\rangle)-nqE_i-\frac{nq}{c}\epsilon_{ijk}B_kv_j=\pm\int d^3v\,v_i\left(\frac{\partial f}{\partial t}\right)_c=\pm K_i$

$\psi=\frac{1}{2}mv^2 \Rightarrow$

since collisions of e^- and p do not change the total density of momentum, we can write $m_p\int d^3v\,v_i\left(\frac{\partial f_p}{\partial t}\right)_c=m_e\int d^3v\,v_i\left(\frac{\partial f_e}{\partial t}\right)_c=K_i$

The same will be for the energy, collisions do not change the total amount: $\frac{1}{2}m_p\int d^3v\,v^2\left(\frac{\partial f_p}{\partial t}\right)_c=\frac{1}{2}m_e\int d^3v\,v^2\left(\frac{\partial f_e}{\partial t}\right)_c=H$.

K_i can be understood as a relative force of 'friction' between the electrons and protons, and H is the speed of the energy transfer from electrons to protons.

General Ohm's law

We have all prepared now, to use the eq.

$$\frac{\partial}{\partial t}(n\langle\psi\rangle)+\frac{\partial}{\partial r_i}(n\langle\psi v_i\rangle)-\frac{qn}{m}\mathcal{E}_i\left\langle\frac{\partial\psi}{\partial v_i}\right\rangle-\frac{qn}{mc}\epsilon_{ijk}B_k\left\langle\frac{\partial\psi}{\partial v_i}v_j\right\rangle = \left(\frac{\partial}{\partial t}(n\langle\psi\rangle)\right)_c$$

$\psi=1$, $(\;)_c=0$: $\dfrac{\partial n}{\partial t}+\dfrac{\partial}{\partial r_i}(n u_i)=0$ conservation eq. for each particle species

$\psi=v_i$ and introducing $K_i = m_p\int d^3v\, v_i\left(\dfrac{\partial f_p}{\partial t}\right)_c = -m_e\int d^3v\, v_i\left(\dfrac{\partial f_e}{\partial t}\right)_c$, since

"friction force between electrons and protons."

collisions do not change the total density of momentum or energy: $H=\frac{1}{2}m_p\int d^3v\, v^2\left(\dfrac{\partial f_p}{\partial t}\right)_c = -\frac{1}{2}m_e\int d^3v\, v_i\left(\dfrac{\partial f_e}{\partial t}\right)_c$

→ speed of transport of energy from electrons to protons

gives, after multiplying with mass m:

$$\frac{\partial}{\partial t}(nm u_i)+\frac{\partial}{\partial r_j}(nm\langle v_i v_j\rangle)-nq\mathcal{E}_i-\frac{nq}{c}\epsilon_{ijk}u_j B_k=\pm K_i$$

→ + for protons, − for electrons

$\psi=\frac{1}{2}mv^2$:

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}nm\langle v^2\rangle\right)+\frac{\partial}{\partial r_i}\left(\tfrac{1}{2}nm\langle v^2 v_i\rangle\right)-nq\mathcal{E}_i u_i=\pm H$$

for the eqs. of transfer of momentum and energy for each particle species.

General Ohm's law

To write those eqs. through fluid variables, we introduce the second and third moments of velocity:

$$\langle v_i v_j \rangle = \langle (U_i + w_i)(U_j + w_j) \rangle = \langle U_i U_j + U_i w_j + w_i U_j + w_i w_j \rangle =$$

$$= U_i U_j + U_i \cancel{(u_j - U_j)} + \cancel{(u_i - U_i)} U_j + \frac{P_{ij}}{nm} = \frac{P_{ij}}{nm} + U_i u_j + U_j u_i - U_i U_j \Rightarrow$$

$$\langle v_i v_j \rangle = \langle v^2 \rangle = \frac{P_{jj}}{nm} + 2 U_i u_i - U^2$$

To have everything through average values

$$\langle v^2 v_i \rangle = \langle v^2 (w_i + U_i) \rangle = \langle v^2 w_i + v^2 U_i \rangle = \dots$$

$$= \langle w^2 w_i \rangle + \frac{2}{nm} P_{ij} U_i + \frac{1}{nm} P_{ii} U_j + 2 U_j U_i u_i + U^2 u_j - 2 U^2 U_j$$

We insert it back to our eq:

$$\frac{\partial}{\partial t}(nm u_i) + \frac{\partial}{\partial r_i}\left[P_{ij} + nm(U_i u_j + U_j u_i - U_i U_j)\right] - nq\mathcal{E}_i - \frac{nq}{c}\epsilon_{ijk} u_j B_k = \pm K_i$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}P_{jj} + nm U_j u_j - \frac{1}{2} nm U^2\right) + \frac{\partial}{\partial r_i}\left(\frac{1}{2} nm \langle w^2 w_i \rangle + \frac{1}{2} P_{jj} U_i + P_{ij} U_j + nm U_i U_j u_j + \frac{1}{2} nm U^2 u_i - nm U^2 U_i\right) -$$
$$- nq\mathcal{E}_i u_i = \pm H$$

The mass continuity is now: $\frac{\partial n}{\partial t} + \frac{\partial}{\partial r_i}(n u_i) = 0 \quad / \cdot m_{p,e}$

$$\frac{\partial (n_p m_p)}{\partial t} + \frac{\partial}{\partial r_i}(n m_p u_{p,i}) = 0$$

$$\frac{\partial (n_e m_e)}{\partial t} + \frac{\partial}{\partial r_i}(n m_e u_{e,i}) = 0 \quad / + \Rightarrow$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r_i}\underbrace{(n m_p u_{p,i} + n m_e u_{e,i})}_{\rho U_i} = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r_i}(\rho U_i) = 0$$

General Ohm's law

For $\varsigma = e(n_p - n_e)$ we can write the continuity eq. $\frac{\partial n}{\partial t} + \frac{\partial}{\partial r_i}(n u_i) = 0$ as

$$\frac{\partial n_p}{\partial t} + \frac{\partial}{\partial r_i}(n_{p,i} u_{p,i}) = 0$$

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial r_i}(n_{e,i} u_{e,i}) = 0$$

$\cdot \frac{e}{c} \Rightarrow \frac{1}{c}\frac{\partial \varsigma}{\partial t} + \frac{\partial j_i}{\partial r_i} = 0$ the charge conservation eq.

We do the same with eqs. of motion for p and e.
U_i and U_j do not carry the p, e index as they are the electron-proton plasma quantities.

$$\frac{\partial}{\partial t}(n_e m_e u_{e,i} + n_p m_p u_{p,i}) + \frac{\partial}{\partial r_i}\Big[P_{e,ij} + P_{p,ij} + \underbrace{n_e m_e(U_i u_{e,j} + U_j u_{e,i} - U_i U_j) + n_p m_p(U_i u_{p,j} + U_j u_{p,i} - U_i U_j)}_{\rho U_i U_j + \rho U_j U_i - \rho U_i U_j = \rho U_i U_j}$$

$$- \underbrace{(n_e + n_p)\mathcal{E}_i q}_{\varsigma \mathcal{E}_i} - \underbrace{\frac{n_e q}{c}\epsilon_{ijk} u_{e,j} B_k - \frac{n_p q}{c}\epsilon_{ijk} u_{p,j} B_k}_{-\epsilon_{ijk} j_j B_k} = 0 \;\Downarrow$$

$$\frac{\partial}{\partial t}(\rho U_i) + \frac{\partial}{r_j}(P_{ij} + \rho U_i U_j) - \varsigma \mathcal{E}_i - \epsilon_{ijk} j_j B_k = 0$$

usually neglected for nonrelativ. motion of fluid.

We use the mass continuity & introduce the convective derivation $\frac{dU_i}{dt} = \frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial r_j}$ to write

$$\frac{\partial}{\partial t}(\rho U_i) + \frac{\partial}{\partial r_i}(\rho U_i U_j) = \frac{\partial \rho}{\partial t} U_i + \rho \frac{\partial U_i}{\partial t} + \frac{\partial(\rho U_i)}{\partial t} U_j + \rho U_j \frac{\partial U_i}{\partial r_i} = \rho\underbrace{\left(\frac{\partial U_i}{\partial t} + \frac{\partial U_i}{\partial r_j} U_j\right)}_{\frac{dU_i}{dt}} + \Big[\underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U_j)}{\partial r_j}}_{=0}\Big] U_i = \rho \frac{dU_i}{dt}$$

We expect this pressure tel. mag. forces!

General Ohm's law

Sure, we do the same with energy eq., summing the eqs. for H for each kind of particles (e, p):

$$\frac{\partial}{\partial t}\Big[\tfrac{1}{2}(P_{e,jj}+P_{p,jj}) + n_e m_e U_j u_{e,j} + n_p m_p U_j u_{p,j} - \tfrac{1}{2}\underbrace{(n_e m_e + n_p m_p)}_{\rho} U^2\Big] +$$

$$+\frac{\partial}{\partial r_i}\Big[\underbrace{\tfrac{1}{2}(n_e m_e \langle w_e^2 w_{e,i}\rangle + n_p m_p \langle w_p^2 w_{p,i}\rangle}_{\text{slide 35, } Q_i} + \tfrac{1}{2}P_{jj}U_i + P_{ij}U_j + \overbrace{n_e m_e U_i U_j u_{e,j} + n_p m_p U_i U_j u_{p,j}}^{\rho U_j U_i U_j = \rho U^2 U_i} +$$

$$+\underbrace{\tfrac{1}{2}n_e m_e U^2 u_{e,i} + \tfrac{1}{2}n_p m_p U^2 u_{p,i}}_{\frac{1}{2}\rho U^2 U_i} - \underbrace{(n_e m_e + n_p m_p)}_{\rho} U^2 u_i\Big] - \underbrace{\underbrace{n_e q \mathcal{E}_i u_{e,i} - n_p q \mathcal{E}_i u_{p,i}}_{e(n_e u_{e,i} - n_p u_{p,i})\mathcal{E}_i}}_{c\, j_i \mathcal{E}_i} = 0$$

Finally we have $\frac{\partial}{\partial t}\left(\tfrac{1}{2}P_{jj} + \tfrac{1}{2}\rho U^2\right) + \frac{\partial}{\partial r_i}\left(Q_i + \tfrac{1}{2}P_{jj}U_i + P_{ij}U_j + \tfrac{1}{2}\rho U^2 U_i\right) - c j_i \mathcal{E}_i = 0$ for the energy flux of fluid, with pressure and el.mag. forces.

with $P_{jj} = 3NkT$, where $N = n_e + n_p \approx 2n_e$ (total number density of particles), T average plasma temperature, so we can write

$$\frac{\partial}{\partial t}\left(\tfrac{3}{2}NkT + \tfrac{1}{2}\rho U^2\right) + \frac{\partial}{\partial r_i}\left(\tfrac{3}{2}NkT U_i + \tfrac{1}{2}\rho U^2 U_i + P_{ij}U_j + Q_j\right) - c j_i \mathcal{E}_i = 0$$

Now we use the expression for k and the def. of convective derivation to write this as

$$\frac{\partial}{\partial t}\left(\tfrac{3}{2}NkT\right) + \tfrac{1}{2}\frac{\partial}{\partial t}(\rho U^2) + U_i\frac{\partial}{\partial r_i}\left(\tfrac{3}{2}NkT\right) + \tfrac{3}{2}NkT\frac{\partial U_i}{\partial r_i} + \tfrac{1}{2}\frac{\partial}{\partial r_i}(\rho U^2 U_i) + \frac{\partial}{\partial r_i}(P_{ij}U_j) + \frac{\partial}{\partial r_i}Q_i - c j_i \mathcal{E}_i = 0$$

$$\frac{d}{dt}\left(\tfrac{3}{2}NkT\right) + \tfrac{3}{2}NkT\frac{\partial U_i}{\partial r_i} + \frac{\partial P_{ij}}{\partial r_i}U_j + P_{ij}\frac{\partial U_j}{\partial r_i} + \frac{\partial}{\partial r_i}Q_i + \tfrac{1}{2}\frac{\partial}{\partial t}(\rho U^2) + \tfrac{1}{2}\frac{\partial}{\partial r_i}(\rho U^2 U_i) - c j_i \mathcal{E}_i = 0 \text{, reorder:}$$

$$\underbrace{\frac{d}{dt}\left(\tfrac{3}{2}NkT\right) + \tfrac{3}{2}NkT\frac{\partial U_i}{\partial r_i} + P_{ij}\frac{\partial U_j}{\partial r_i} + \frac{\partial Q_i}{\partial r_i}}_{A} + \underbrace{\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho U^2\right) + \frac{\partial}{\partial r_i}\left(\tfrac{1}{2}\rho U^2 U_i\right)} + \frac{\partial P_{ij}}{\partial r_i}U_j - c j_i \mathcal{E}_i = 0$$

$$\tfrac{1}{2}\left(\frac{\partial \rho}{\partial t}U^2 + \rho\frac{\partial U^2}{\partial t} + \frac{\partial(\rho U_i)}{\partial r_i}U^2 + \rho U_i \frac{\partial U^2}{\partial r_i}\right) = \tfrac{1}{2}\rho\frac{\partial U^2}{\partial t} + \tfrac{1}{2}\rho U_i\frac{\partial U^2}{\partial r_i} = \rho U_i\frac{\partial U_i}{\partial t} + \rho U_j U_i \frac{\partial U_i}{\partial r_j}.$$

$\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r_i}\rho U_i\right)U^2 = 0$ since $() = 0$

General Ohm's law

$$A+\underbrace{\rho U_i\frac{\partial U_i}{\partial t}+\rho U^{i}\frac{\partial u_j}{\partial r_i}}+\frac{\partial P_{ij}}{\partial r_i}U_j-c\,j_i\mathcal{E}_i=0$$

$$\underbrace{\frac{\partial U_i}{\partial t}+\underbrace{U_i\frac{\partial U_j}{\partial r_i}}_{U_j\frac{\partial U_i}{\partial r_j}}}_{\frac{dU_i}{dt}}$$

since mute indices, we can exchange them

$$A+\rho U_i\frac{dU_i}{dt}+\frac{\partial P_{ij}}{\partial r_i}U_j-c\,j_i\mathcal{E}_i=0$$

, $\rho\frac{dU_i}{dt}$ we got 2 slides back, so

$$A+U_i\left(-\frac{\partial P_{ij}}{\partial r_j}+\xi\mathcal{E}_i+\epsilon_{ijk}j_jB_k\right)+\frac{\partial P_{ij}}{\partial r_i}U_j-c\,j_i\mathcal{E}_i=0$$

$$A\underbrace{-\frac{\partial P_{ij}}{\partial r_j}U_i+\frac{\partial P_{ij}}{\partial r_i}U_j}_{=0}+\xi\mathcal{E}_iU_i+\epsilon_{ijk}j_jB_kU_i-c\,j_i\mathcal{E}_i=0$$

$$A+(\xi U_i-c\,j_i)\mathcal{E}_i+\epsilon_{ijk}j_jB_kU_i=0$$

and we have

$$\frac{d}{dt}\left(\frac{3}{2}nkT\right)+\frac{3}{2}nkT\frac{\partial U_i}{\partial r_i}+P_{ij}\frac{\partial U_j}{\partial r_i}+\frac{\partial Q_i}{\partial r_i}+(\xi U_i-c\,j_i)\mathcal{E}_i+\epsilon_{ijk}j_jB_kU_i=0$$

General Ohm's law

Combining the equations for K for each kind of particles we obtained the equation of motion in the middle of slide 39, but the current density is coupled with the first moment of two mediums by eqs.

If we multiply the eq. for protons with $\frac{e}{m_p}$, and for electrons with $\left(-\frac{e}{m_e}\right)$ and add the eqs. for K:

$$+\begin{cases} \frac{\partial}{\partial t}(n_p e u_{p,i}) + \frac{\partial}{\partial r_j}\left[P_{ij}\frac{e}{m_p} + n_p e\left(U_i u_{p,j} + U_j u_{p,i} - \frac{1}{c}U_i U_j\right)\right] - \frac{n_p e^2}{m_p}\mathcal{E}_i - \frac{n_p e^2}{c m_p}\epsilon_{ijk} u_{p,j} B_k = \pm K_i \frac{e}{m_p} \\ \frac{\partial}{\partial t}(-n_e e u_{e,i}) + \frac{\partial}{\partial r_j}\left[-P_{e,ij}\frac{e}{m_e} - n_e e\left(U_i u_{e,j} + U_j u_{e,i} - \frac{1}{c}U_i U_j\right)\right] - \frac{n_e e^2}{m_e}\mathcal{E}_i - \frac{n_e e^2}{c m_e}\epsilon_{ijk} u_{e,j} B_k = \pm K_j\left(-\frac{e}{m_e}\right) \end{cases}$$

$$-\frac{\partial}{\partial t}(n_p u_{p,i} - n_e u_{e,i}) + \frac{\partial}{\partial r_j}\left\{e\left(\frac{P_{p,ij}}{m_p} - \frac{P_{e,ij}}{m_e}\right) + e\left[U_i(n_p u_{p,j} - n_e u_{e,j}) + U_j(n_p u_{p,i} - n_e u_{e,i}) -\right.\right.$$

$$\left.\left. - \frac{1}{c}U_i U_j (n_p n_e)\right]\right\} - e^2\mathcal{E}_i\left(\frac{n_p}{m_p} + \frac{n_e}{m_e}\right) - \frac{e^2}{c}\epsilon_{ijk}B_k\left(\frac{n_p}{m_p}u_{p,j} + \frac{n_e}{m_e}u_{e,j}\right) = ceK_r\left(\frac{1}{m_p} + \frac{1}{m_e}\right)$$

We ignore the terms quadratic in velocities U_p, $u_{p,i}$ and $u_{e,i}$; it does not do without this:

$$c\frac{\partial j_i}{\partial t} + \frac{e}{m_p}\frac{\partial P_{p,ij}}{\partial r_j} - \frac{e}{m_e}\frac{\partial P_{e,ij}}{\partial r_j} - e^2\mathcal{E}_i\left(\frac{n_p}{m_p} + \frac{n_e}{m_e}\right) - \frac{e^2}{c}\epsilon_{ijk}B_k\left(\frac{n_p u_{p,j}}{m_p} + \frac{n_e u_{e,j}}{m_e}\right) = e\left(\frac{1}{m_p} + \frac{1}{m_e}\right)K_r \qquad (*)$$

Since $m_e \ll m_p$, $\frac{e}{m_p}\frac{\partial P_{p,ij}}{\partial r_j} \ll \frac{e}{m_e}\frac{\partial P_{e,ij}}{\partial r_j}$, $\frac{1}{m_p} + \frac{1}{m_e} \simeq \frac{1}{m_e}$, $\rho = n_p m_p + n_e m_e / \frac{1}{m_p} \Rightarrow n_e \simeq n_p \simeq \rho/m_p$

$$U_i = \frac{n_p m_p u_{p,i} + n_e m_e u_{e,i}}{n_p m_p + n_e m_e} = \frac{n_p u_{p,i} + n_e \frac{m_e}{m_p} u_{e,i}}{n_p + n_e \underbrace{\frac{m_e}{m_p}}_{\to 0}} =$$

$$\doteq \frac{n_p u_{p,i}}{n_p} = u_{p,i}$$

$$j_i = \frac{e}{c}(n_p u_{p,i} - n_e u_{e,i}) \Rightarrow$$

$$\Rightarrow \frac{c j_i}{e} = n_p U_i - \underbrace{n_e}_{= \frac{\rho}{m_p}} u_{e,i} = n_p U_i - \frac{\rho}{m_p}u_{e,i} \Rightarrow u_{e,i} = \frac{m_p}{\rho} n_p U_i - \frac{m_p c}{\rho e}$$

$$\boxed{u_{e,i} = U_i - \frac{c m_p}{e\rho} j_i}$$

Choosing the indices in $\mathcal{E}_i$ and B_k to match $\frac{\partial}{\partial r_j}$, i.e. changing $i \to j$, we can write

$$c\frac{\partial j_i}{\partial t} = \frac{e}{m_e}\frac{\partial P_{e,ij}}{\partial r_j} + \frac{e^2\rho}{m_p m_e}\mathcal{E}_i + \frac{e^2\rho}{c m_p m_e}\epsilon_{ijl} U_j B_k - \frac{e}{m_e}\epsilon_{ijk} j_j B_k + \frac{e}{m_e} k_i \quad \Big/ \cdot \frac{m_e m_p}{\rho e^2}$$

In vector notation: $$c\frac{\partial \vec{j}}{\partial t} = \frac{e}{m_e}\nabla P_e + \frac{e^2\rho}{m_p m_e}(\vec{\mathcal{E}} + U\times\vec{B}) - \frac{e}{m_e}(\vec{j}\times\vec{B}) + \frac{e}{m_e}\vec{k}$$

If $|\vec{u}_p - \vec{u}_e| \ll$, we can write $k_i = \nu m_e n (u_{e,i} - u_{p,i})$

$\hookrightarrow$ momentum collision frequency for scattering of electrons on protons.

Introducing the resistivity coefficient η and conductivity coefficient $\sigma = \frac{1}{\eta} = \frac{n e^2}{m_e c \nu}$, $k_i = -\nu m_e n(u_{p,i} - u_{e,i}) = -\nu m_e \frac{c}{e} j_i = -\frac{\eta n e^2}{m_e c} m_e \frac{c}{e} j_i$

$$\nu = \frac{\eta n e^2}{m_e c} \qquad j_i = \frac{e}{c}(n_p u_{p,i} - n_e u_{e,i}) \qquad = \frac{\rho e}{m_p}\eta j_i$$

$$n_e = n_p = n$$

$$j_i \approx \frac{ne}{c}(u_{p,i} - u_{e,i})$$

We insert k_i into ($\star$) and multiply with $\frac{m_e m_p}{\rho e^2}$

$$(6) \quad \boxed{\frac{c\, m_e m_p}{\rho e^2}\frac{\partial j_i}{\partial t} = \mathcal{E}_i + \frac{1}{c}\epsilon_{ijk} U_j B_k - \frac{m_p}{\rho e}\epsilon_{ijk} j_j B_k - \frac{m_p}{\rho e}\frac{\partial P_{e,ij}}{\partial r_j} - \eta j_i}$$

Generalized Ohm's law.

It defines the current density through the electric and mag. field, "mass speed" and electronic pressure gradient. We neglected the quadratic terms in velocity and approximated all in such a way to obtain linear equation in j_i. ///

General Ohm's law

In the frequency of collisions ν, consideration for relaxation times give approximations from which for electron-proton plasma $\nu=10^2\, nT^{-3/2}$.

From the relation in the middle of previous slide we could compute η, but for a more precise value we need to include the electron-electron collisions. Since the isotropization because of such collisions is faster than for electron-ion collisions, which we therefore can neglect.

Inclusion of electron-electron collisions increases the resistivity for a factor of 2.7 , so $\nu=10^{3\cdot 7}\, nT^{-3/2}$ giving $\eta=1/\sigma=10^{3\cdot 7}\, nT^{-3/2}$ in modified Gauss units, with current expressed in emu and electric field in esu.

It is interesting that the plasma number density is not present in the expression. It is because of the proportionality of both the plasma number carriers and the frequency of collisions with n, so the two effects cancel in the equations.

Now we embark on various simplification of the general expression G, to obtain the Ohm's law usually used in practice.

Ohm's law in ideal MHD systems

$$\frac{c m_e m_p}{\rho e^2}\frac{\partial j_i}{\partial t} = \mathcal{E}_i + \frac{1}{c}\epsilon_{ijk}U_j B_k - \frac{m_p}{\rho e}\epsilon_{ijk} j_j B_k + \frac{m_p}{\rho e}\frac{\partial P_{e,ij}}{\partial r_j} - \eta j_i \qquad (6)$$

The ratio of and , with perturbations of the frequency ω, $j_j = \nu A \sin\omega t$, $\frac{\partial j_j}{\partial t} = A\omega\cos\omega t$

and $\sigma = \frac{1}{\eta} = \frac{ne^2}{m_e c\nu}$, $R_1 = \frac{\left|\frac{c m_p m_e}{\rho e^2}\frac{\partial j_i}{\partial t}\right|}{\left|\frac{m_e c\nu}{ne^2} j_i\right|} = \frac{|A\omega\cos\omega t|}{|\nu A\sin\omega t|} = \frac{\omega}{\nu}$

$\frac{m_p}{\rho} \simeq \frac{1}{n}$ from slide 42

We can neglect the l.h.s. term for the low frequency ω small with respect to e-p collisions, then $R_1 \ll 1$ and we can write $\mathcal{E}_i + \frac{1}{c}\epsilon_{ijk}U_jB_k - \frac{m_p}{\rho e}\epsilon_{ijk}j_jB_k + \frac{m_p}{\rho e}\frac{\partial P_{e,ij}}{\partial r_j} - \eta j_i = 0$, $\eta = \frac{1}{\sigma}$

$$j_i + \frac{m_p}{\rho e}\sigma\epsilon_{ijk}j_jB_k = \sigma\left(\mathcal{E}_i + \frac{1}{c}\epsilon_{ijk}U_jB_k\right) + \frac{\sigma m_p}{\rho e}\frac{\partial P_{e,ij}}{\partial r_j} \qquad (*)$$

Taking the ratio of two last terms

$R_2 = \frac{\left|\frac{\sigma m_p}{\rho e}\frac{\partial P_{e,ij}}{\partial r_j}\right|}{\frac{\sigma}{c}|\epsilon_{ijk}U_jB_k|}$ and $\left|\frac{\partial P_{e,ij}}{\partial r_j}\right| \simeq \frac{P_e}{L_j}$ → lengthscale for the pressure gradient

$\frac{m_p}{\rho} = \frac{1}{n}$

$P_e = n_e kT_e = \frac{1}{3}n_e m_e v_{th,e}^2$ thermal

$$R_2 = \frac{m_e|v_{th,e}^2|}{3L\frac{e}{c}|\underbrace{\epsilon_{ijk}U_jB_k}_{\vec U\times\vec B}|} = \frac{m_e|v_{th,e}^2|}{3L\frac{e}{c}U\frac{m_e c v_{th,e}}{e r_g}} = \frac{1}{3}\frac{v_{th,e}}{U}\frac{r_g}{L}$$

giromagnetic

e^- and ions tend to move in circular or helicoidal orbits in B, with orbital frequencies $\omega_B = \frac{eB}{mc}$. Since $\vec v = \vec\omega\times\vec r$, for the circular orbit $\frac{v}{r_g} = \frac{eB}{mc}$, $B = \frac{mcv}{er_g}$, $v\sim v_{th,e}$

If $\frac{L}{r_g} \gg \frac{v_{th}}{U} \Rightarrow R_2 \ll 1$, we can neglect the last term in (*):

$$\boxed{j_i + \frac{m_p}{\rho e}\sigma\epsilon_{ijk}j_jB_k = \sigma\left(\mathcal{E}_i + \frac{1}{c}\epsilon_{ijk}U_jB_k\right)} \qquad (**)$$

For the ratio of and j_i: $R_3 = \frac{\left|\frac{m_p}{\rho e}\sigma\epsilon_{ijk}j_jB_k\right|}{|j_i|} = \frac{\left|\frac{m_p}{\rho e}\frac{\rho}{m_p}\frac{e^2}{m_e c\nu}\epsilon_{ijk}j_jB_k\right|}{|j_i|} = \frac{\left|\frac{e}{m_e c\nu}\vec j\times\vec B\right|}{|\vec j|} \simeq \frac{\frac{e}{m_e c\nu}j\frac{m_e c v}{e r_g}}{j} = \frac{\omega_{g,e}}{\nu}$, $\frac{v}{r_g} = \omega_{g,e}$

With the dense enough plasma we will have high enough ν to have $R_3 \ll 1$ and $j_i = \sigma\left(\mathcal{E}_i + \frac{1}{c}\epsilon_{ijk}U_jB_k\right) \Rightarrow \boxed{\vec j = \sigma\left(\vec{\mathcal{E}} + \frac{1}{c}\vec U\times\vec B\right)}$

This is the usually used eq. for ideal MHD cases.

Ohm's law in ideal MHD systems

From the charge conservation $\frac{1}{c}\frac{\partial \xi}{\partial t} + \frac{\partial j_i}{\partial r_i} = 0$ using the Poisson's eq.
(actually Maxwell's eq.) $\nabla\cdot\vec{\mathcal{E}} = 4\pi\xi$
$\equiv \frac{\partial \mathcal{E}_i}{\partial r_i} = 4\pi\xi$, $\frac{\partial}{\partial t}(\nabla\cdot\vec{\mathcal{E}}) = \frac{\partial}{\partial t}\frac{\partial \mathcal{E}_i}{\partial r_i} = \frac{\partial}{\partial r_i}\frac{\partial \mathcal{E}_i}{\partial t} = 4\pi\frac{\partial \xi}{\partial t}$

$\frac{\partial}{\partial r_i}\left(\frac{1}{4\pi c}\frac{\partial \mathcal{E}_i}{\partial t} + j_i\right) = 0$, ratio of terms inside ( ) is $R_4 = \frac{\left|\frac{1}{4\pi c}\frac{\partial \vec{\mathcal{E}}}{\partial t}\right|}{|\vec{j}|}$.

In the limit of high conductivity σ, in the Ohm's law $\vec{j} = \sigma\left(\vec{\mathcal{E}} + \frac{1}{c}\vec{U}\times\vec{B}\right)$ we can take only $\frac{1}{c}\vec{U}\times\vec{B}$ since then $\frac{\vec{j}}{\sigma} = \vec{\mathcal{E}} + \frac{1}{c}\vec{U}\times\vec{B}$, $\sigma \gg \Rightarrow \vec{\mathcal{E}} = -\frac{1}{c}\vec{U}\times\vec{B}$

If we take the wave-like perturbation of the frequency ω and the wave number k, we can write, from Maxwell's, $\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j} + c\frac{\partial\vec{\mathcal{E}}}{\partial t}$ that $\vec{j} = \frac{c}{4\pi}\nabla\times\vec{B} - \frac{c^2}{4\pi}\frac{\partial\vec{\mathcal{E}}}{\partial t}$

This into $R_4 = \frac{\left|\frac{1}{4\pi}\frac{\partial\vec{\mathcal{E}}}{\partial t}\right|}{\left|\frac{c}{4\pi}\nabla\times\vec{B} - \frac{c^2}{4\pi}\frac{\partial\vec{\mathcal{E}}}{\partial t}\right|} = \frac{1}{\left|c\frac{\nabla\times\vec{B}}{\frac{\partial\vec{\mathcal{E}}}{\partial t}} - c^2\right|} =$ $\left[\begin{array}{l}\vec{\mathcal{E}} \simeq -\frac{1}{c}\vec{U}\times\vec{B} \simeq -\frac{1}{c}\vec{U}\cdot\vec{B}\,\vec{n} \to \text{a normal to } \vec{U}\times\vec{B} \\ \frac{\partial\vec{\mathcal{E}}}{\partial t} = -\frac{1}{c}\vec{U}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{i}{c}\vec{U}\omega\vec{B}, \text{ since } \vec{B} = \vec{B}e^{i(\vec{k}\vec{r}-\omega t)} \\ \nabla\times\vec{B} = i\vec{k}\vec{B}\end{array}\right]$

$$R_4 = \frac{1}{\left|c\frac{i\vec{k}\cdot\vec{B}}{\frac{i}{c}\omega\vec{U}\vec{B}} - c^2\right|} = \frac{1}{\left|\frac{c^2 k}{\omega U} - c^2\right|} = \frac{1}{c^2}\frac{1}{\left|\frac{k}{\omega U} - 1\right|}$$

If we do the Taylor expansion for the subrelativistic cases in small ω: $R_4 = \frac{1}{c^2}\left|\frac{1}{\frac{k}{\omega U} - 1}\right| \underset{\omega\to 0}{\approx} \frac{1}{c^2}\frac{1}{\frac{k}{\omega U}} = \frac{\omega U}{kc^2}$

For the subrelativistic motion of both the fluid and phase velocity, $R_4 \ll 1$ and the eq. of charge conservation reduces to $\frac{\partial \mathcal{E}_i}{\partial t} \ll j_i$, $\frac{\partial j_i}{\partial r_i} \simeq 0 \Rightarrow$ we can neglect the Maxwell's displacement current $\frac{1}{c}\frac{\partial\vec{\mathcal{E}}}{\partial t}$ in comparison with the current $\vec{j}$, and we remain with the $\underline{\underline{\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j}}}$ (actually a pre-Maxwell eq.!)

Ohm's law in ideal MHD systems

We obtained before

$$(\star)\quad \rho\frac{dU_i}{dt} = -\frac{\partial P_{ij}}{\partial r_j} + \varsigma\mathcal{E}_i + \epsilon_{ijk} j_j B_k,$$

If we take the ratio of the 2^{nd} and 3^{rd} terms on r.h.s. $R_\varsigma = \frac{|\varsigma\vec{\mathcal{E}}|}{|\vec{j}\times\vec{B}|}$, from $\nabla\cdot\vec{\mathcal{E}} = 4\pi\varsigma$, $\vec{\mathcal{E}} = \vec{\mathcal{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, $\nabla\cdot\vec{\mathcal{E}} = i\vec{k}\cdot\vec{\mathcal{E}} \Rightarrow |\varsigma| = \frac{1}{4\pi}k\mathcal{E}$

with previously found $\vec{\mathcal{E}} \sim -\frac{1}{c}\vec{U}\times\vec{B}$

$$R_\varsigma = \frac{|\frac{1}{4\pi}k\mathcal{E}\,\frac{1}{c}UB|}{|\vec{j}\times\vec{B}|} = \frac{|\frac{1}{4\pi c^2}kU^2B|}{|\vec{j}\times\vec{B}|} = \frac{1}{4\pi}\left|\frac{U}{c}\right|^2.$$

For sub-relativistic motion, $R_\varsigma \ll 1$ so the eq. of motion $(\star)$ can be simplified to

$$\rho\frac{dU_i}{dt} = -\frac{\partial P_{ij}}{\partial r_j} + \epsilon_{ijk} j_j B_k .$$

From the energy eq. at the bottom of slide 41, with $\sigma \gg$ we can neglect the last 2 terms:

$$\ldots + \varsigma U_i\mathcal{E}_i - c j_i\mathcal{E}_i + \underbrace{\epsilon_{ijk} j_j B_k U_i}_{(\vec{j}\times\vec{B})\cdot\vec{U}} = 0 \Rightarrow \ldots + \varsigma\vec{U}\cdot\vec{\mathcal{E}} - c\vec{j}\cdot\vec{\mathcal{E}} + (\vec{j}\times\vec{B})\cdot\vec{U}$$

For $\frac{j_i}{\sigma}\to 0 \Rightarrow \vec{\mathcal{E}} \sim -\frac{1}{c}\vec{U}\times\vec{B}$ and $\ldots - \varsigma\frac{\vec{U}\times\vec{B}}{c} + \underbrace{\vec{j}\cdot(\vec{U}\times\vec{B}) + \underbrace{(\vec{j}\times\vec{B})\cdot\vec{U}}_{-\vec{j}(\vec{U}\times\vec{B})}}_{=0} = 0$

$\ldots - \varsigma\frac{\vec{U}\times\vec{B}}{c} = 0$. Since $|\varsigma| = \frac{1}{4\pi}\vec{k}\cdot\vec{\mathcal{E}} = \frac{1}{4\pi c}\vec{k}\,\vec{U}\times\vec{B}$

$\ldots - \frac{k}{4\pi c^2}(\vec{U}\times\vec{B})^2 = 0$, $\frac{1}{c^2}$ means that we can neglect this term too!

Neglecting the heat flux and assuming the isotropic tensor of pressure, $P = nkT$ we can write

$$\frac{d}{dt}\left(\frac{3}{2}P\right) + \frac{3}{2}P\frac{\partial U_i}{\partial r_i} + P\frac{\partial U_j}{\partial r_i} = 0 \quad j=i$$

$$\Rightarrow \frac{d}{dt}\left(\frac{3}{2}P\right) + \frac{5}{2}P\frac{\partial U_i}{\partial r_i} = 0 \qquad (\triangle)$$

The continuity eq. $\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial r_i}(\rho U_i) = 0$ can be written as

$$\underbrace{\frac{\partial\rho}{\partial t} + U_i\frac{\partial\rho}{\partial r_i}}_{\frac{d\rho}{dt}} + \rho\frac{\partial U_i}{\partial r_i} = 0 \quad \left(\text{since } \frac{dA_i}{dt} \overset{\text{def}}{=} \frac{\partial A_i}{\partial t} + U_j\frac{\partial A_i}{\partial r_j}\right)$$

$\frac{d\rho}{dt} = -\rho\frac{\partial U_i}{\partial r_i}$ and we can write $(\triangle)$ as $\frac{d}{dt}\left(\frac{3}{2}P\right) + \frac{5}{2}P\left(-\frac{1}{\rho}\frac{d\rho}{dt}\right) = 0$, $\frac{1}{P}\frac{dP}{dt} = \frac{5}{3}\frac{1}{\rho}\frac{d\rho}{dt}$

For such „ideal" case we obtain $\frac{dP}{P} = \frac{5}{3}\frac{d\rho}{\rho}\ /\int$, $\ln P = \frac{5}{3}\ln\rho + \text{const} = \ln\rho^{5/3} + \text{const} \Rightarrow P = \text{const}\,\rho^{5/3}$

adiabatic equation of state!

We obtained eqs. for highly conductive plasma and isotropic pressure:

$\frac{\partial\rho}{\partial t} + \nabla(\rho\vec{U}) = 0$, $\rho\frac{d\vec{U}}{dt} = -\nabla P + \vec{j}\times\vec{B}$, $P\rho^{-5/3} = \text{const}$, two Maxwell's eqs: $\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{U}\times\vec{B})$, $\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j}$

from $\nabla\times\vec{\mathcal{E}} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$, $\vec{\mathcal{E}} = -\frac{1}{c}\vec{U}\times\vec{B}$, $-\frac{1}{c}\nabla\times(\vec{U}\times\vec{B}) + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$

MHD, lecture 4

- Alfven waves
- Toroidal and poloidal fields
- Helmholtz decomposition

Waves in magnetized plasmas

In nonmagnetic hydrodynamics, except surface waves, the only other possible waves of small amplitude are longitudinal, sound waves. Speed of propagation of such waves is related to derivation of pressure with respect to density, with constant entropy. At the end of the previous lecture, we found the pressure & density relation which is relevant in most astrophysical plasmas. It characterizes hydrodynamical waves. In magnetic plasmas, there is another kind of waves, related with transverse motion of the lines of magnetic induction B. Tension in those lines, which is of magnetic origin, tends to straighten them, and this produces transverse oscillations.

Analogous to usual sound waves soundspeed^2=pressure/density, we expect v=B^2/(8πρ), where B^2/(8π) is the magnetic pressure and v is the Alfven speed, the speed of propagation of such waves.

In addition to them, there exist also longitudinal magnetic wave=magnetosonic wave. Let's derive those waves in magnetized plasmas.

Alfven waves

We consider magnetized, conducting fluid without gravitational field. The eqs. describing it are:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) = 0 \quad (E1)$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\nabla \cdot \vec{v})\vec{v} = -\nabla P - \frac{1}{4\pi}\vec{B}\times(\nabla\times\vec{B})$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{v}\times\vec{B})$$

$$P = k\rho^{\gamma}$$

We assume the equilibrium velocity $\vec{v}_0 = 0$, uniform, static mag. induction $\vec{B}_0$ and $\rho_0 = \text{const}$.

Perturbation method: small departures from equilibrium

$$\vec{B} = \vec{B}_0 + \vec{B}_1(\vec{r},t)$$
$$\rho = \rho_0 + \rho_1(\vec{r},t)$$
$$\vec{v} = \vec{v}_0 + \vec{v}_1(\vec{r},t)$$

For adiabatic change $P = k(\rho_0 + \rho_1)^{\gamma}$

$$\nabla P = \gamma k(\rho_0+\rho_1)^{\gamma-1}\cdot\nabla(\rho_0+\rho_1)$$

$$\frac{\partial P}{\partial \rho} = \gamma k(\rho_0+\rho_1)^{\gamma-1}\cdot\underbrace{\frac{\partial \rho_1}{\partial \rho}}_{=1}$$

$$\underbrace{\gamma k(\rho_0+\rho_1)^{\gamma-1}}_{\frac{\partial P}{\partial \rho} = c_s^2}, \quad \nabla\rho_0 \to 0 \text{ as } \rho_0 = \text{const} \implies \boxed{\nabla P = c_s^2 \nabla\rho_1}$$

The continuity eq. becomes $\frac{\partial \rho_1}{\partial t} + \nabla[(\rho_0+\rho_1)\cdot\vec{v}_1] = 0$, $\boxed{\frac{\partial \rho_1}{\partial t} + \rho_0\nabla\cdot\vec{v}_1 = 0}$ (E1')

$$\underbrace{(\nabla\rho_0)}_{=0}\vec{v}_1 + \rho_0\nabla\cdot\vec{v}_1 + \nabla(\rho_1\vec{v}_1)$$

$\to 0$ as it is the 2nd order in small quantities

$$\frac{\partial(\vec{B}_0+\vec{B}_1)}{\partial t} = \nabla\times\vec{v}_1\times(\vec{B}_0+\vec{B}_1)$$

$$\boxed{\frac{\partial \vec{B}_1}{\partial t} = \nabla\times(\vec{v}_1\times\vec{B}_0)} \quad (E3)$$

$$(\rho_0+\rho_1)\frac{\partial \vec{v}}{\partial t} + (\rho_0+\rho_1)(\nabla\cdot\vec{v}_1)\underbrace{\vec{v}_1}_{\to 0} = -\nabla P - \frac{1}{4\pi}(\vec{B}_0+\vec{B}_1)\times[\nabla\times(\vec{B}_0+\vec{B}_1)]$$

$$\boxed{\rho_0\frac{\partial \vec{v}_1}{\partial t} = -c_s^2\nabla\rho_1 - \frac{1}{4\pi}\vec{B}_0\times(\nabla\times\vec{B}_1)} \quad (E2)$$

$$\vec{B}_0\times\nabla\times(\vec{B}_0+\vec{B}_1) + \vec{B}_1\times[\nabla\times(\vec{B}_0+\vec{B}_1)] = \vec{B}_0\times(\nabla\times\vec{B}_1) + \underbrace{\vec{B}_1\times(\nabla\times\vec{B}_1)}_{=0}$$

From E1, E2 & E3 we can find expression for $\vec{v}_1$. (E2) / $\frac{\partial}{\partial t}$

$$\rho_0\frac{\partial^2\vec{v}_1}{\partial t^2} = -c_s^2\nabla\frac{\partial\rho_1}{\partial t} - \frac{\vec{B}_0}{4\pi}\times\left(\nabla\times\frac{\partial\vec{B}_1}{\partial t}\right) \quad (E3) \;/\; \frac{1}{\rho_0}$$

(E1') $-c_s^2\nabla(-\rho_0\nabla\cdot\vec{v}_1)$

$c_s^2\rho_0\nabla(\nabla\cdot\vec{v}_1)$

$$\boxed{\frac{\partial^2\vec{v}_1}{\partial t^2} + c_s^2\nabla(\nabla\cdot\vec{v}_1) + \frac{\vec{B}_0}{4\pi\rho_0}\times\nabla\times[\nabla\times(\vec{v}_1\times\vec{B}_0)] = 0} \quad (E)$$

(E) we can write

$$\frac{\partial^2\vec{v}_1}{\partial t^2} + c_s^2\nabla(\nabla\cdot\vec{v}_1) + \frac{\vec{B}_0}{\sqrt{4\pi\rho_0}}\times\nabla\times\left[\nabla\times\left(\vec{v}_1\times\frac{\vec{B}_0}{\sqrt{4\pi\rho_0}}\right)\right] = 0$$

$\vec{v}_A = \frac{\vec{B}_0}{\sqrt{4\pi\rho_0}}$ "Alfvén velocity"

$$\boxed{\frac{\partial^2\vec{v}_1}{\partial t^2} + c_s^2\nabla(\nabla\cdot\vec{v}_1) + \vec{v}_A\times\nabla\times[\nabla\times(\vec{v}_1\times\vec{v}_A)] = 0} \quad \boxed{A}$$

Complicated, but for waves $\parallel$ or $\perp$ to $\vec{B}_0$ we can solve.

Longitudinal Alfven waves

We proceed usual way: let $\vec{v}_1(\vec{r},t)$ be a continuous wave with wave vector $\vec{k}$ and frequency ω.

Then $\vec{v}_1(\vec{r},t) = \vec{v}_{10}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ ← const amplitude; $\vec{k}$ → const direction of propagation.

$\nabla\cdot\vec{v}_1 = i\vec{k}\cdot\vec{v}_1$, $\nabla(\nabla\cdot\vec{v}_1) = -(\vec{k}\cdot\vec{v}_1)\cdot\vec{k}$

$\frac{\partial \vec{v}_1}{\partial t} = -i\omega\vec{v}_1$, $\frac{\partial^2 \vec{v}_1}{\partial t^2} = -\omega^2\vec{v}_1$

important direction $\vec{k}$, not $\nabla^2\vec{v}_1$!

$\nabla\times\vec{v}_1 = \nabla\times(\vec{v}_{10}e^{(\,)}) = \nabla(e^{(\,)}\times\vec{v}_{10}) + e^{(\,)}\underbrace{(\nabla\times\vec{v}_{10})}_{=0 \text{ as } \vec{v}_{10}=\text{const}} = i\vec{k}\times\vec{v}_1$, with $\nabla e^{(\,)}\times\vec{v}_{10} = i\vec{k}e^{(\,)}\times\vec{v}_{10}$; that means $(\nabla\nabla)\vec{v}_1$

Inserting into $\boxed{A}$ $\frac{\partial^2\vec{v}_1}{\partial t^2} + c_s^2\nabla(\nabla\cdot\vec{v}_1) + \vec{v}_A\times\nabla\times[\nabla\times(\vec{v}_1\times\vec{v}_A)] = 0$ we have

$$-\omega^2\vec{v}_1 + c_s^2(\vec{k}\cdot\vec{v}_1)\vec{k} + \vec{v}_A\times\nabla\times[\nabla\times(\vec{v}_1\times\vec{v}_A)] = 0 \qquad \boxed{A'}$$

$$\nabla\times(\vec{v}_1\times\vec{v}_A) = (\vec{v}_A\nabla)\vec{v}_1 - \vec{v}_A(\nabla\vec{v}_1) + \vec{v}_1\underbrace{(\nabla\cdot\vec{v}_A)}_{=0} - \underbrace{(\vec{v}_1\cdot\nabla)\vec{v}_A}_{=0} = i(\vec{v}_A\cdot\vec{k})\vec{v}_1 - i(\vec{k}\cdot\vec{v}_1)\vec{v}_A$$

$$= \vec{v}_A\times\nabla\times[(\vec{v}_A\cdot\vec{k})\vec{v}_1 - (\vec{k}\cdot\vec{v}_1)\vec{v}_A]$$

$$\nabla(\vec{v}_A\vec{k})\times\vec{v}_1 + (\vec{v}_A\cdot\vec{k})(\nabla\times\vec{v}_1)$$

$$\nabla(\vec{k}\cdot\vec{v}_1)\times\vec{v}_A + (\vec{k}\vec{v}_1)\underbrace{(\nabla\times\vec{v}_A)}_{=0}$$

$\boxed{A'}$ becomes:

$$-\omega^2\vec{v}_1 + c_s^2(\vec{k}\cdot\vec{v}_1)\vec{k} + i\vec{v}_A\times\{[\nabla(\underbrace{\vec{v}_A\cdot\vec{k}}_{\text{const}})\times\vec{v}_1] + (\vec{v}_A\cdot\vec{k})\, i(\vec{k}\times\vec{v}_1) - \underbrace{\nabla(\vec{k}\cdot\vec{v}_1)}_{i(\vec{k}\cdot\vec{v}_1)\vec{k}}]\times\vec{v}_A\} = 0$$

[careful, $(\vec{a}\cdot\vec{b})\vec{c} \neq \vec{a}(\vec{b}\cdot\vec{c})$ only for $\vec{k}\parallel\vec{v}_A$ it will become simpler]

$$-\omega^2\vec{v}_1 + c_s^2(\vec{k}\cdot\vec{v}_1)\vec{k} - (\vec{v}_A\cdot\vec{k})[(\vec{v}_A\cdot\vec{v}_1)\vec{k} - (\vec{v}_A\cdot\vec{k})\vec{v}_1] + (\vec{k}\cdot\vec{v}_1)[(\vec{v}_A\cdot\vec{v}_A)\vec{k} - (\vec{v}_A\cdot\vec{k})\vec{v}_A] = 0$$

$$-\omega^2\vec{v}_1 + c_s^2(\vec{k}\cdot\vec{v}_1)\vec{k} - (\vec{v}_A\cdot\vec{k})(\vec{v}_A\cdot\vec{v}_1)\vec{k} + (\vec{v}_A\cdot\vec{k})(\vec{v}_A\cdot\vec{k})\vec{v}_1 + (\vec{k}\cdot\vec{v}_1)\underbrace{(\vec{v}_A\cdot\vec{v}_A)}_{v_A^2}\vec{k} - (\vec{k}\cdot\vec{v}_1)(\vec{v}_A\cdot\vec{k})\vec{v}_A = 0$$

$$-\omega^2\vec{v}_1 + (c_s^2 + v_A^2)(\vec{k}\cdot\vec{v}_1)\vec{k} + (\vec{v}_A\cdot\vec{k})[(\vec{v}_A\cdot\vec{k})\vec{v}_1 - (\vec{v}_A\cdot\vec{v}_1)\vec{k} - (\vec{k}\cdot\vec{v}_1)\vec{v}_A] = 0 \qquad \boxed{\star}$$

For $\vec{k}\perp\vec{v}_A$, sound waves $\perp$ to $\vec{B}_0$, $\vec{v}_A\cdot\vec{k}=0$, in $\boxed{\star}$ we have only $-\omega^2\vec{v}_1 + (c_s^2+v_A^2)(\vec{k}\cdot\vec{v}_1)\vec{k} = 0$

Such solutions are, with $\vec{v}_1 = \vec{v}_1 e^{i(\vec{k}\vec{r}-\omega t)}$, $\partial_t^2\vec{v}_1 = -\omega^2\vec{v}_1$, $\partial_r^2\vec{v}_1 = -k^2\vec{v}_1$, solution of the equation

$$\partial_t^2\vec{v}_1 - (c_s^2+v_A^2)\partial_r^2\vec{v}_1 = 0,$$

which are longitudinal magnetosonic waves of the phase velocity $u_{long} = \sqrt{c_s^2+v_A^2}$

This wave propagates with a speed which is a sum of hydrostatic and magnetic pressures. ←

Transverse Alfven waves

For the case with $\vec{k} \parallel \vec{B}_0$ (i.e. $\vec{k} \parallel \vec{v}_A$) in $\boxed{\star}$ from the previous slide

$$\vec{v}_A \cdot \vec{k}\left[v_A k \vec{v}_1 - (\vec{v}_A \cdot \vec{v}_1)\vec{k} - (\vec{k}\cdot\vec{v}_1)\vec{v}_A\right] = v_A^2 k^2 \vec{v}_1 - (\vec{v}_A \cdot \vec{k})(\vec{v}_A\cdot\vec{v}_1)\cdot\vec{k} - (\vec{v}_A\cdot\vec{k})\underbrace{(\vec{k}\cdot\vec{v}_1)\vec{v}_A}_{\substack{\text{for } \vec{v}_A \parallel \vec{k} \\ = (\vec{v}_A \cdot \vec{v}_1)\vec{k}}} = v_A^2 k^2 \vec{v}_1 - 2(\vec{v}_A\cdot\vec{k})(\vec{v}_A\cdot\vec{v}_1)\vec{k}$$

$$\underbrace{- (\vec{v}_A \cdot \vec{k})(\vec{v}_A\cdot\vec{v}_1)\cdot\vec{k} - (\vec{v}_A\cdot\vec{k})(\vec{k}\cdot\vec{v}_1)\vec{v}_A}_{-2(\vec{v}_A\cdot\vec{k})(\vec{v}_A\cdot\vec{v}_1)\vec{k}}$$

Now $\boxed{\star}$ becomes:

$$-\omega^2 \vec{v}_1 + (c_s^2 + v_A^2)(\vec{k}\cdot\vec{v}_1)\vec{k} + v_A^2 k^2 \vec{v}_1 - 2(\vec{v}_A\cdot\vec{k})(\vec{v}_A\cdot\vec{v}_1)\vec{k} = 0$$

$$(k^2 v_A^2 - \omega^2)\vec{v}_1 + (c_s^2 + v_A^2)(\vec{k}\cdot\vec{v}_1)\vec{k} - 2\underbrace{\underbrace{(\vec{v}_A\cdot\vec{k})}_{k v_A}(\vec{v}_A\cdot\vec{v}_1)\vec{k}}_{\substack{v_A^2 k \;\; \hat{k}k \\ -2v_A^2(\vec{k}\cdot\vec{v}_1)\vec{k}}} = 0,$$

$$(k^2 v_A^2 - \omega^2)\vec{v}_1 + (c_s^2 + v_A^2 - 2v_A^2)(\vec{k}\cdot\vec{v}_1)\vec{k} = 0$$

$$(k^2 v_A^2 - \omega^2)\vec{v}_1 + (c_s^2 - v_A^2)(\vec{k}\cdot\vec{v}_1)\vec{k} = 0, \quad \hat{k} = \hat{v}_A = \frac{\vec{v}_A}{v_A}$$

$$(k^2 v_A^2 - \omega^2)\vec{v}_1 + (c_s^2 - v_A^2)\underbrace{\left(k\frac{\vec{v}_A}{v_A}\vec{v}_1\right)\hat{k}k}_{\left(k^2 \frac{\vec{v}_A}{v_A^2}\vec{v}_1\right)\vec{v}_A} = 0 \quad \text{a clever trick}$$

Here we can distinguish two kinds of waves: ⟵

$$\boxed{(k^2 v_A^2 - \omega^2)\vec{v}_1 + \left(\frac{c_s^2}{v_A^2} - 1\right)k^2(\vec{v}_A\cdot\vec{v}_1)\vec{v}_A = 0} \quad (\Delta)$$

1) Longitudinal wave, $\vec{v}_1 \parallel \vec{k}$ ($\& \vec{v}_A$) with phase velocity c_s.

2) Transverse wave, $\vec{v}_1 \cdot \vec{v}_A = 0$ with phase velocity v_A. Such Alfvén wave is purely MHD phenomenon, depending only on B (tension) and density (inertia)

In all field strengths in laboratory $v_A \ll c_s$. In astrophysical plasmas it can be $v_A \gg$, since the densities are very small.

Magnetosonic and Alfven waves

Magnetic field values in the different kinds of waves we can find from Eq in slide 50:

$$\frac{\partial \vec B_1}{\partial t} = \nabla\times(\vec v_1\times\vec B_0) = \underbrace{(\vec B_0\cdot\nabla)\vec v_1}_{} - \underbrace{(\vec v_1\cdot\nabla)\vec B_0}_{\vec v_1\cdot 0} + \vec v_1\underbrace{(\nabla\cdot\vec B_0)}_{=0} - \vec B_0(\overbrace{\nabla\cdot\vec v_1}^{i\vec k\vec v_1})$$

$$\vec B_0\left(\vec i\frac{\partial}{\partial x}+\vec j\frac{\partial}{\partial y}+\vec k\frac{\partial}{\partial z}\right)\vec v_1 = \left(B_{0x}\partial_x + B_{0y}\partial_y + B_{0z}\partial_z\right)\vec v_1 = (\;)\vec v_{1,0}e^{i(\vec k\cdot\vec r-\omega t)} = (\vec B_0\cdot\vec k)i\vec k e^{(\;)} = (\vec B_0\cdot\vec k)i\vec v_1$$

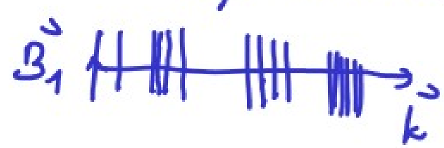
$$\frac{\partial \vec B_1}{\partial t} - i(\vec B_0\cdot\vec k)\vec v_1 + i\vec B_0(\vec k\cdot\vec v_1) = 0 \;\Big/\int dt$$

$$\vec B_1 = i\vec k[\vec B_0(\vec v_1 dt)] - i\vec B_0(\vec k\int\vec v_1 dt) = i\vec k\left[\vec B_0\left(-\frac{1}{i\omega}\right)\vec v_1\right] - i\vec B_0\left[\vec k\left(-\frac{1}{i\omega}\right)\vec v_1\right] = -\frac{\vec k\vec B_0}{\omega}\vec v_1 + \frac{\vec B_0}{\omega}(\vec k\cdot\vec v_1)$$

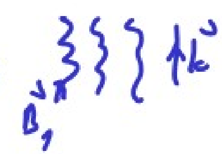
gives direction

For $\vec k\perp\vec B_0$, $\vec B_0\cdot\vec k=0$, $\vec B_1 = \frac{k v_1}{\omega}\vec B_0$

Such magnetosonic waves cause changes in the density of force lines, without change in their direction.



For $\vec k\parallel\vec B_0$ transverse waves, $\vec k\perp\vec v_1$, $\vec k\cdot\vec v_1=0$, we remain with $\vec B_1 = -\frac{\vec k\vec B_0}{\omega}\vec v_1 = -\frac{k}{\omega}B_0\vec v_1$, such waves create oscillations of the force lines:



In both these cases, the force lines are "frozen" in fluid and move with it.

For $\vec k\parallel\vec B_0$ & longitudinal waves, $\vec k\parallel\vec v_1$, $\vec B_0(\vec k\vec v_1) - (\vec B_0\cdot\vec k)\vec v_1 = \vec k(\vec B_0\vec v_1) - (\vec B_0\vec v_1)\vec k = 0 \Rightarrow \vec B_1 = 0$

Purely $\underbrace{\text{longitudinal}}_{\vec k\parallel\vec v_1}$ waves are called magnetosonic, and $\underbrace{\text{transverse}}_{\vec k\perp\vec v_1}$ are called Alfven waves.

Magnetosonic and Alfven waves in non-ideal fluids

For non-ideal cases, when conductivity is not infinitely large or viscous effects become important, dissipative losses will damp the plasma oscillations. The eqs for $\partial_t \vec{B}_1$ and $\partial_t \vec{v}_1$ from slide 50 become

$$\rho_0 \frac{\partial \vec{v}_1}{\partial t} = -c_s^2 \nabla \rho_1 - \frac{\vec{B}_0}{4\pi} \times (\nabla \times \vec{B}_1) + \underbrace{\nu \nabla^2 \vec{v}_1}_{\text{viscosity}}$$

, with dissipative terms added.

$$\frac{\partial \vec{B}_1}{\partial t} = \nabla \times (\vec{v}_1 \times \vec{B}_0) + \underbrace{\frac{c^2}{4\pi\sigma} \nabla^2 \vec{B}_1}_{\text{conductivity}}$$

for incompressible fluid, it is actually not correct physically, but here we think it effectively only as a diffusion coefficient, should give a reasonably good qualitative description.

we further write it as:

E''

$$\frac{\partial \vec{v}_1}{\partial t} - \frac{\nu}{\rho_0} \nabla^2 \vec{v}_1 = \frac{1}{\rho_0} \vec{A}_1$$

$$\frac{\partial \vec{B}_1}{\partial t} - \frac{c^2}{4\pi\sigma} \nabla^2 \vec{B}_1 = \vec{A}_2$$

Inhomogeneous diffusion eqs., solutions are given as plane waves.

$\vec{v}_1 = \vec{v}_{10} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$, $\nabla^2 \vec{v}_1 = -k^2 \vec{v}_1$

$\partial_t \vec{v}_1 = -i\omega \vec{v}_1$

$\Rightarrow \nabla^2 \vec{v}_1 = -k^2 \frac{1}{-i\omega} \partial_t \vec{v}_1$

$= -\frac{ik^2}{\omega} \partial_t \vec{v}_1$

$\partial_t \vec{v}_1 - \frac{\nu}{\rho_0}\left(-i\frac{k^2}{\omega}\right)\partial_t \vec{v}_1 = \frac{1}{\rho_0} \vec{A}_1$

$\partial_t \vec{v}_1 \left(1 + i\frac{\nu k^2}{\rho_0 \omega}\right) = \frac{1}{\rho_0} \vec{A}_1$

The same as → for $\vec{B}_1$:

$\vec{B}_1 = \vec{B}_{10} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$\nabla^2 \vec{B}_1 = -k^2 \vec{B}_1$

$\partial_t \vec{B}_1 = -i\omega \vec{B}_1$

$\Rightarrow \nabla^2 \vec{B}_1 = -i\frac{k^2}{\omega} \partial_t \vec{B}_1$

$\partial_t \vec{B}_1 \left(1 + i\frac{\frac{c^2}{4\pi\sigma} k^2}{\omega}\right) = \vec{A}_2$ →

$$\boxed{\begin{aligned} \rho_0 \frac{\partial \vec{v}_1}{\partial t} &= \frac{1}{1 + i\frac{\nu k^2}{\rho \omega}} \vec{A}_1 \\ \frac{\partial \vec{B}_1}{\partial t} \left(\frac{1}{1 + i\frac{c^2 k^2}{4\pi\sigma\omega}}\right) &= \vec{A}_2 \end{aligned}}$$

Some call this "obvious" from E'' well, ... maybe.

Magnetosonic and Alfven waves in non-ideal fluids

The equation $\boxed{\star}$ from slide 51

$$-\omega^2\vec{v}_1+(c_s^2+v_A^2)(\vec{k}\cdot\vec{v}_1)\vec{k}+(\vec{v}_A\cdot\vec{k})\left[(\vec{v}_A\cdot\vec{k})\vec{v}_1-(\vec{v}_A\cdot\vec{v}_1)\vec{k}-(\vec{k}\cdot\vec{v}_1)\vec{v}_A\right]=0$$

will change, c_s^2 becomes multiplied with $\left(1+i\frac{c^2k^2}{4\pi\sigma\omega}\right)$ and ω^2 with $\left(1+i\frac{\nu k^2}{\rho_0\omega}\right)$.

From $\rho_0\partial_t\vec{v}_1=\frac{1}{1+i\frac{\nu k^2}{\rho_0\omega}}\left[-c_s^2\nabla\rho_1-\frac{\vec{B}_0}{4\pi}\times(\nabla\times\vec{B}_1)\right]/\partial_t$

$$\rho_0\partial_t^2\vec{v}_1=\frac{1}{1+i\frac{\nu k^2}{\rho_0\omega}}\left[-c_s^2\nabla\partial_t\rho_1-\frac{\vec{B}_0}{4\pi}\times(\nabla\times\partial_t\vec{B}_1)\right],\quad \text{for } \partial_t\vec{B}=\frac{1}{1+i\frac{c^2k^2}{4\pi\sigma\omega}}\left[\nabla\times(\vec{v}_1\times\vec{B}_0)\right]$$

Since $\vec{v}_1=\vec{v}_{1_0}e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, $\partial_t\vec{v}_1=-i\omega\vec{v}_1$, $\partial_t^2\vec{v}_1=-\omega^2\vec{v}_1$ and we can write

from

$$\rho_0(-\omega^2\vec{v}_1)=\frac{1}{1+i\frac{\nu k^2}{\rho_0\omega}}\left[-c_s^2\nabla(-\rho_0\nabla\vec{v}_1)-\frac{\vec{B}_0}{4\pi}\times\left\{\nabla\times\frac{1}{1+i\frac{c^2k^2}{4\pi\sigma\omega}}\left[\nabla\times(\vec{v}_1\times\vec{B}_0)\right]\right\}\right]$$

$$-\omega^2\rho_0\underbrace{\left(1+i\frac{\nu k^2}{\rho_0\omega}\right)}\left(1+i\frac{c^2k^2}{4\pi\sigma\omega}\right)\vec{v}_1=\rho_0c_s^2\nabla(\nabla\cdot\vec{v}_1)\underbrace{\left(1+i\frac{c^2k^2}{4\pi\sigma\omega}\right)}-\frac{\vec{B}_0}{4\pi}\nabla\times\left\{\nabla\times\left[\nabla\times(\vec{v}_1\times\vec{B}_0)\right]\right\}$$

→ the additional multipliers ←
in comparison with ideal case.

Alfven waves at higher frequencies

We neglected Maxwell's displacement current, so our solutions are valid only for low frequencies. At higher frequencies charge separation increases in influence, but even if we ignore this, the displacement current changes the solutions.

The complete Ampere's law is $\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$, with infinite conductivity

$$\frac{j}{\sigma} = \vec{E} + \frac{\vec{v}}{c}\times\vec{B} = 0\,,\quad \vec{E}\sim -\frac{\vec{v}}{c}\times\vec{B}\,,\quad \nabla\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{v}\times\vec{B})$$

$$\vec{j} = \frac{c}{4\pi}\left[\nabla\times\vec{B} - \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{v}\times\vec{B})\right]$$ is the current.

Now the equation of motion $\rho\frac{d\vec{v}}{dt} = -\nabla p + \frac{1}{c}(\vec{j}\times\vec{B})$ becomes,

$$(\underbrace{\rho_0 + \rho_1}_{\text{2nd order}})(\partial_t\vec{v}_1 + \underbrace{\vec{v}_1\nabla\vec{v}_1}_{\to 0}) = -c_s^2\nabla\rho_1 + \frac{1}{c}\left\{\frac{c}{4\pi}\left[\nabla\times(\underbrace{\vec{B}_0}_{=0} + \vec{B}_1)\right] + \frac{1}{c^2}\partial_t\left[\vec{v}_1\times(\vec{B}_0 + \underbrace{\vec{B}_1}_{=0})\right]\right]\times(\vec{B}_0 + \underbrace{\vec{B}_1}_{=0})$$

$$\rho_0\partial_t\vec{v}_1 + c_s^2\nabla\rho_1 = \frac{1}{4\pi}\left[\nabla\times\vec{B}_1 + \frac{1}{c^2}\partial_t(\vec{v}_1\times\vec{B}_0)\right]\times(\vec{B}_0 + \vec{B}_1) = \frac{1}{4\pi}\left[(\nabla\times\vec{B}_1)\times\vec{B}_0 + \frac{1}{c^2}(\partial_t\vec{v}_1\times\vec{B}_0)\times\vec{B}_0\right] =$$

$\vec{B}_0 = \text{const}$

$$= \underbrace{-\frac{\vec{B}_0}{4\pi}\times(\nabla\times\vec{B}_1)}_{\text{without displ. current}} + \underbrace{\frac{\vec{B}_0}{4\pi c^2}\times(\vec{B}_0\times\partial_t\vec{v}_1)}_{\text{additional part because of } \frac{1}{c^2}\partial_t(\vec{v}\times\vec{B})}$$

$$\rho_0\partial_t\vec{v}_1 + \frac{\vec{B}_0}{4\pi c^2}\times(\partial_t\vec{v}_1\times\vec{B}_0) = -c_s^2\nabla\rho_1 - \frac{\vec{B}_0}{4\pi}\times(\nabla\times\vec{B}_1)\,,$$ we introduce $\vec{v}_A = \frac{\vec{B}_0}{\sqrt{4\pi\rho_0}}$,

$$\rho_0\partial_t\vec{v}_1 + \frac{\vec{v}_A\sqrt{4\pi\rho_0}}{4\pi c^2}\times(\partial_t\vec{v}_1\times\vec{v}_A\sqrt{4\pi\rho_0}) = -c_s^2\nabla\rho_1 - \frac{\vec{B}_0}{4\pi}\times(\nabla\times\vec{B}_1)$$

$$\rho_0\left[\partial_t\vec{v}_1 + \frac{1}{c^2}\vec{v}_A\times(\partial_t\vec{v}_1\times\vec{v}_A)\right] = -c_s^2\nabla\rho_1 - \frac{\vec{B}_0}{4\pi}\times(\nabla\times\vec{B}_1)\ /\partial_t$$

$$\rho_0\left[\partial_t^2\vec{v}_1 + \frac{1}{c^2}\vec{v}_A\times(\partial_t^2\vec{v}_1\times\vec{v}_A)\right] = -c_s^2\nabla\partial_t\rho_1 - \frac{\vec{B}_0}{4\pi}\times(\nabla\times\partial_t\vec{B}_1)\,,$$ since $\partial_t\rho_1 + \rho_0\nabla\vec{v}_1 = 0$, $\partial_t\vec{B}_1 = \nabla\times(\vec{v}_1\times\vec{B}_0)$ from slide 50

we have
$$\underbrace{\partial_t^2\vec{v}_1 + \frac{1}{c^2}\vec{v}_A\times(\partial_t^2\vec{v}_1\times\vec{v}_A)}_{\text{additional term compared to before}} = \frac{c_s^2}{\rho_0}\nabla(\rho_0\nabla\vec{v}_1) - \frac{1}{\rho_0}\frac{\vec{B}_0}{4\pi}\times\{\nabla\times[\nabla\times(\vec{v}_1\times\vec{B}_0)]\} =$$

$$= c_s^2\nabla(\nabla\vec{v}_1) - \frac{1}{\rho_0}\frac{\vec{v}_A\sqrt{4\pi\rho_0}}{4\pi}\times\{\nabla\times[\nabla\times(\vec{v}_1\times\vec{v}_A\sqrt{4\pi\rho_0})]\} =$$

$$= c_s^2\nabla(\nabla\cdot\vec{v}_1) - \vec{v}_A\times\{\nabla\times[\nabla\times(\vec{v}_1\times\vec{v}_A)]\}$$

$$\partial_t^2\big[\vec{v}_1 + \frac{1}{c^2}\underbrace{\vec{v}_A\times(\vec{v}_1\times\vec{v}_A)}_{\vec{v}_1(\vec{v}_A\cdot\vec{v}_A) - \vec{v}_A(\vec{v}_A\cdot\vec{v}_1)}\big] - c_s^2\nabla(\nabla\cdot\vec{v}_1) + \vec{v}_A\times\{\nabla\times[\nabla\times(\vec{v}_1\times\vec{v}_A)]\} = 0$$

$$\partial_t^2\left[\vec{v}_1\left(1 + \frac{v_A^2}{c^2}\right) - \frac{\vec{v}_A}{c^2}(\underbrace{\vec{v}_A\cdot\vec{v}_1}_{=0})\right] \underbrace{- c_s^2\nabla(\nabla\cdot\vec{v}_1) + \vec{v}_A\times \text{ — // — }}_{=0}$$

With $\vec{v}_1 \parallel \vec{B}_0$ $(= \vec{v}_1 \parallel \vec{v}_A)$ the eq. remains the same as before since $\vec{v}_A\vec{v}_1 = \vec{v}_1\cdot\vec{v}_A$ and $[\ ] = \vec{v}_1$

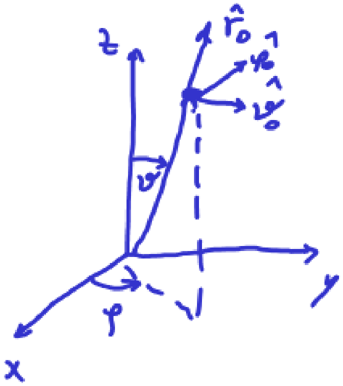
For transverse $\vec{v}_1$ (or magnetosonic with $\vec{k}\perp\vec{B}_0$ or Alfvén waves with $\vec{k}\parallel\vec{B}_0$, ω^2 will be multiplied with $\left[1 + \frac{v_A^2}{c^2}\right]$ since $\vec{v}_A\cdot\vec{v}_1 = 0$ and /. This means that phase velocity of Alfvén waves becomes $u_A = \frac{v_A}{\sqrt{1 + \frac{v_A^2}{c^2}}} = \frac{c\,v_A}{\sqrt{c^2 + v_A^2}}$

For usual limit $v_A \ll c$, $u_A \approx v_A$, displacement current is not important. But, if $v_A \gg c$, $u_A = c$, from the system connecting with el.-mag. waves, transverse Alfvén waves can be considered as a wave in a medium with refraction index $n = \frac{c}{u_A}$, $n^2 = 1 + \frac{c^2}{v_A^2} = 1 + \frac{4\pi\rho c^2}{B_0^2}$ This result is valid only if charge separation does not influence the flow much.

Toroidal and poloidal fields

It is often useful, especially in axisymmetric cases, to represent vector fields as a sum of two perpendicular vector fields, so called poloidal and toroidal vector fields. There are some useful properties of such fields, which enable reducing of the vector equations to the systems of scalar equations. Such methods are often useful in the theory of turbulent dynamos, relevant in astrophysics.

Consider a vector field $\vec V$ defined in the simply-connected Euclidean space. We assume $\vec V$ spatially differentiable to the 2nd order. We work in the spherical coordinates (r, ϑ, φ) with unit vectors $(\hat r_0, \hat\vartheta_0, \hat\varphi_0)$ and radius-vector $\vec r$, components of $\vec V$ are $V_r, V_\vartheta, V_\varphi$.



For any scalar field we can write the spatial derivatives

$$\nabla F = \frac{\partial F}{\partial r}\hat r_0 + \frac{\partial F}{\partial \vartheta}\hat\vartheta_0 + \frac{1}{r\sin\vartheta}\frac{\partial F}{\partial \varphi}\hat\varphi_0$$

$$\nabla^2 F = \Delta F = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 F}{\partial\varphi^2} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial F}{\partial\vartheta}\right)\right]$$

We introduce for every $r = \text{const}$ surface Laplace's operator $\frac{1}{r^2}\Omega$

we can write $\Omega F = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial F}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 F}{\partial\varphi^2}$ (i.e. $\Omega = r^2\nabla^2$)

Integrating on such $r = \text{const}$ surfaces can be written as $\oint F\,d\tilde\omega = \int_{\vartheta=0}^{\pi}\int_{\varphi=0}^{2\pi} F\sin\vartheta\, d\vartheta\, d\varphi$

We can write $\nabla\cdot\vec V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta V_\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial V_\varphi}{\partial\varphi}$

$$\nabla\times\vec V = \frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}(\sin\vartheta V_\varphi) - \frac{\partial V_\vartheta}{\partial\varphi}\right]\hat r_0 + \frac{1}{r}\left[\frac{1}{\sin\vartheta}\frac{\partial V_r}{\partial\varphi} - \frac{\partial}{\partial r}(rV_\varphi)\right]\hat\vartheta_0 + \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\vartheta) - \frac{\partial V_r}{\partial\vartheta}\right]\hat\varphi_0$$

For axially-symmetric $\vec V$ with polar axis $\parallel$ with the symmetry axis) the components V_r, V_ϑ and V_φ are not depending on φ, so we can decompose $\vec V = \vec V_p + \vec V_t$, a sum of poloidal and toroidal fields, such that

$$\boxed{\vec V_t = V_\varphi\hat\varphi_0, \quad \vec V_p = V_r\hat r_0 + V_\vartheta\hat\vartheta_0}$$

Toroidal and poloidal fields

1) If in a point it is $\vec{V}=0 \Rightarrow \vec{V}_p=\vec{V}_t=0$
2) For a scalar a, $a\vec{V}_p$ is poloidal and $a\vec{V}_t$ is toroidal.
3) For any axially symmetric vector field $\vec{V}^+$, in any fixed point $\vec{V}_t^+\times\vec{V}_t=0$, $\vec{V}_t\times\vec{V}_p$ and $\vec{V}_p^+\times\vec{V}_t$ are poloidal, $\vec{V}_p^+\times\vec{V}_p$ is toroidal. In particular $\vec{r}\times\vec{V}_t$ is poloidal and $\vec{r}\times\vec{V}_p$ is toroidal.
4) $\nabla\cdot\vec{V}_t=0$
5) $\nabla\times\vec{V}_t$ is poloidal, $\nabla\times\vec{V}_p$ is toroidal
6) If in a considered volume $\nabla\times\vec{V}_t=0 \Rightarrow \vec{V}_t=0$
7) In any fixed point $\vec{V}_t\perp\vec{V}_p$, $\vec{V}_t\cdot\vec{V}_p=0$

Most of 1)-7) is obvious, 4) and 5) are clear if we think in spherical coords., as $\partial_\varphi=0$ and we have

$$\nabla\cdot\vec{V}_{t,p}=\frac{1}{r^2}\partial_r(r^2 V_{t,p_r})+\frac{1}{r\sin\vartheta}\partial_\vartheta(\sin\vartheta\, V_{t,p_\vartheta})$$

$$\nabla\times\vec{V}_{t,p}=\frac{1}{r\sin\vartheta}\partial_\vartheta(\sin\vartheta\, V_{t,p_\varphi})\hat{r}_0+\frac{1}{r}\left[-\partial_r(r V_{t,p_\varphi})\right]\hat{\vartheta}_0+\frac{1}{r}\left[\partial_r(r V_{t,p_\vartheta})-\partial_\vartheta V_{t,p_r}\right]\hat{\varphi}_0$$

$\vec{V}_t=V_\varphi\hat{\varphi}_0$, $\vec{V}_p=V_r\hat{r}_0+V_\vartheta\hat{\vartheta}_0$, it is obvious that $\nabla\cdot\vec{V}_t=0$, $\nabla\times\vec{V}_t$ has component only in $\hat{r}_0$ and $\hat{\vartheta}_0$ directions $\Rightarrow$ it is poloidal, $\nabla\times\vec{V}_p$ has only a $\hat{\varphi}_0$ component $\Rightarrow$ it is toroidal.

6) follows, for smoothly connected space, $\nabla\times\vec{V}_t=0 \Rightarrow \vec{V}_t$ is an $\underbrace{\text{(axially-symmetric)}}_{}$ gradient of some potential.
$\Rightarrow$ it is poloidal $\Rightarrow$ from 1) $\vec{V}_t=0$

If we consider, from 4) and 5), $\Delta\vec{V}=\Delta\vec{V}_t+\Delta\vec{V}_p$.

In general, $\Delta\vec{V}=\nabla(\nabla\cdot\vec{V})-\nabla\times(\nabla\times\vec{V})$, since $\nabla\cdot\vec{V}_t=0 \Rightarrow \Delta\vec{V}_t=-\overbrace{\nabla\times\underbrace{(\nabla\times\vec{V}_t)}_{\text{pol}}}^{\text{tor}} \Rightarrow$ toroidal

$\Delta\vec{V}_p=\underbrace{\nabla(\nabla\cdot\vec{V}_p)}_{\text{pol}}-\underbrace{\nabla\times\underbrace{(\nabla\times\vec{V}_p)}_{\text{tor}}}_{\text{pol}} \Rightarrow$ poloidal (difference of 2 poloidal vectors)

For solenoidal axisymmetric field $\nabla\cdot\vec{V}\overset{\text{def}}{=}0$,
since $\nabla\cdot\vec{V}_t=0 \Rightarrow \vec{V}_p$ has to be solenoidal $\Rightarrow \nabla\cdot\vec{V}_p=0$, since $\nabla\cdot\vec{V}=\nabla\cdot(\vec{V}_p+\vec{V}_t)=0 \Rightarrow$ we can set $\nabla\times\vec{V}_p=\vec{A}$, with $\vec{A}$ a "vector potential", $\vec{A}=\vec{A}_p+\vec{A}_t$, $\nabla\times\vec{A}_t$ is poloidal, $\nabla\times\vec{A}_p$ is toroidal

so $\nabla\times\vec{A}_p=0$
$\Rightarrow \vec{A}_p=0$ and we can write
$\vec{V}_p=\nabla\times\vec{A}_t$

Both $\vec{V}_t$ and $\vec{V}_p$ we can represent with a scalar quantity, equal to rotation of some vector potential.
(If $\nabla\times\vec{V}_p=0$ everywhere, $\exists$ scalar function ϕ for which $\vec{V}=\nabla\phi$)

Helmholtz decomposition

Helmholtz theorem, or fundamental theorem of vector calculus: any sufficiently smooth, rapidly decaying vector field in 3D can be resolved into the sum of curl-free (=irrotational) and a divergence free (=solenoidal) vector fields. Equivalently, any vector field can be generated by a combination of a vector potential and a scalar potential:

$$\vec{F} = -\nabla\phi + \nabla\times\vec{A}, \quad \text{with} \quad \phi(\vec{r}) = \frac{1}{4\pi}\int_V \frac{\nabla'\cdot\vec{F}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dV' - \frac{1}{4\pi}\oint_S \frac{\vec{F}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\cdot d\vec{S}\,'$$

$$\vec{A}(\vec{r}) = \frac{1}{4\pi}\int_V \frac{\nabla'\times\vec{F}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dV' + \frac{1}{4\pi}\oint_S \frac{\vec{F}(\vec{r}\,')\times d\vec{S}\,'}{|\vec{r}-\vec{r}\,'|}$$

$\to 0$ for $\vec{F}$ vanishing sufficiently fast at infinity

Another useful general relation:

Any continuous differentiable vector field $\vec{V}$ can be represented as

$$\vec{V} = \vec{r}A_1 + \nabla A_2 + \vec{r}\times\nabla A_3,$$

where A_1, A_2, A_3 are three uniquely determined <u>scalar</u> fields.

– it is actually a variant of a fact that any 3D vector can be represented by 3 mutually non-coplanar vectors ($\equiv$ 3 linearly independent vectors).

[This one I found in S. Chandrasekhar, ApJ, 124, 232 (1956), needed to write down the axisymmetric eqs. of MHD]

MHD, lecture 5

- Conductivity tensor
- Reconnection
- Dynamo

Conductivity tensor

On slide 45 we obtained a version of the Ohm's law which is often used. What are its implications on the current and electric field?

$$j_i + \frac{m_p}{\rho e}\sigma\epsilon_{ijk} j_j B_k = \sigma\left(\mathcal{E}_i + \frac{1}{c}\epsilon_{ijk} U_j B_k\right), \quad \text{with } \frac{m_p}{\rho} \simeq \frac{1}{n} \text{ from slide 42}$$

this can be written $\vec j + \frac{\sigma}{ne}\vec j\times\vec B = \sigma\vec{\mathcal{E}} + \underline{\frac{\sigma}{c}\vec U\times\vec B}$

with $\frac{1}{\eta} = \sigma = \frac{ne^2}{m_e c\nu}$ we can write

we neglect it now, but it can easily be restored

$$\vec j + \frac{e}{m_e c\nu}\vec j\times\vec B = \sigma\vec{\mathcal{E}}, \quad \text{with } \hat b \stackrel{def}{=} \frac{\vec B}{B}$$

$$\vec j + \underbrace{\frac{eB}{m_e c\nu}}_{\omega_{g,e}}\vec j\times\hat b = \sigma\vec{\mathcal{E}} \Rightarrow \vec j + \omega_{g,e}\tau\,\vec j\times\hat b = \sigma\vec{\mathcal{E}},$$

$\nu = \frac{1}{\tau}$, τ is average time between collisions

which can be written in general as

$$\vec j = \alpha\vec{\mathcal{E}} + \beta\hat b\times\vec{\mathcal{E}} + \gamma(\hat b\cdot\vec{\mathcal{E}})\hat b$$

we have

$$\alpha\vec{\mathcal{E}} + \beta\hat b\times\vec{\mathcal{E}} + \gamma(\hat b\cdot\vec{\mathcal{E}})\hat b + \omega_{g,e}\tau\left[\left(\alpha\vec{\mathcal{E}} + \beta\hat b\times\vec{\mathcal{E}} + \underbrace{\gamma(\hat b\cdot\vec{\mathcal{E}})\hat b\right]\times\hat b}_{=0} = \sigma\vec{\mathcal{E}}$$

$$\alpha\vec{\mathcal{E}}\times\hat b = -\alpha\hat b\times\vec{\mathcal{E}}$$

$$\beta(\hat b\times\vec{\mathcal{E}})\times\hat b = \beta(\hat b\cdot\hat b)\vec{\mathcal{E}} - (\hat b\cdot\vec{\mathcal{E}})\hat b$$

$\alpha\vec{\mathcal{E}} + \beta\hat b\times\vec{\mathcal{E}} + \gamma(\hat b\cdot\vec{\mathcal{E}})\hat b + \omega_{g,e}\tau\left[-\alpha\hat b\times\vec{\mathcal{E}} + \beta\vec{\mathcal{E}} - \beta(\hat b\cdot\vec{\mathcal{E}})\hat b\right] = \sigma\vec{\mathcal{E}}$. We consider separately the coefficients near $\vec{\mathcal{E}}$, $\hat b\times\vec{\mathcal{E}}$ and $\hat b$ to obtain 3 eqs:

$$\left.\begin{aligned}&\alpha + \beta\omega_{g,e}\tau - \sigma = 0\\ &\beta - \alpha\omega_{g,e}\tau = 0\end{aligned}\right\} \rightarrow \beta = \alpha\omega_{g,e}\tau \qquad \underline{\alpha + \alpha\omega_{g,e}^2\tau^2 = \sigma} \qquad \alpha = \frac{\sigma}{1+\omega_{g,e}^2\tau^2},\ \beta = \frac{\sigma\omega_{g,e}\tau}{1+\omega_{g,e}^2\tau^2},$$

$$\gamma(\hat b\cdot\vec{\mathcal{E}}) - \beta\omega_{g,e}\tau(\hat b\cdot\vec{\mathcal{E}}) = 0 \Rightarrow \gamma = \beta\omega_{g,e}\tau \qquad \gamma = \frac{\sigma\omega_{g,e}^2\tau^2}{1+\omega_{g,e}^2\tau^2}$$

We can write the equivalent $\vec j = \sigma_\perp\vec{\mathcal{E}} + (\sigma_\parallel - \sigma_\perp)(\hat b\cdot\vec{\mathcal{E}})\hat b + \sigma_H\hat b\times\vec{\mathcal{E}}$, where

$$\sigma_\perp(=\alpha) = \frac{\sigma}{1+\omega_{g,e}^2\tau^2}, \quad \sigma_H(=\beta) = \frac{\sigma\omega_{g,e}\tau}{1+\omega_{g,e}^2\tau^2} \text{ — Hall}, \quad \sigma_\parallel - \sigma_\perp(=\gamma) = \frac{\sigma\omega_{g,e}^2\tau^2}{1+\omega_{g,e}^2\tau^2} \Rightarrow$$

$$\sigma_\parallel = \frac{\sigma}{1+\omega_{g,e}^2\tau^2} + \frac{\sigma\omega_{g,e}^2\tau^2}{1+\omega_{g,e}^2\tau^2} = \frac{\sigma}{\cancel{1+\omega_{g,e}^2\tau^2}}\cancel{(1+\omega_{g,e}^2\tau^2)} = \underline{\underline{\sigma}}$$

If $\vec{\mathcal{E}}\parallel\vec B \Rightarrow \vec j = \sigma\vec{\mathcal{E}}$

If $\vec{\mathcal{E}}\perp\vec B \Rightarrow \vec j = \vec j_\parallel + \vec j_\perp = \sigma_\perp\vec{\mathcal{E}} + \sigma_H\hat b\times\vec{\mathcal{E}}$

Choosing the axes so that $\hat b = (0,0,1)$ we can write $\vec j$ as a matrix

$$\begin{bmatrix} j_1\\ j_2\\ j_3\end{bmatrix} = \begin{bmatrix}\sigma_\perp & -\sigma_H & 0\\ \sigma_H & \sigma_\perp & 0\\ 0 & 0 & \sigma_\parallel\end{bmatrix}\begin{bmatrix}\mathcal{E}_1\\ \mathcal{E}_2\\ \mathcal{E}_3\end{bmatrix}$$

For high collision frequencies, with $\omega_{g,e}\tau\to 0$, $\sigma_\parallel\to\sigma$, $\sigma_\perp\to\sigma$ and $\sigma_H\to 0$, σ becomes isotropic, as expected.

For low collision frequencies, with $\omega_{g,e}\tau\to\infty$, $\sigma_\perp\to 0$, $\sigma_H\to 0$, current of e^- moves only in the direction of $\vec B$, driven by the component of $\vec{\mathcal{E}}$ which is $\parallel$ to $\vec B$.

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PRVO SPOMINJAME MEHANIZMA REKONEKCIJE

early brick building in England but also the most beautiful of English baronial buildings. The mouldings and dressed work are for the most part executed in stone, a greensand, which gives great sharpness of detail and which has weathered well.

After 1740 the Castle fell into neglect, and in 1777 the interior, including the buildings in the court within the main rectangular structure, were demolished and the materials used to build the mansion now known as Herstmonceux Place. Little remained of the original fabric beyond the outer walls, with their towers, and portions of the inner walls. In 1911 the Castle was purchased by Colonel Claude Lowther, who commenced the restoration. After his death it was acquired in 1932 by Sir Paul Latham, Bart., under whose direction the restoration was completed by Mr. W. H. Godfrey. The outer walls and towers were carefully repaired and restored to their original condition; the ivy was removed from the walls; and the moat, on the east, south and part of the west fronts, was refilled with water. The reflexion of the Castle in the water of the moat, emphasizing the vertical lines of the towers, adds greatly to the beauty and impressiveness of the Castle.

The situation of the Castle was thus described by Francis Grose ("Antiquities of England and Wales", **5**, 157–8), a century and a half ago:

"The Castle of Herstmonceux stands in a pleasant park, well diversified by hill and vale, finely wooded with old trees, and well watered by clear pools, and from it there is a fine view over the adjacent rich level of Pevensey (in the midst of which, on a little rise, is the town and ancient ruined Castle of Pevensey). The sea appears in front, southward of the hills towards Hastings to the east; and the South Downs rise mountain-like at some distance to the west. The Castle is seated near the southern edge of the park, and rather in the lowest part of it; the soil is, however, very dry."

In this noble and dignified castle, rich with historical associations, the Observatory will in the future find a home befitting its long traditions. In the grounds which have been acquired surrounding the Castle there is ample space for setting up the present instrumental equipment and future additions to it. The conditions for astronomical observation are about as good as can be found in England and, freed from the hampering conditions under which it has worked for many years, the Observatory will have new opportunities for making contributions of importance to the promotion of astronomical science.

Because of the long association of the Observatory with Greenwich and the selection in 1884, by international agreement, of the Greenwich meridian as the prime meridian of longitude, in recognition of the great contribution of the Observatory to astronomical and nautical science, the Royal Observatory will be known in its new site as the Royal Greenwich Observatory. The Observatory will no longer remain on the prime meridian, but there will be a sufficient overlap in observations at Greenwich and at Herstmonceux for the longitude of the new site to be determined with the necessary degree of accuracy.

The Nautical Almanac Office, which it has not been possible to accommodate with the observational departments of the Royal Observatory on the restricted site at Greenwich, will also be housed at Herstmonceux Castle. The closer association between the observational and computational branches of the Royal Observatory, which will thus become possible, will be to their mutual advantage.

A THEORY OF CHROMOSPHERIC FLARES

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IT has been established from observation that chromospheric flares are closely associated with sunspots, and that the probability of a flare occurring near a spot increases with the size of the latter. The probability is higher when the group is increasing its size than when it is stationary, and it is also higher for the magnetically complex βγ- and γ-groups than for the simpler α- and β-type groups[1]. The flares themselves are short-lived phenomena, of mean life about thirty minutes, and are quite localized. It is generally accepted that they show no velocity either in height or across the surface of the sun.

A mechanism is proposed here for the production of these flares based on the energies acquired by charged particles moving in induced electric fields associated with sunspots.

Strong magnetic fields exist in sunspots, and there is a large magnetic flux from the spots. It is usually agreed that the field due to a spot extends to an appreciable distance from it[2]. During the growth of a sunspot, there must be electric fields induced in its neighbourhood, and if the axis of the spot be vertical, the lines of electric force will be circles coaxial with the spot and parallel to the sun's surface. The magnitude of the electric field at a given point will depend not only on the rate of growth of the spot, but also on the conductivity of the surrounding medium. The existence of a magnetic field away from the spot, however, indicates that the conductivity does not prevent the magnetic field from being established in a time at least comparable with that required for the spot to grow. It is reasonable, therefore, to compute the magnetic and induced electric fields for a surrounding medium of zero conductivity and apply these results to determine the order of magnitude of the actual fields; provided that polarization charges do not become serious.

For a sunspot growing uniformly, in 50 hours, to a diameter of 7×10^8 cm., that is, one-hundredth of that of the sun, and a maximum magnetic field of 2,000 gauss, the magnetic and induced electric fields may be computed by treating the sunspot as a circular coil of the same radius carrying a current such as to produce, at any time, the same axial magnetic field. The electric and magnetic fields, which are mutually perpendicular, are given below for radial distances, in the plane of the coil, 2·5 and 5 times the spot radius, that is, in the region where flares frequently occur.

Distance from centre of sunspot (cm.)	Magnetic field (gauss)	Electric field (volt/cm.)
$3{\cdot}5 \times 10^9$	8	$1{\cdot}55 \times 10^{-3}$
$1{\cdot}75 \times 10^9$	64	$6{\cdot}2 \times 10^{-3}$

Chapman and Cowling have shown[3] that, in crossed electric and magnetic fields, charged particles undergo a drift which has a component velocity in the direction of the electric field amounting to

$$v = \frac{Ee\tau}{m(1 + \omega^2\tau^2)},$$

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where τ is the mean time between collisions and ω is eH/m.

Considering the motion of electrons in these fields, they will on the average acquire energy between collisions, and if this be greater than the loss due to elastic collisions, which with hydrogen atoms is about one-thousandth of the electron energy, the average energy will increase until excitation of the atoms can occur. The increase of energy will be much less with protons, owing to their greater mass, and they are therefore afterwards neglected.

For an electron to acquire energy equal to the first ionization potential of hydrogen atoms it can be shown that

$$\frac{E^2\lambda^2}{1 + 8{\cdot}8 \times 10^{-3}H^2\lambda^2} \geq 2 \times 10^{15},$$

where λ is the mean free path and E is in E.M.U., so that if $H = 0$, then $\lambda \geq 4{\cdot}5 \times 10^7/E$. If E is 10^{-3} volt./cm., that is, 10^5 E.M.U./cm., then $\lambda \geq 450$ cm.

This discussion has neglected the distribution of velocities about the mean, so that, clearly, some excitation of hydrogen atoms will take place for shorter mean free paths than given above. When Cillié and Menzel's results for electron distribution in the chromosphere[4] are combined with Cowling's figures for cross-sections of protons[5], it is found that a mean free path of 450 cm. occurs about the middle of the chromosphere, some 6,000 km. above its base. If the induced field exceeds the above value, then the excitation occurs down to lower levels in the chromosphere.

If, in the equation above, $8{\cdot}8 \times 10^{-3} H^2\lambda^2$ is large compared with unity, then $H \leq E/1{\cdot}32 \times 10^6$ and for the same value of E as above $H \leq 7{\cdot}5 \times 10^{-2}$ gauss. Thus excitation can occur with mutually perpendicular electric and magnetic fields only if the magnetic field is very small, and it is of interest to see whether such conditions can exist in the chromosphere.

The presence of a general magnetic field on the sun was announced by Hale in 1913, its magnitude being given as about 25 gauss. While there is still some doubt expressed as to the actual existence of the field owing to its small size, recent measurements by Thiessen[6] confirms its reality.

Apart from a general magnetic field, fields from other sunspots may still be of appreciable size in the neighbourhood of the spot under consideration. It is thus to be expected that there will be places where actual neutral points exist and where conditions are thus suitable for the excitation of atoms by collision.

It is not essential that the magnetic field be small for electrons to acquire high energies: this is only the requirement so long as the electric and magnetic fields are assumed to be mutually perpendicular. If, owing to an external magnetic field, the electric field has an appreciable component in the direction of the resultant magnetic field, the electrons will have a component of their drift velocity in that direction, and this component will not be affected by the magnitude of the magnetic field.

In the above discussion, polarization charges have not been considered, and it remains to be shown whether they can influence the electric field to a significant extent. Cowling[5] has developed equations giving the conductivity of the material in the sun's atmosphere in the form

$$\sigma^{\rm I} + i\sigma^{\rm II} = \{6{\cdot}8 \times 10^{13}\bar{Z}T^{-3/2} - i\, 8{\cdot}6 \times 10^{3}HT/\rho_e\}^{-1} \text{ E.M.U.},$$

where $\sigma^{\rm I}$ and $\sigma^{\rm II}$ are the direct and transverse conductivities for mutually perpendicular electric and magnetic fields, $\bar{Z}$ is the mean degree of ionization, T the (electron) temperature and ρ_e the electron pressure. Throughout the chromosphere the direct conductivity is of the order of 10^{-8} E.M.U. for very small magnetic fields, while for magnetic fields greater than 0·1 gauss both conductivities are very much less, becoming negligible in comparison with the conductivity for zero magnetic fields and, for non-perpendicular fields, with the conductivity along the lines of force. Thus in the chromosphere, except near neutral points, electrons are constrained to move in the direction of the magnetic field. However, in and below the reversing layer, conditions are such that magnetic fields of the order of hundreds of gauss will have little effect on the conductivity, owing to the much higher electron pressure.

If the external magnetic field be inclined to the surface, the neutral point will be either higher or lower than the spot, according to the direction of the external field and the polarity of the spot. If the neutral point be in the chromosphere, then electrons moving away from this point under the influence of the electric field will eventually reach a region where the magnetic field is strong enough to confine movement to the lines of force. These lead down to the reversing layer where the conductivity is high enough to prevent the accumulation of space charges.

Along lines of force which pass into the sunspot, the question of the magnitude of any space charges which may be built up in these regions depends at least in part on whether currents will flow transversely through a sunspot. It is not necessary, however, to indicate here whether or not this is the case, as any polarization fields built up in these regions will tend to increase rather than decrease the electric intensity in the neighbourhood of the neutral point, and thus cause no blockage to the mechanism. A similar result follows from any space charges due to the movement of electrons along the lines of force of the external magnetic field.

The above discussion has shown that it is possible to have localized regions in the chromosphere where electrons acquire sufficient energy to cause excitation of atoms by collision, and where there will thus be an increase in the radiation emitted. Many of the features associated with this radiation—its elevation in the chromosphere, location with respect to spots, stationary nature and association with changing sunspots—are similar to those of chromospheric flares, and it is therefore suggested that the above mechanism is responsible for these flares.

The transient nature of the flares may have several explanations. It may be due to temporary increases in the rate of growth of a spot, or to changes in the magnetic field, for example, location of neutral points, in the vicinity of a spot.

Similarly, the more frequent occurrence of flares in γ- than in β-groups may be attributed to either the greater number of suitable locations, such as neutral points, which must occur in these groups or greater fluctuations in the individual spots which make up the group.

The implications of the mechanism proposed above will be discussed in greater detail elsewhere.

[1] Giovanelli, R. G., *Astrophys. J.*, **89**, 555 (1939).
[2] Chapman, S., *Mon. Not. Roy. Ast. Soc.*, **103**, 117 (1943).
[3] Chapman, S., and Cowling, T. G., "The Mathematical Theory of Non-Uniform Gases", 327 (Cambridge University Press, 1939).
[4] Cillié, G. G., and Menzel, D. H., Harvard College Observatory Circular 410 (1935).
[5] Cowling, T. G., *Proc. Roy. Soc.*, **183**, 453 (1945).
[6] *Observatory*, **66**, 230 (1946).

Reconnection

The first mention of reconnection in astrophysics was in solar physics, with a problem: how it is that energy of the magnetic field is released from the fluid when the lines of magnetic field are "frozen" in it? All the other sources of energy in the Sun flares were too small, only magnetic field from the solar corona seemed possible.

- Theory of Chromospheric Flares, **Ronald Gordon Giovanelli,** Nature volume 158, pages 81–82 (1946)

Abstract: It has been established from observation that chromospheric flares are closely associated with sunspots, and that the probability of a flare occurring near a spot increases with the size of the latter. The probability is higher when the group is increasing its size than when it is stationary, and it is also higher for magnetically complex βγ and γ-groups than for the simpler α- and β-type groups. The flares themselves are short-lived phenomena, of mean life about thirty minutes, and are quite localized. It is generally accepted that they show no velocity either in height or across the surface of the sun.

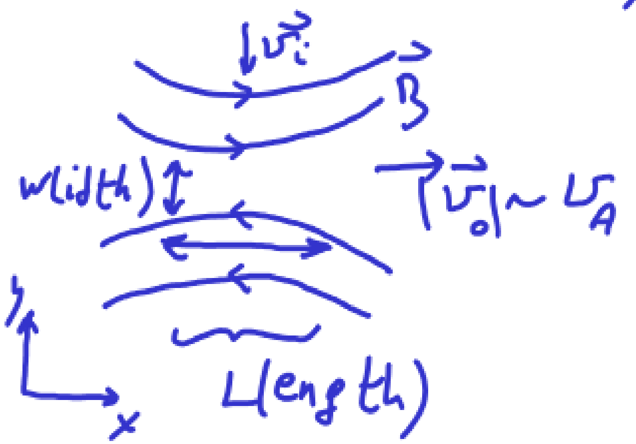
A mechanism is proposed here of the production of these flares based on the energies acquired based on charged particles moving in induced electric fields associated with sunspots.

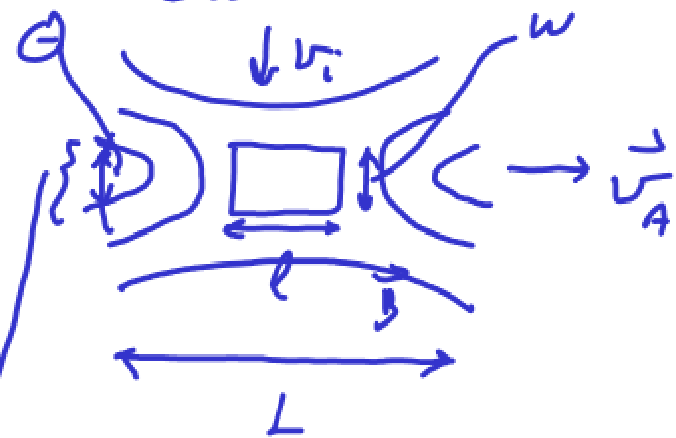
- But how there could be flares of magnetic energy from the hot, and therefore, perfectly ionized=conducting plasma=>no magnetic diffusion. For a 10^6 K corona, magnetic diffusion η=10 000 cm^2/s , which means that in a region of 10^4 km typical for flares a typical timescale $\tau=L^2/\eta$ would be τ=10^14 s , but it is usually τ=1000 s.
- Soon it was shown by Sweet & Parker that it is not as bad: contrary to the case in a solid conductor, in plasma the changes of B stirs the fluid into motion. Plasma, carrying the frozen B, may generate steep gradients of B typically located in shell-like structures, which leads to much shorter diffusion times-this is concept of resistive current sheets (Sweet-Parker sheets): distance_transverse distance δ~sqrt(τ_A η) , since the typical speed for MHD is Alfven speed. This defines the timescale τ_A~1 s. The dynamics is limited by the rate of convective field transport toward the sheet which is about 10^7 s. It is much shorter than before, but still few orders of magnitude larger than measured value.

Reconnection

assumed: steady state, incompressibility

Sweet-Parker (1958) | Petchek (1964)





Slow mode MHD shock

mass continuity:

$L v_i = w v_0$

$\partial_t \vec B = \nabla\times(\vec v\times\vec B - \eta\,\underbrace{\nabla\times\vec B}_{\frac{4\pi}{c} j})$

$= 0$ in the direction $\perp$ to $\vec B$: $v_B = \frac{4\pi}{c}\eta j$, $j = \frac{v_B c}{4\pi\eta}$

$\parallel$ to $\vec B$; eq. of motion becomes $\rho v_0 \frac{dv_0}{dy} = -\frac{dP}{dy} \,/ \int dy$

$\rho \frac{v_0^2}{2} = P \simeq B^2/8\pi \Rightarrow \boxed{v_0 = v_A}$

for $\theta \ll 1$, small current sheet,

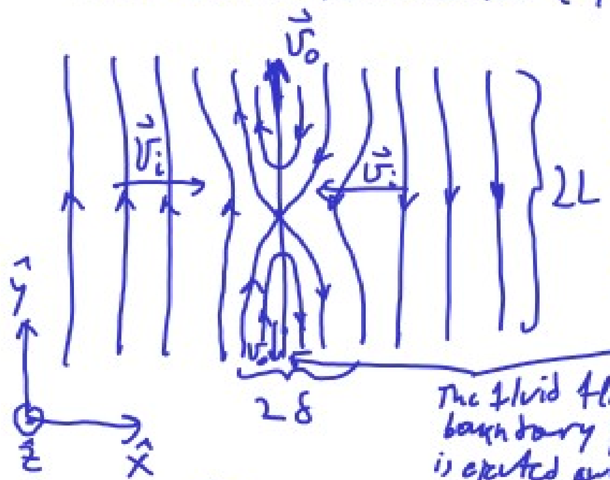
$v_i = \frac{w}{L} v_0 = \frac{L\theta}{L} v_0 \approx \theta v_0$

$\boxed{v_i = \theta v_A}$

At a symposium on solar fields Petchek presented even faster mechanism for energy exchange, where the speed depends only logarithmically on magnetic diffusivity η.

Such models were not real models of reconnection, but rather MHD configurations set-ups assuming the presence of an efficient reconnection mechanism-they assumed resistivity provided such a mechanism. In 1980-ies it was shown that Petchek's configuration is not valid in resistive MHD, not shortened, but extended current sheets are formed.

Sweet-Parker reconnection (by Petschek 1964)



Mass conservation

$v_i L = v_0 \delta$

$\hat{y}$ Bernoulli along the y axis in the middle of current sheet $\dfrac{\rho v_0^2}{2} = p_i - p_0$

p_0: at large distance from the boundary, we ignored $\dfrac{B_{y0}^2}{8\pi} + \dfrac{\rho v_0^2}{2}$

p_i: in the middle of the boundary between two B directions

The fluid flows towards the boundary from both sides and is ejected away to both 'up' and 'down' directions

$\hat{x}$ In the x-direction, across the boundary, the pressure balance is hydrostatic since the inflow into the boundary is very small, from Bernoulli $p = p_i + \dfrac{B_i^2}{8\pi} + \dfrac{\rho v_i^2}{2}$

$p = p_i + \dfrac{B_i^2}{8\pi}$ $\Leftarrow$ $B_i \gg B_0$ $v_i \ll v_0$

with $p_i = p_0$,

$$\frac{\rho v_0^2}{2} = p_i + \frac{B_i^2}{8\pi} - p_0 = \frac{B_i^2}{8\pi}, \quad v_0^2 = \frac{B_i^2}{4\pi\rho} = v_A^2$$

Since B and v are small near X point, Ohm's law $j_z = \sigma \mathcal{E}_z$. For a steady flow $\nabla \times \vec{\mathcal{E}} = 0 \Rightarrow \mathcal{E}_z = \text{const}$. Outside the boundary, current $=0$, $\vec{\mathcal{E}} = \dfrac{\vec{j}}{\sigma} - \dfrac{\vec{v}\times\vec{B}}{c} = -\dfrac{\vec{v}\times\vec{B}}{c} / \hat{z}$

$\mathcal{E}_z = -\dfrac{v_i B_{y0}}{c}$, $v_i = -\dfrac{c\mathcal{E}_z}{B_{y0}}$, $v_i = -\dfrac{c\, j_z}{B_{y0}\sigma}$

Across the boundary change in B is $2B_{y0}$, so from Ampere's law $\dfrac{4\pi j_z}{c} = (\nabla\times\vec{B})_z = \dfrac{\partial B_y}{\partial x} - \underbrace{\dfrac{\partial B_x}{\partial y}}_{=0} = \dfrac{\partial B_y}{\partial x}$

and

$\dfrac{4\pi j_z}{c}\, dx = dB_y \Rightarrow j_z = \dfrac{c B_{y0}}{4\pi\delta}$
$\quad\;\; 2\delta \quad\;\;\; 2B_{y0}$

$$v_i L = v_0 \delta = v_0 \frac{c B_{y0}}{4\pi j_z} = \frac{v_0}{v_i}\frac{c^2}{4\pi}\frac{v_i B_{y0}}{B_{y0}\sigma} \ldots = \frac{v_0}{v_i}\frac{c^2}{4\pi\sigma}$$

$$\underline{\underline{v_i^2}} = v_0 \frac{c^2}{4\pi\sigma L} = \frac{c^2}{4\pi\sigma L}\frac{B_i}{\sqrt{4\pi\rho}} = \frac{c^2}{4\pi\sigma L}\frac{B_{y0}}{\sqrt{4\pi\rho}} \Leftarrow$$

$$= \frac{c^2}{4\pi\sigma L}\underline{\underline{v_A}}$$

Reconnection

In magnetospheric physics, the problem of reconnection is approached in a different way. Except for the Earth ionosphere, the magnetospheric plasma is so dilute that Coulomb collisions practically do not occur=>classical resistivity vanishes. Can magnetic reconnection occur in a collisionless plasma?This problem was already recognized by James Dungey (who coined the term 'magnetic reconnection' in his PhD Thesis 1950; in 1970-ies also "magnetic field merging" was used, but today is phased out) who investigated magnetospheric convection. The usual approach was to consider the small scale turbulence excited by some microinstability, so that the scattering of electrons by charge fluctuations effectively mimics the Coulomb collisions=effective or anomalous resistivity.

For reconnection, timescales (τ) are most important

$$\tau = \frac{L}{v} = \frac{\text{spatial characteristic scale}}{\text{average velocity generated during the process}}$$

$$v \propto v_A = \frac{\delta B}{\sqrt{4\pi\rho}} \rightarrow \text{change of B is playing the main role}$$

$\downarrow$

because we expect a non-ideal process R in Ohm's law.

$$\tau \simeq \left[\frac{L}{v_A}\right] f(R) = \tau_A f(R)$$

; finding or at least approximating f is the main task in reconnection theory.

$\downarrow$

Lazarian & Vishniac, whole 'new' school of reconnection from 1990-ies.

For Sweet-Parker $f = \sqrt{\frac{\tau_\eta}{\tau_A}}$.

None of the many proposed causes of such resistivity was satisfying for Earth magnetotail.

Fast quasi-collisionless reconection became relevant also during the tokamak experiments, in explanation of sawtooth oscillations, an internal relaxation oscillation, where in Ohm's law nonlinear terms other than resistivity grow in importance.

Today it seems that collisionless reconnection is significantly more efficient than resistive diffusion and allows fast quasi-Alfvenic reconnection velocities.

Reconnection

In addition to time scales, another important feature for reconnection is energy partition. Electromagnetic energy is eruptively released in reconnection into:

-Bulk plasma motion, often generates a strong shock, the explosive blast wave;

-Electron and ion heating;

-Acceleration of a certain number of electrons and ions to superthermal energies.

- For reconnection, threshold conditions are also important: a certain amount of free energy has to be accumulated before rapid relaxation sets in→sudden onset of energy release.

Diffusion timescale for any quantity ψ

$$\frac{\partial \psi}{\partial t} = D \frac{\partial^2 \psi}{\partial x^2}$$ diffusion eq

can be $D \nabla^2 \psi$

coefficient of diffusion

1st approx

$$\frac{d}{dt} = \frac{1}{\tau}, \quad \frac{d}{dx} = \frac{1}{L}$$

$$\frac{1}{\tau} = D \frac{1}{L^2}, \quad \tau = \frac{L^2}{D}, \quad \text{for } D = \eta$$

$$\tau = \frac{L^2}{\eta}$$

PHYSICS OF PLASMAS **21**, 032121 (2014)

Magnetic reconnection in a comparison of topology and helicities in two and three dimensional resistive magnetohydrodynamic simulations

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Through a direct comparison between numerical simulations in two and three dimensions, we investigate topological effects in reconnection. A simple estimate on increase in reconnection rate in three dimensions by a factor of $\sqrt{2}$, when compared with a two-dimensional case, is confirmed in our simulations. We also show that both the reconnection rate and the fraction of magnetic energy in the simulations depend linearly on the height of the reconnection region. The degree of structural complexity of a magnetic field and the underlying flow is measured by current helicity and cross-helicity. We compare results in simulations with different computational box heights. [http://dx.doi.org/10.1063/1.4869333]

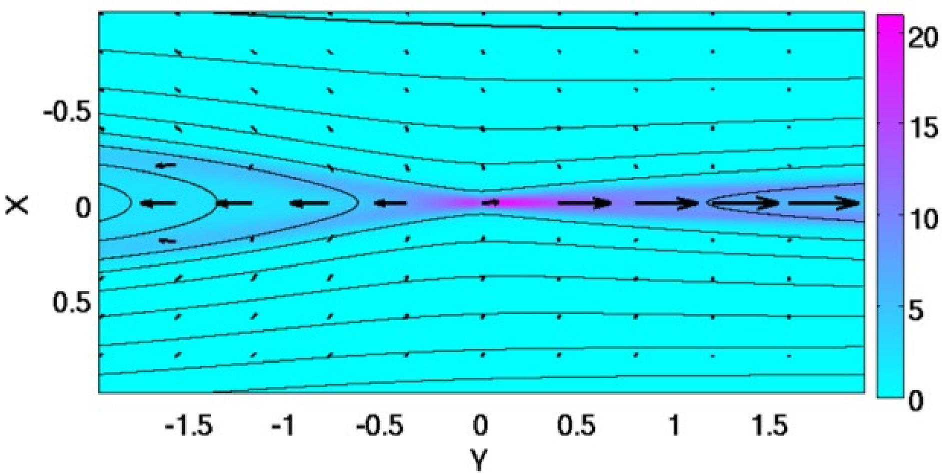


FIG. 2. Reconnection in two dimensions at T = 30 in code units, with current density shown in color grading, magnetic field contour lines in solid lines, and arrows showing velocity.

Reconnection rate in 3D

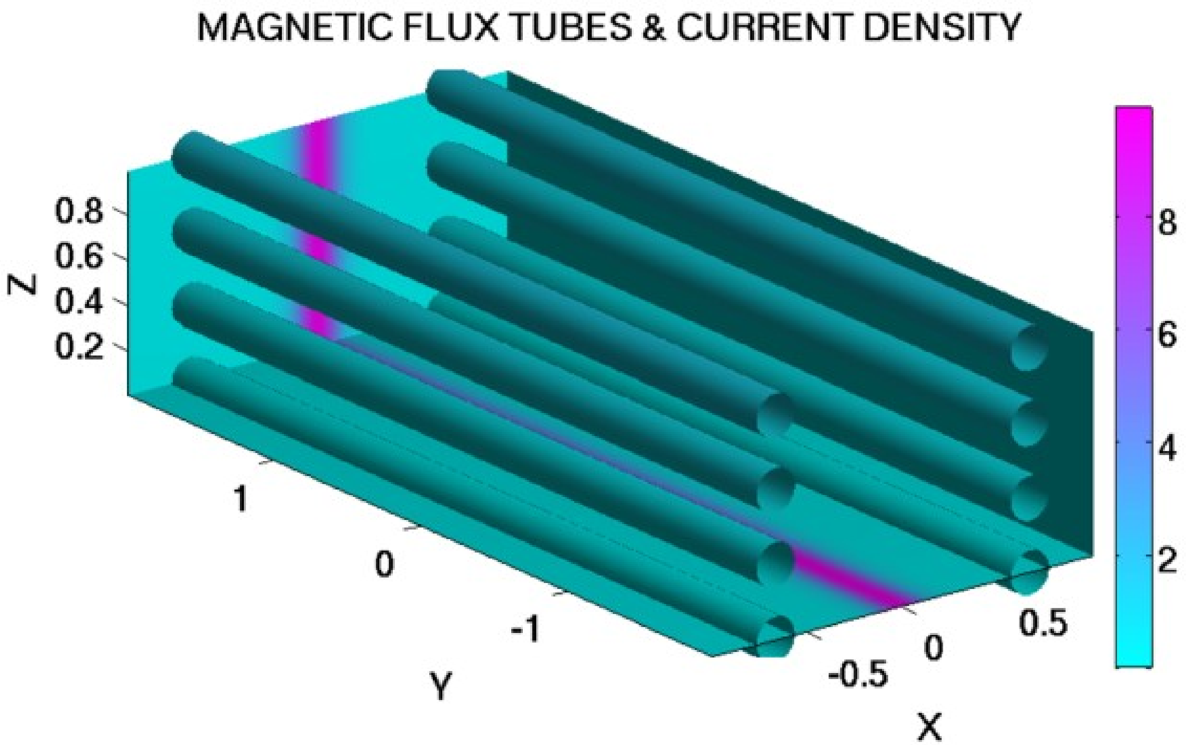


FIG. 1. Setup of initial and boundary conditions in our simulations in three dimensions. In two dimensional simulations, Z = 0. Color grading is showing a current density in code units, at the boundary planes; the diameter of the magnetic flux tube is set proportional to the magnetic field strength. We start with a 2D simulation in Cartesian coordinates $X \times Y$. Increasing the height of a box in Z direction, we compare the reconnection rates and other interesting quantities in the flow.

Reconnection rate in 3D

032121-4 M. Čemeljić and R.-Y. Huang Phys. Plasmas **21**, 032121 (2014)

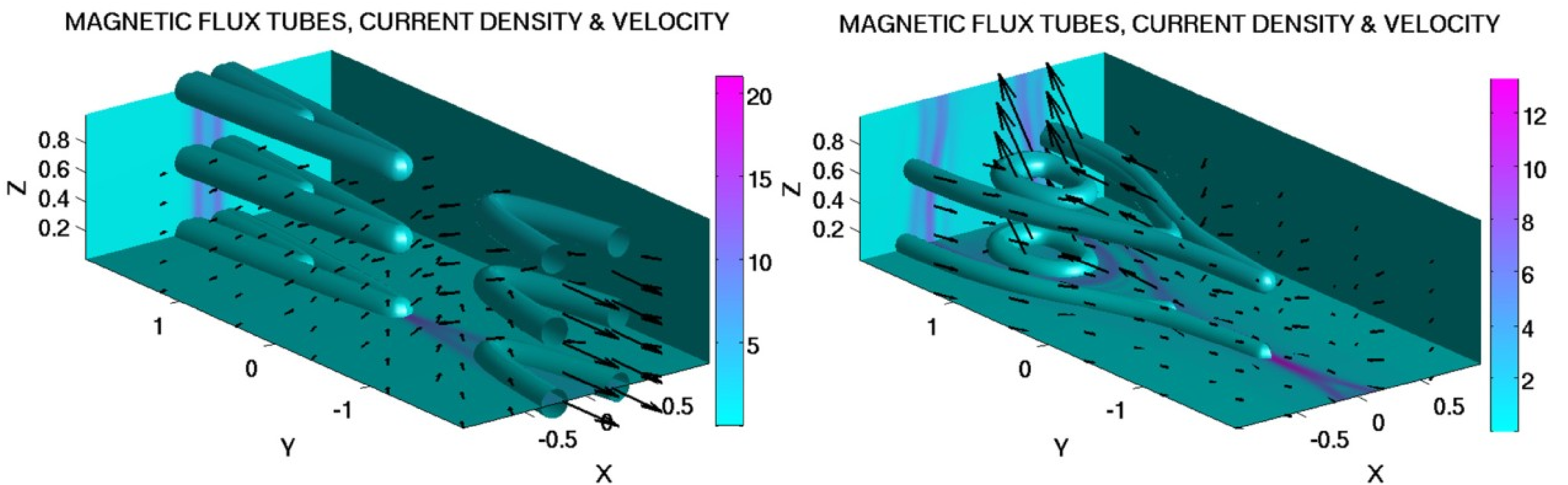


FIG. 3. Solutions in 3D in the first case, without the asymmetry in resistivity in the Z direction at T = 30 (left panel), and in the second case, with the asymmetry in the Z direction at T = 70 (right panel). Color grading is showing the toroidal current density at the boundary planes; tubes show a choice of the magnetic flux tubes, with the diameter of the tube set proportional to the magnetic field strength; arrows show velocity. A change in connectivity of the magnetic flux tubes in 3D, triggered by the asymmetry in resistivity in the vertical plane, additionally changes, and complicates, the topology of magnetic field.

Reconnection rate in 3D

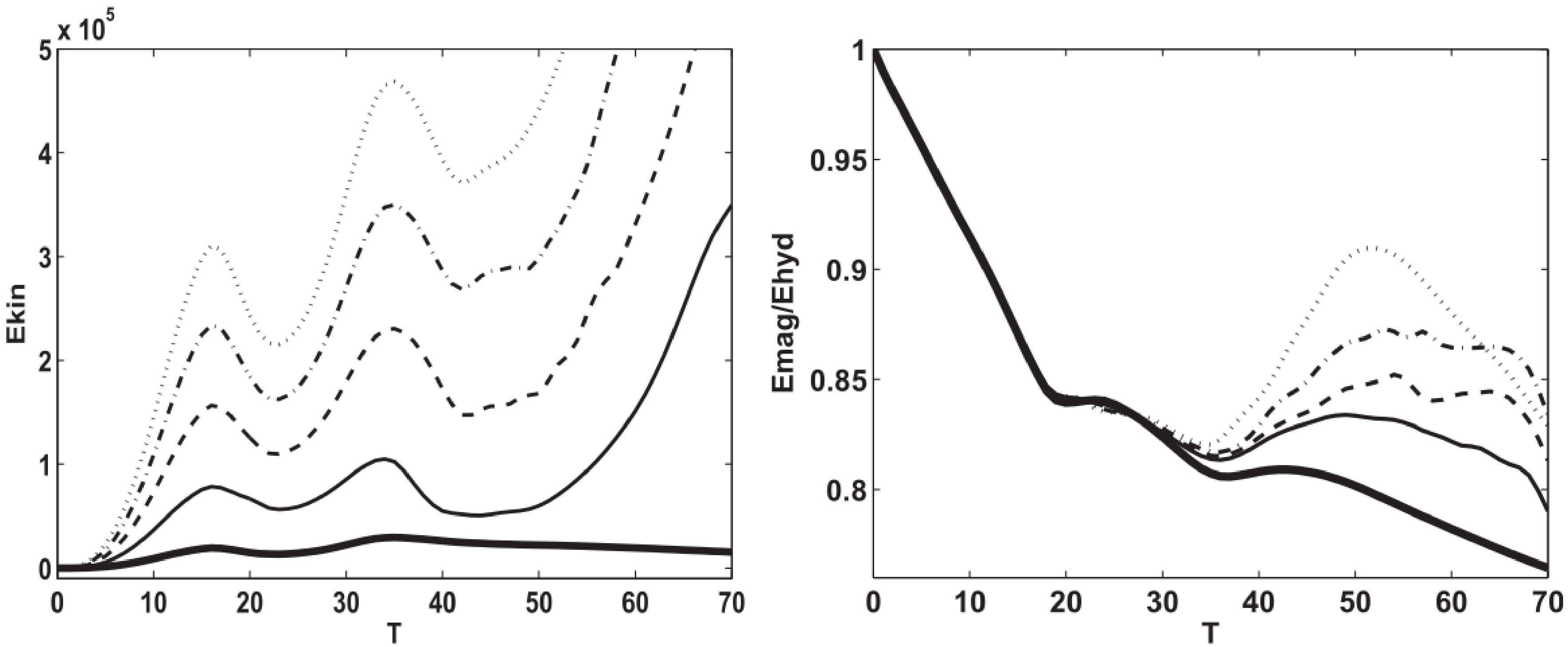


FIG. 4. Time dependence of the energy with different heights of the computational box in our simulations with reconnection in all three directions. In the left panel is shown the kinetic energy, and in the right panel is shown the ratio of magnetic to the sum of internal and kinetic energy. Results with different heights of the box h = 1, 2, 3, and 4, are shown in solid, dashed, dotted-dashed, and dotted lines, respectively. In thick solid line is shown the result with height h = 0.25 in the case with reconnection only in the X-Y plane, which is our reference 2D case. Kinetic energy during the build-up of reconnection is linearly increasing with height of the computational box, with the factor of proportionality about 2. The fraction of magnetic energy is steadily decreasing with time, until the reconnection in the third direction starts; then it increases proportionally with height.

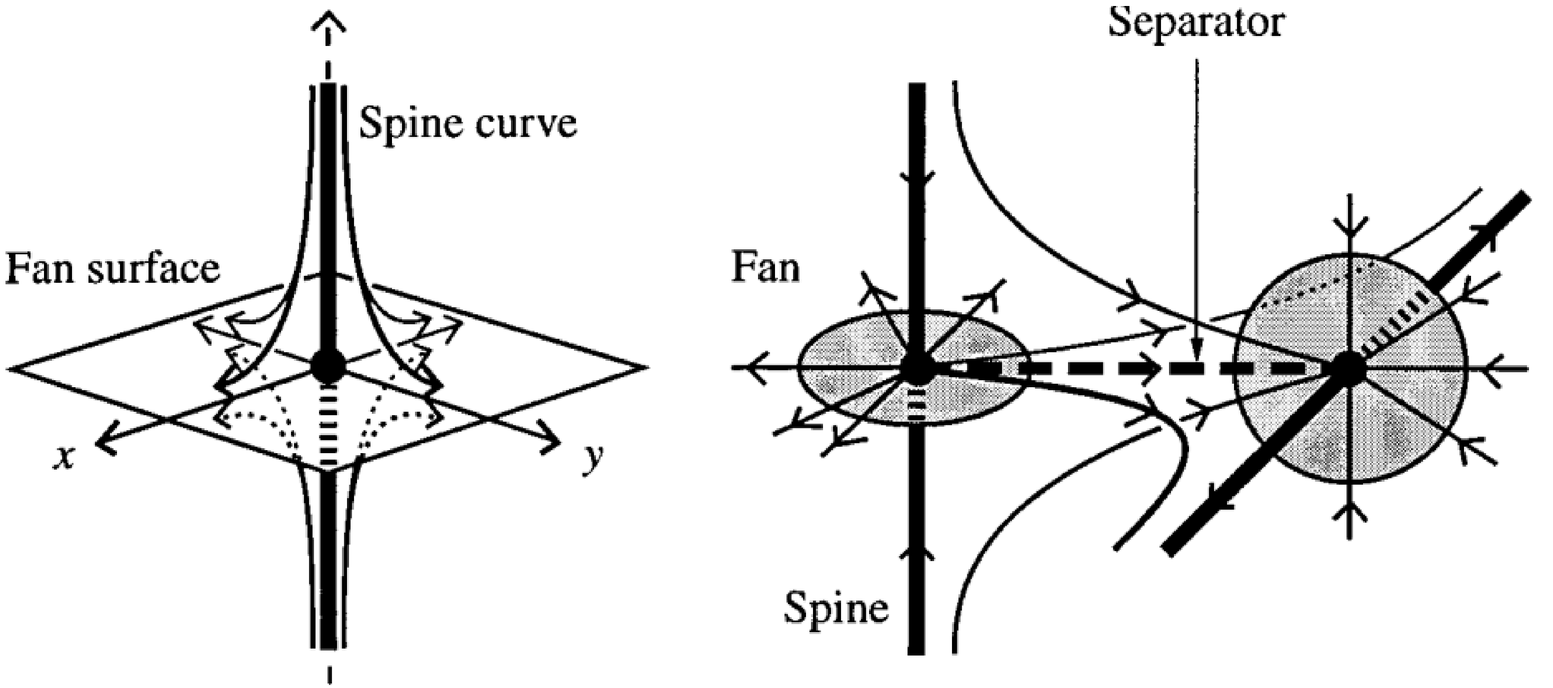


Figure 3. The structure of (a) a three-dimensional null point and (b) the separator joining two nulls. The fan surfaces are indicated schematically by shaded disks.

3.1. Spine and Fan Reconnection

Surround a null point by a cylindrical surface with its axis parallel to the spine and consider first what happens in a vertical plane through the null where the magnetic field lines have a simple X-type topology. If the footpoints of the field lines on one side move down continuously, while those on the other side move up, then the field lines in that plane will just reconnect in the classical two-dimensional manner (Figure 4(a)). A similar process takes place in all the other planes through the spine, but what happens to the flux surfaces? You form a flux surface by taking a series of footpoints on a curve and constructing the field lines through the footpoints.

MHD, lecture 6

Boussinesq approximation

Dynamo mechanism:
-kinematic, nonlinear (hydromagnetic) dynamo
-alpha and alpha-omega dynamos

Cylindric Taylor-Couette flow & dynamo

Nonlinear (hydromagnetic) dynamo

Magnetorotational instability (MRI)

Boussinesq approximation

Where the fluid varies in temperature (or composition) from one location to another, driving a flow of fluid and heat or mass transfer, if we can neglect the variations of density, then we can write, from the mass continuity:

$$\underbrace{\frac{\partial \rho}{\partial t}}_{\to 0} + \nabla\cdot(\rho\cdot\vec{v}) = 0 \quad , \quad \underline{\nabla\cdot\vec{v} = 0} \quad \text{(A)}$$

Boussinesq approximation

In addition, conservation of momentum of an incompressible, newtonian fluid (the Navier-Stokes eq.) is

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \underbrace{\nu}_{\text{kinematic viscosity}}\nabla^2\vec{v} + \frac{1}{\rho}\underbrace{\vec{F}}_{\text{total of forces}}$$

and the density variations are assumed to consist of a fixed part and a part linearly proportional to temperature,

$$\rho = \rho_0 - \underbrace{\alpha}_{\text{the coefficient of thermal expansion}}\rho_0(T - T_0)$$

Boussinesq: the density variation is important only in the buoyancy term.

with $\vec{F} = \rho\vec{g}$ for gravity only, we can write

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho_0}\nabla(p - \rho_0\vec{g}\cdot\vec{z}) + \nu\nabla^2\vec{v} - \vec{g}\alpha(T - T_0) \quad \text{(B)}$$

the heat capacity per unit volume ρc_p in the eq. for heat flow in a temperature gradient is assumed const and dissipation term negligible, so we stay with

$$\frac{\partial T}{\partial t} + \vec{v}\cdot\nabla T = \frac{\overbrace{k}^{\text{thermal conductivity}}}{\rho c_p}\nabla^2 T + \frac{J}{\rho c_p} \quad \text{(C)}$$

J: the rate per unit volume of internal heat production

A, B and C are the basic convection equations in Boussinesq approximation.

Boussinesq flows look the same when viewed upside-down, provided that the identities of the fluids are reversed. The Boussinesq approximation is inaccurate when the dimensionless density difference $\Delta\rho/\rho$ is approximately 1, i.e. $\Delta\rho \approx \rho$.

Dynamo mechanism

-Dynamo theory started with Joseph Larmor idea about the origin of the Earth magnetic field in 1919. With the development of the MHD, dynamos are considered for magnetic fields in astrophysics. Most of the work on analytic theories and numerical simulations, but from the beginning of 21st ct. also various liquid metal experiments have been performed.

Dynamos are divided into:

-kinematic dynamos, the flow is prescribed

-nonlinear dynamos (or "hydromagnetic dynamos"), with the flow affected by the magnetic field through the Lorentz force.

- Before Larmor, William Gilbert, the first to write about magnetism (1600), thought Earth as a permanent magnet. The next was Ampere's theory about "internal currents". Today we follow Elsasser's theory which states the fluid outer core of Earth as the site of Earth dynamo.

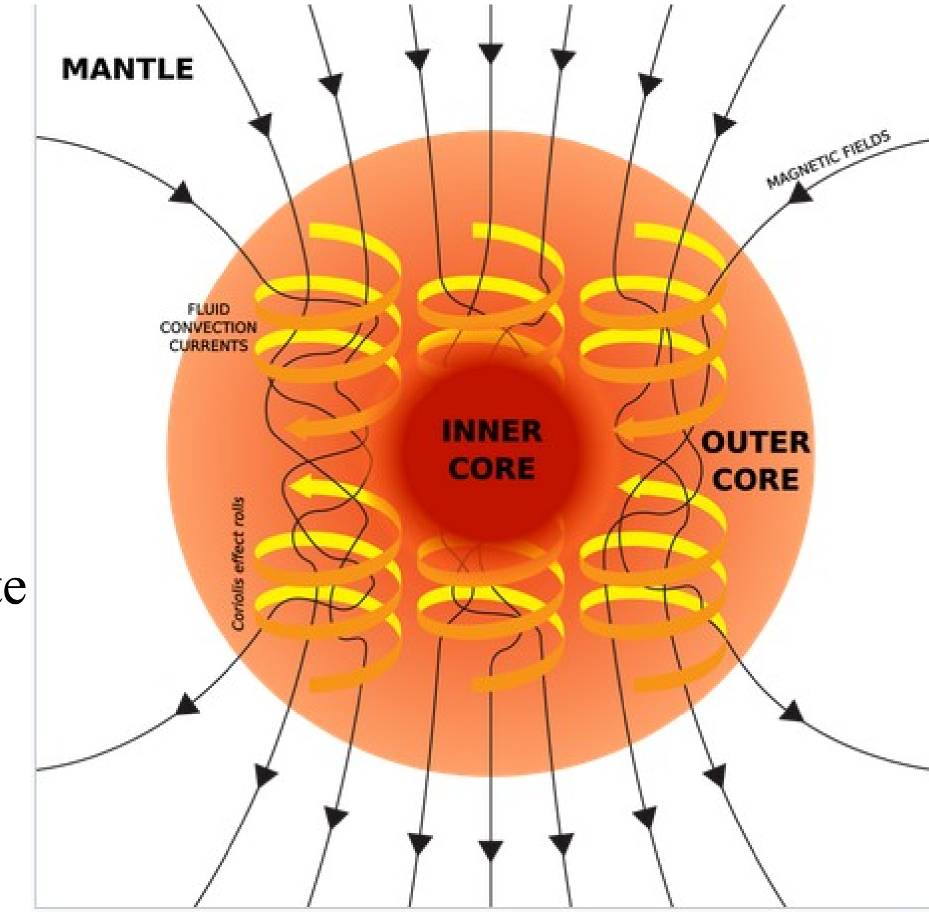


Illustration of the dynamo mechanism that generates the Earth's magnetic field: convection currents of fluid metal in the Earth's outer core, driven by heat flow from the inner core, organized into rolls by the Coriolis force, generate circulating electric currents, which supports the magnetic field.[1]

Dynamo mechanism

Without dynamo, the magnetic field of Earth, created by any current captured in the mantle would disappear, because of ohmic decay, in about 20 000 years.

Elsasser investigated the history of the Earth's magnetic field, studying the magnetic orientation of minerals in rocks. The Earth field has existed at roughly its present intensity (except possibly during rapid reversals) on a geological time-scale of order 10^9 years.

If the dynamo is to work, the outer core must be convecting likely some combination of thermal and compositional convection. The rate of westward drift (0.18" per year) suggests velocities of order u== 4×10^{-4} m/s near the core-mantle interface. A characteristic length-scale for magnetic perturbations associated with the variation is 1000 km. This gives a magnetic Reynolds number of ~150, which is not infinite, but allows, in the first approximation, the frozen-field assumption for magnetic field perturbations.

Dynamo mechanism

- With the development of the MHD, dynamos are considered for magnetic fields in astrophysics. Most of the work on analytic theories and numerical simulations, but from the beginning of 21st ct. also various liquid metal experiments have been performed.
- Dynamo theory today mostly relies on **mean field theory**, in which small scale motions are exciting a large scale field.
- Dynamos are divided into:

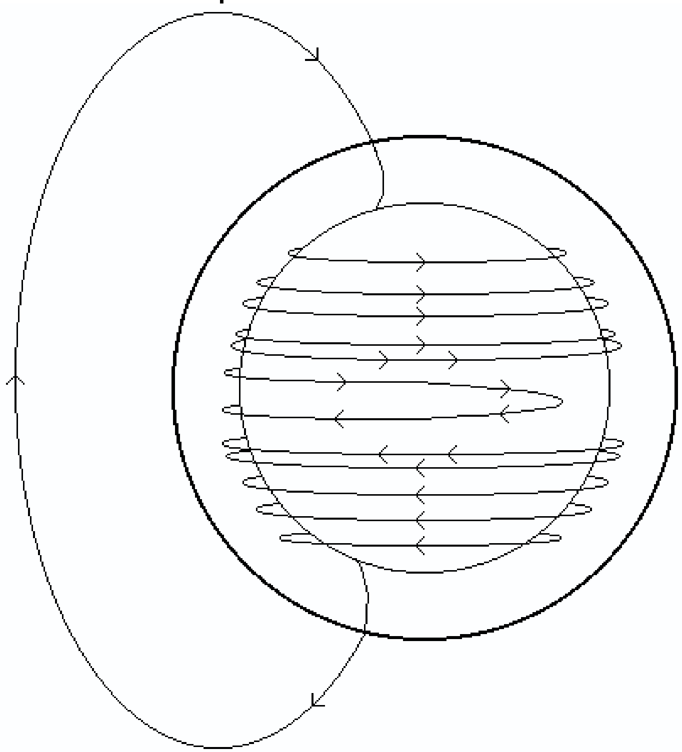
  -kinematic dynamos, with the field considered negligible and therefore the flow can be considered as given. They can be **small** and **large** scale dynamos. Kinematic theory is usually used to test if the given flow can produce dynamo effect.

  -nonlinear dynamos (or "hydromagnetic dynamos"), with the flow affected by the magnetic field through the Lorentz force. They are the saturations of the corresponding kinematic dynamos.

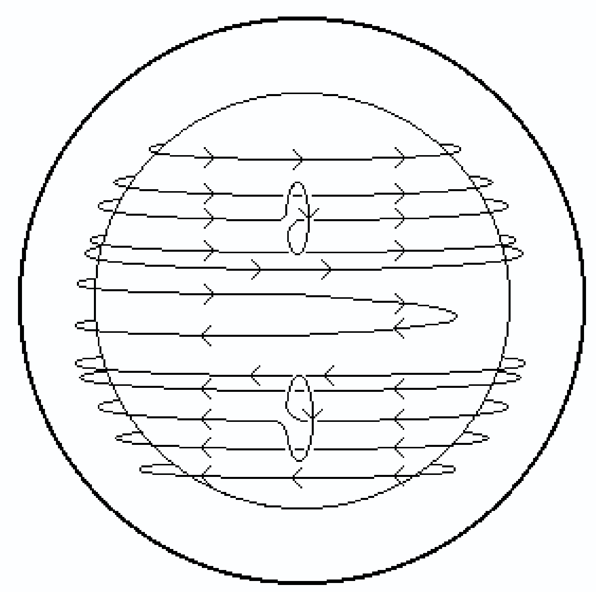
Alpha and omega effects

The Omega Effect

Magnetic fields within the Sun are stretched out and wound around the Sun by differential rotation - the change in rotation rate as a function of latitude and radius within the Sun is called the omega-effect (Greek Omega is usual symbol for rotation rate). The Sun's differential rotation with latitude can take a north-south oriented magnetic field line and wrap it once around the Sun in about 8 months.



The ω-effect



The α-effect

The Alpha Effect

Twisting of the magnetic field lines caused by the effects of the Sun's rotation is called the alpha-effect after the Greek letter that looks like a twisted loop. Early models of the Sun's dynamo assumed that the twisting is produced by the effects of the Sun's rotation on very large convective flows that carry heat to the Sun's surface. One problem with this assumption is that the expected twisting is far too much and it produces magnetic cycles that are only a couple years in length.

Alpha dynamo

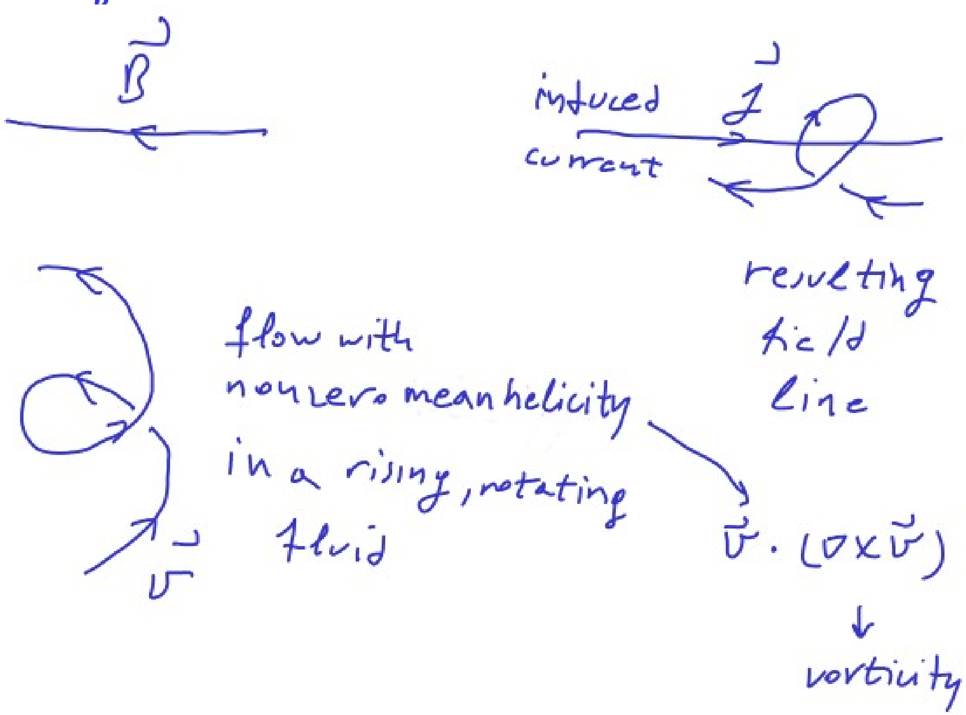
Most simple flows do not produce dynamos irrespective of their magnetic Reynolds number, which should be of the order of 10 for dynamo to work. Alpha-dynamo:

$$\frac{\partial \vec{B}}{\partial t} = \lambda \nabla^2 \vec{B} + \nabla \times (\vec{v} \times \vec{B})$$ in α-dynamo case

becomes $$\frac{\partial \vec{B}}{\partial t} = \lambda \nabla^2 \vec{B} + \tilde{\alpha} \cdot \vec{B},$$

and if we assume the tensor $\tilde{\alpha}$ to be a scalar (diagonal terms only, all the same), it is the simplest α-dynamo.

It is mostly used in stellar physics, in planets it is not working well.



Helicity is defined as the dot product of the flow and vorticity.

Dynamo

Example of computations Braginskii style, see the book by Moffatt, or with averages, for large scale kinematic dynamos:

$$\vec{v} = \sum_{\vec{q}} \vec{v}(\vec{q})\, e^{i\vec{q}\cdot\vec{r}}$$

$$\vec{B} = \vec{B}_0\, e^{i\vec{k}\cdot\vec{r}+\sigma t} + \vec{b}(\vec{r})\, e^{\sigma t}$$

with assumption $q \gg k$

$$\vec{v}\times\vec{B} = e^{\sigma t}\sum_{\vec{q}}\left[\vec{v}(\vec{q})\times\vec{B}_0\right] e^{i(\vec{q}+\vec{k})\cdot\vec{r}}$$

$$\vec{b}(\vec{r}) = \sum_{\vec{q}} \vec{b}(\vec{q})\, e^{i(\vec{q}+\vec{k})\cdot\vec{r}}$$

$q \gg k$ used

$$\sigma\vec{b}(\vec{q}) = -\lambda q^2\vec{b}(\vec{q}) + i\vec{q}\times\left[\vec{v}(\vec{q})\times\vec{B}_0\right]$$

$$\vec{v}\times\vec{b} = \sum_{\vec{q},\vec{q}'} \frac{1}{(\sigma+\lambda q^2)}\cdot i\vec{v}(\vec{q}')\times\left\{\vec{q}\times\left[\vec{v}(\vec{q})\times\vec{B}_0\right]\right\} e^{i(\vec{q}+\vec{q}'+\vec{k})}$$

Mean field theory: we choose $\vec{q}' = -\vec{q}$, contributions that affect the large scale field. $\Rightarrow$ $\vec{v}\times\vec{b}$ can be written as $\tilde{\alpha}\cdot\vec{\bar{B}}$, $\tilde{\alpha} = \sum_{\vec{q}} \frac{1}{(\sigma+\lambda q^2)}\, i\left[\vec{v}(-\vec{q})\times\vec{v}(\vec{q})\right]\vec{q}$

(using $\vec{q}\cdot\vec{v} = 0$ for incompressible flow)

$$\frac{\partial\vec{\bar{B}}}{\partial t} = \nabla\times\left(\vec{\bar{V}}\times\vec{\bar{B}} + \underbrace{\overline{\vec{v}\times\vec{b}}}_{\vec{\bar{\mathcal{E}}}} - \eta\mu_0\vec{\bar{J}}\right)$$

$\vec{\bar{\mathcal{E}}} = \alpha\vec{\bar{B}} - \eta_t\mu_0\vec{\bar{J}}$ $\vec{\bar{\mathcal{E}}}$ mean turbulent electromotive force (emf)

is the simplest, isotropic case.

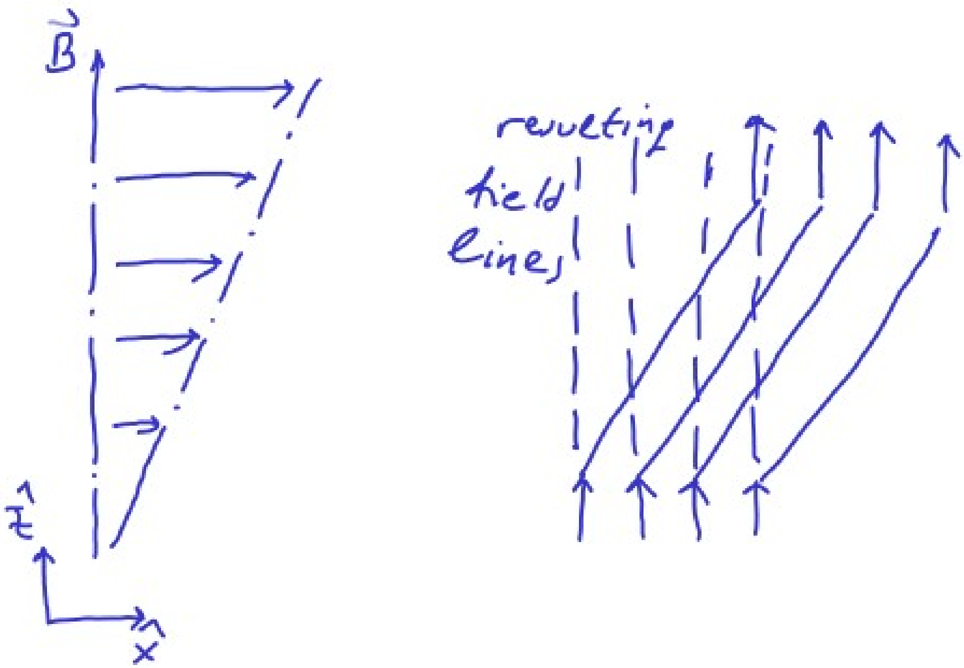
turbulent diffusivity

If not, more terms can be added

Alpha-omega dynamo

If we have a shear flow in the form **ωz** in the x direction, and the initial field is purely in the z direction, the induction effect is as written below, with the x-component of the field linearly growing with time. Omega effect is a strong effect but is not a dynamo as it just converts one field component into another.

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\omega z \hat{x} \times B \hat{z}) = \omega B \hat{x}$$



If we combine an α-effect (which is usually very anisotropic and is most probable to convert e.g. a horizontal into vertical field) to convert an x-component of the field into the z-component with the ω-effect to convert it back to the x-component, we are completing the regenerative cycle and we can have an alpha-omega dynamo.

Alpha-omega dynamo

We consider $\vec{B} = (B_x, 0, B_z)e^{\sigma t + iky}$

The x and z components of the induction eq. :

$\sigma B_x = -\lambda k^2 B_x + \omega B_z$

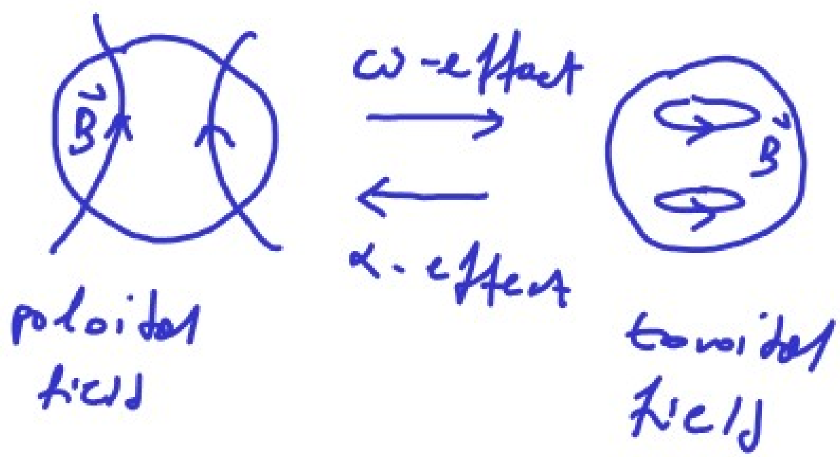
$\sigma B_z = -\lambda k^2 B_z - \alpha i k B_x$

We can write $B_x = \frac{\omega}{\sigma + \lambda k^2} \cdot B_z = \frac{\omega}{(\sigma+\lambda k^2)} \frac{-i\alpha k}{(\sigma + \lambda k^2)} B_x$

to obtain $\sigma = -\lambda k^2 \pm \frac{(1-i)}{\sqrt{2}}\sqrt{\alpha\omega k}$

Real part of $\sigma > 0$ if $\frac{\alpha\omega}{\lambda k\,\lambda k^2} > 2$

This is overstability ≡ growing wave propagation (Parker, 1955), such solution is applicable for Solar cycle. It works like this :



Cylindric Taylor-Couette flow

A magnetized incompressible fluid is contained between two rotating finite cylinders – Fig.1 from Rüdiger & Zhang (2001).

$\Omega(R)=a+b/R^2$, with a and b two constants related to rotation rates at two radii.

$$\Omega_{\mathrm{in}} = a + b/R_{\mathrm{in}}^2, \qquad \Omega_{\mathrm{out}} = a + b/R_{\mathrm{out}}^2. \tag{2}$$

Solving for a and b in terms of Ω_{in} and Ω_{out} we have

$$a = \Omega_{\mathrm{in}} \frac{\hat{\mu} - \hat{\eta}^2}{1 - \hat{\eta}^2}, \qquad b = \Omega_{\mathrm{in}} \frac{{R_{\mathrm{in}}}^2 (1 - \hat{\mu})}{1 - \hat{\eta}^2}, \tag{3}$$

where $\hat{\mu} = \Omega_{\mathrm{out}}/\Omega_{\mathrm{in}}$ and $\hat{\eta} = R_{\mathrm{in}}/R_{\mathrm{out}}$.

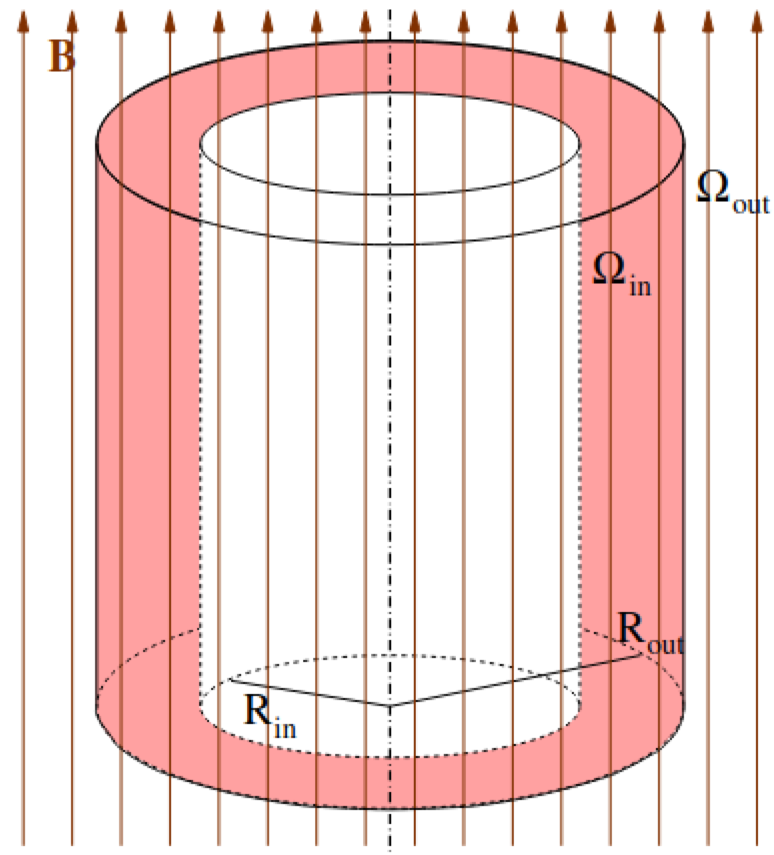


Fig. 1. Geometry of the cylindric Taylor-Couette flow.

Cylindric Taylor-Couette flow

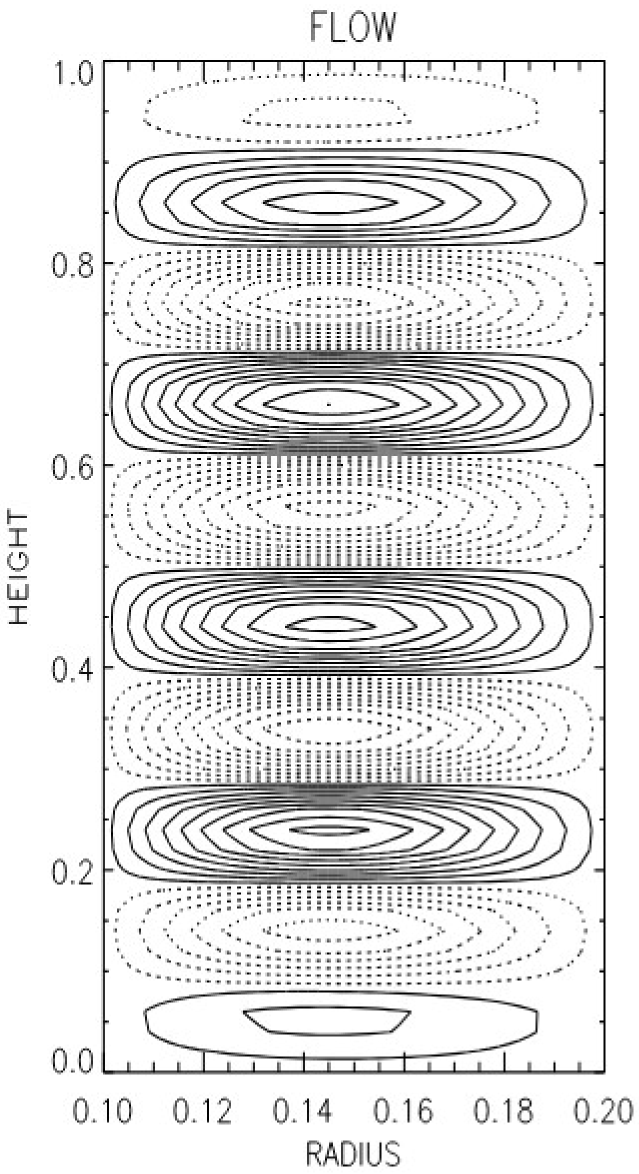


Fig. 3. The pattern for the the azimuthal flow without magnetic field (Ha $= 0$) but for a resting outer cylinder ($\hat{\mu} = 0$). $\hat{\eta} = 0.5$, $\mathrm{Pm} = 1$, $m = 0$.

Cylindric Taylor-Couette flow

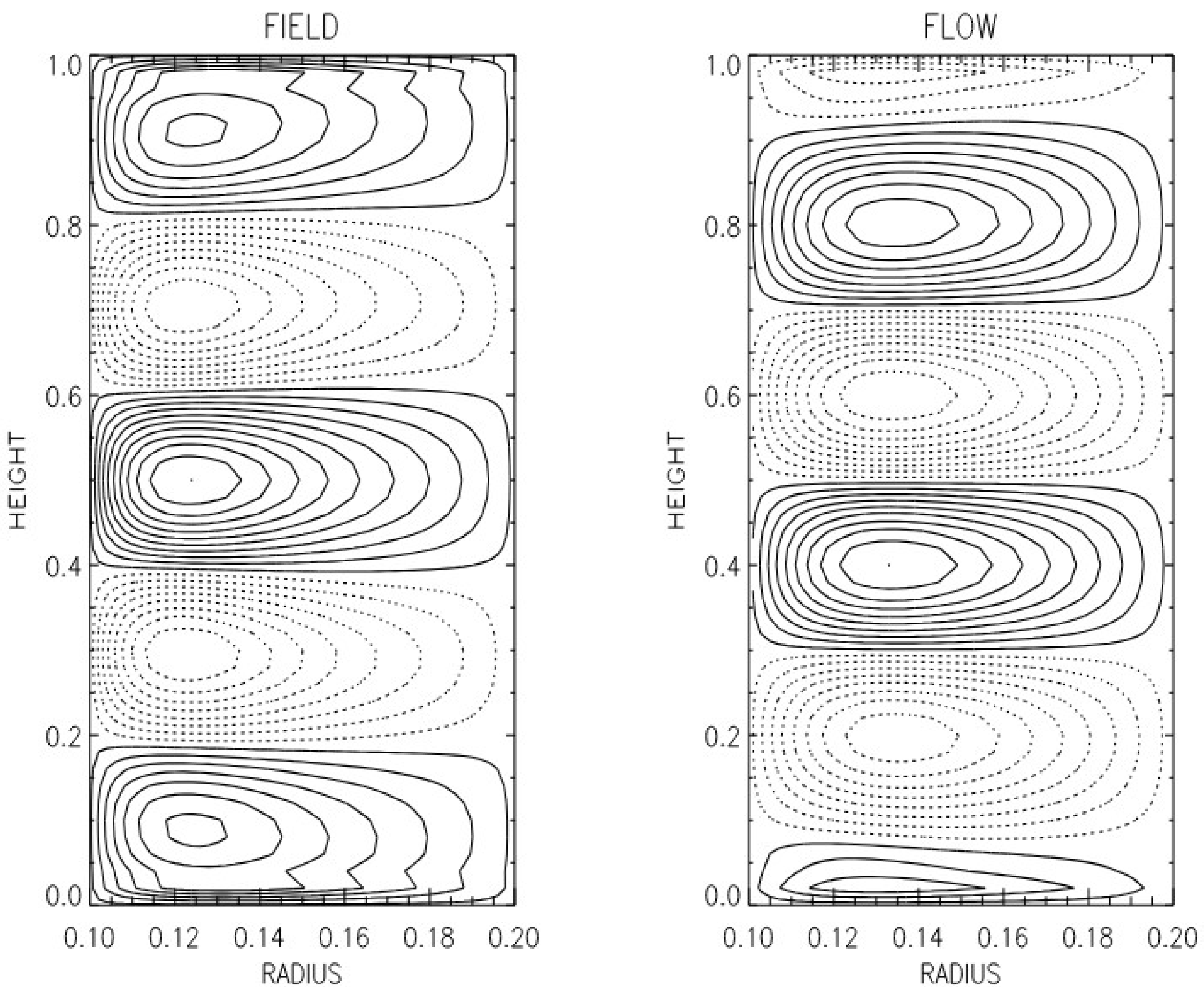


Fig. 4. The eigenfunctions for the toroidal magnetic field (left) and the azimuthal flow (right) in the gap between the cylinders. Ha = 200, Pm = 0.1, $m = 0$

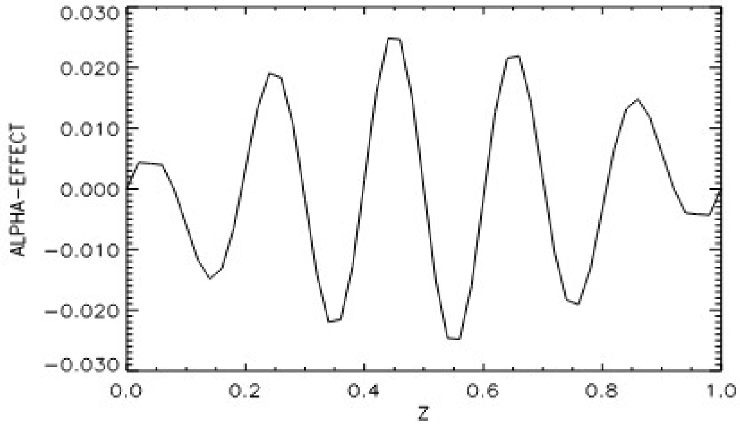


Fig. 6. The nonlinear quantity $\hat{\alpha}$ non-averaged as z-profile for $m = 0$ and at a given radius. Pm = 0.1, Ha = 200, $\hat{\mu} = 0.33$, $\hat{\eta} = 0.5$.

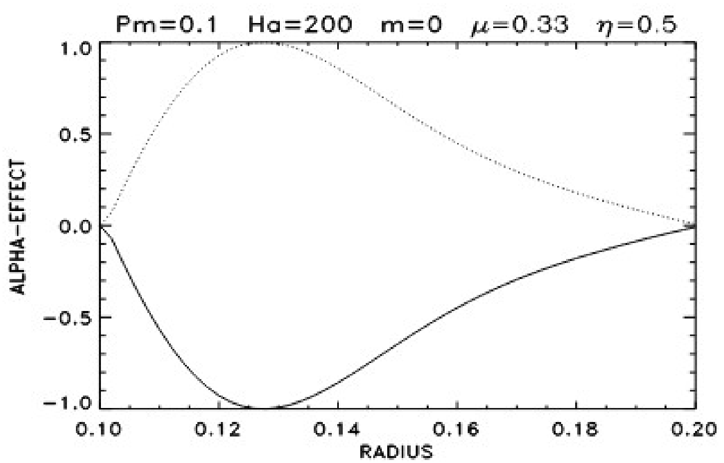


Fig. 7. The (vertical) dynamo-alpha is negative in the upper hemisphere (solid) and positive in the lower hemisphere (dashed).

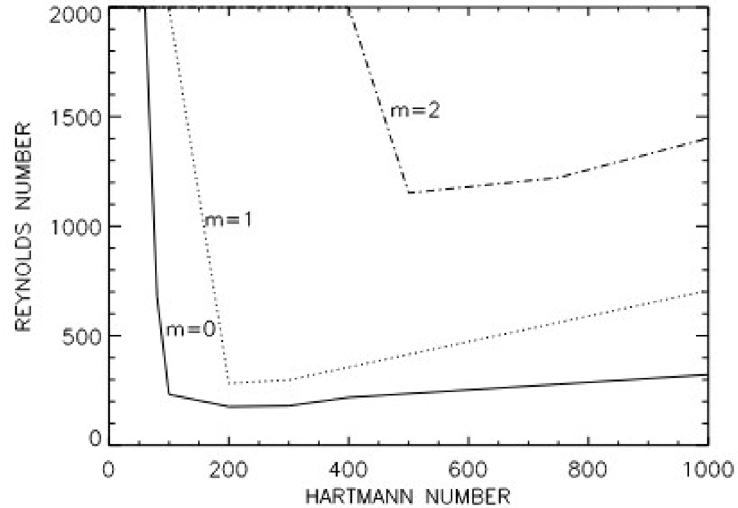


Fig. 8. Neutral-stability lines for various azimuthal wave numbers ($m = 0, 1, 2$) and rigid boundary conditions with Pm = 0.1. Hydrodynamic instability does not exist. The minimum Reynolds number is about 176. $\hat{\mu} = 0.33$, $\hat{\eta} = 0.5$.

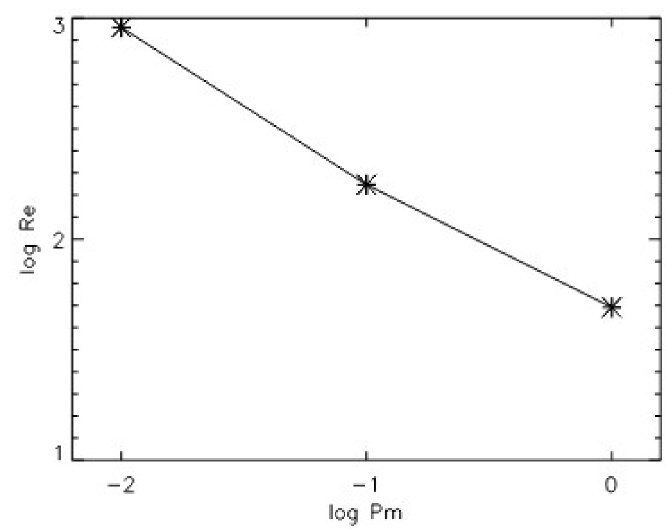


Fig. 9. Critical Reynolds numbers for various magnetic Prandtl numbers. The Reynolds number increases for decreasing Prandtl number. $m = 0$.

Simulations with Pm = 0.1 for a flat basic rotation law are given in detail in Fig. 8. The lowest eigenvalue is now

We consider an incompressible fluid contained between two coaxial cylinders of inner radius R_1 and outer radius R_2 which rotate at prescribed angular velocities Ω_1 and Ω_2. The velocity and magnetic fields, $\boldsymbol{V}(r,\theta,z,t)$ and $\boldsymbol{B}(r,\theta,z,t)$ are determined by the MHD equations which we write in dimensionless form as

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V} = -\boldsymbol{\nabla} p + \nabla^2 \boldsymbol{V} + (\boldsymbol{\nabla}\wedge\boldsymbol{B})\wedge\boldsymbol{B}, \tag{1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \frac{1}{P_m}\nabla^2\boldsymbol{B} + \boldsymbol{\nabla}\wedge(\boldsymbol{V}\wedge\boldsymbol{B}), \tag{2}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \quad \boldsymbol{\nabla}\cdot\boldsymbol{V} = 0, \tag{3}$$

Cylindric Taylor-Couette dynamo

where p is the pressure. In writing (1) and (2) we used $\delta = R_2 - R_1$ as unit of length, δ^2/ν as unit of time and $(\mu_0\rho)^{\frac{1}{2}}\nu/\delta$ as unit of magnetic field, where the kinematic viscosity ν, the density ρ and the permeability μ_0 are constant. The governing dimensionless parameters of the system are the Reynolds numbers Re_1, Re_2, the radius ratio η, and rotation ratio μ,

$$Re_1 = \frac{R_1\Omega_1\delta}{\nu}, \quad Re_2 = \frac{R_2\Omega_2\delta}{\nu}, \quad \eta = \frac{R_1}{R_2}, \quad \mu = \frac{\Omega_2}{\Omega_1}, \tag{4}$$

together with the magnetic Prandtl number P_m,

$$P_m = \frac{\nu}{\lambda}, \tag{5}$$

where $\lambda = 1/(\mu_0\sigma)$ is the magnetic diffusivity and σ is the electrical conductivity.

We assume no-slip boundary conditions for $\boldsymbol{V}$, and electrically insulating boundary conditions for $\boldsymbol{B}$. For modes of the form $\boldsymbol{B}(r,t,z) = \boldsymbol{B}(r)\,\mathrm{e}^{\mathrm{i}(\alpha z + m\theta)}$ these conditions are

$$\begin{aligned}
&\alpha = m = 0: \;\; B_\theta = B_z = 0; \\
&\alpha = 0,\, m \neq 0: \;\; \partial_\theta B_r = \pm m B_\theta, \quad B_z = 0; \\
&\alpha \neq 0: \;\; \partial_z B_r = \frac{\partial_r \mathcal{B}_m}{\mathcal{B}_m} B_z, \quad \frac{1}{r}\partial_\theta B_z = \partial_z B_\theta,
\end{aligned} \tag{6}$$

where for $\pm$ we take $+$ at R_1, $-$ at R_2. The symbol $\mathcal{B}_m(r)$ indicates the modified Bessel functions, $I_m(\alpha r)$ at R_1 and $K_m(\alpha r)$ at R_2. The axial wavelength of a pair of Taylor-vortices is $2\pi/\alpha$.

Our numerical method for timestepping the 3D nonlinear MHD equations is detailed in Willis & Barenghi (2002). The formulation is based on representing $\boldsymbol{V}$ and $\boldsymbol{B}$ with the toroidal-poloidal decomposition

$$\boldsymbol{A} = \psi_0\,\hat{\boldsymbol{\theta}} + \phi_0\,\hat{\boldsymbol{z}} + \boldsymbol{\nabla}\wedge(\psi\boldsymbol{r}) + \boldsymbol{\nabla}\wedge\boldsymbol{\nabla}\wedge(\phi\boldsymbol{r}), \tag{7}$$

where $\psi(r,t,z)$, $\phi(r,t,z)$ and $\psi_0(r)$, $\phi_0(r)$ contain the periodic and non-periodic parts of the field respectively. The potentials are expanded spectrally over Fourier modes in the azimuthal and axial directions and over Chebyshev polynomials in the radial direction.

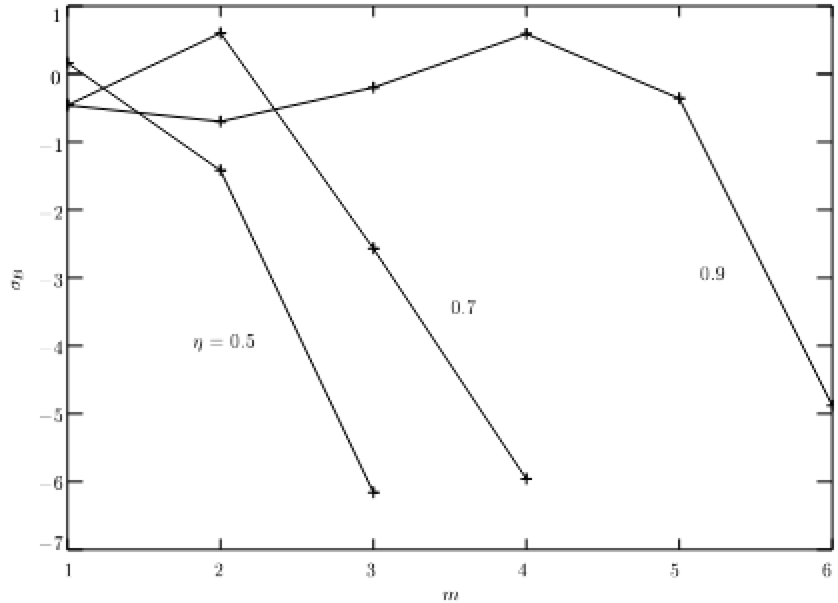


Fig. 1. Growth rates σ_B for various η and m, where $Re_1 = 2Re_{1c}(\eta)$, $P_m = 2$.

axisymmetric Taylor vortices (Chandrasekhar 1961). We also assume $\alpha = 3.14$, which corresponds to almost square cells. The nonlinear Taylor-vortex flow $\boldsymbol{V}$ thus obtained is used when solving the induction Eq. (2) for $\boldsymbol{B}$, starting from a small magnetic seed field of wavelength α_B.

Note that $\boldsymbol{V}$ being fixed, (2) is linear and has eigenfunction solutions $\boldsymbol{B}$ which grow or decay exponentially. If the real part of the growth rate σ_B is positive then the magnetic field grows (kinematic dynamo action). Following Laure et al. (2000) we assume $\alpha_B = \frac{1}{2}\alpha$, the characteristic length for the magnetic field is the length of two *pairs* of Taylor-vortices. In several calculations with $\alpha_B = \alpha$ we found the magnetic seed field decayed quickly. The axisymmetric flow cannot generate an $m = 0$ magnetic field. In Fig. 1 we see that in narrower gaps the dynamo prefers larger m. The dynamo is local here in the sense that the characteristic length scale for the magnetic field comply with the scale of the flow. As the Taylor-vortex flow is itself unstable to non-axisymmetric perturbations in narrower gaps (see Jones 1985), we usually consider the case $\eta = 0.5$

Cylindric Taylor-Couette dynamo

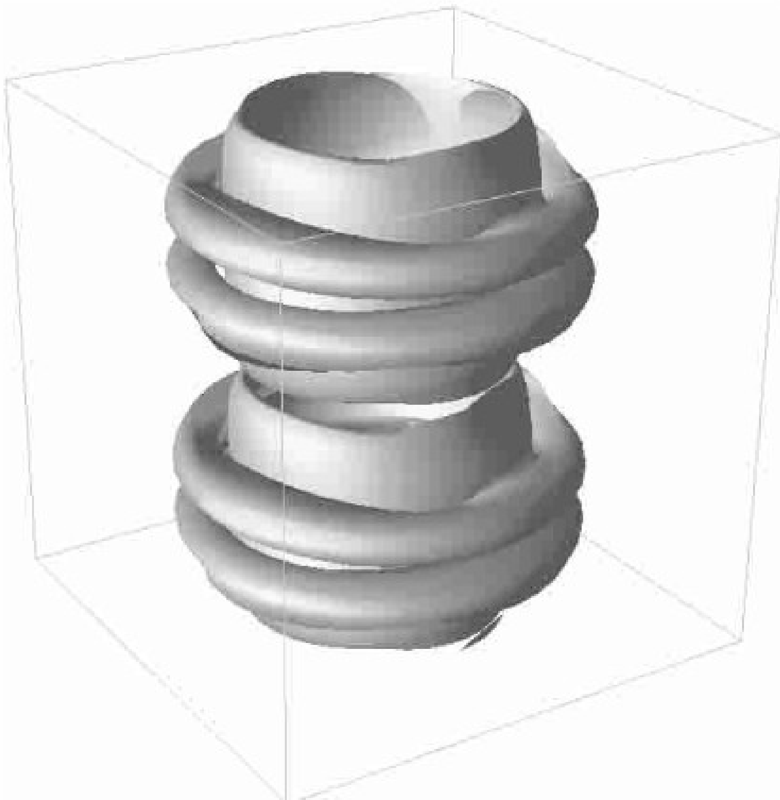


Fig. 10. Isosurface of helicity at magnetic field saturation. The flow is $m = 2$, looking the same if rotated by 180°. Parameters as in Fig. 7a.

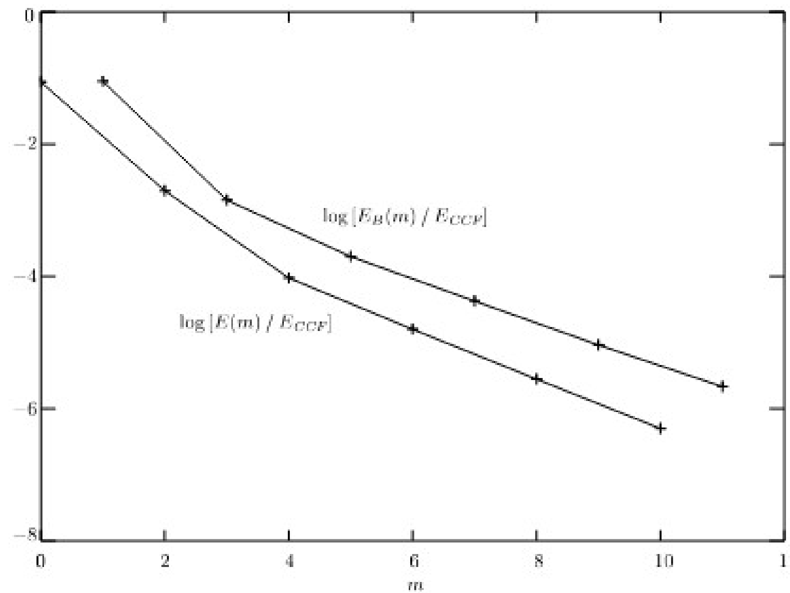


Fig. 11. Kinetic energy of the velocity disturbance and the magnetic energy of the various azimuthal modes. Parameters as in Fig. 7a.

rate is not a monotonic increasing function of the Reynolds number. This is not seen in Dudley & James (1989), most likely due to the prescribed form for the driving flow patterns.

In the Taylor-vortex flow the best growth rates have been obtained with co-rotation. The relative magnitude of the shear and roll in the flow plays an important part in the success of the dynamo mechanism.

Solving the full MHD equations we have demonstrated the existence of a fully self-consistent nonlinearly saturated dynamo. Hopefully these results will stimulate experimental work on the problem. Future theoretical work will investigate dynamo action in hydrodynamically stable flows and address the nature of the magnetic field structure when the dynamo is driven harder – our dynamo is laminar. Most of the present work is concerned with wider gaps.

References

Balbus, S. A., & Hawley, J. F. 1991, ApJ, 376, 214
Barenghi, C. F. 1991, J. Comput. Phys., 95, 175
Chandrasekhar, S. 1961, Hydrodynamic and Hydromagnetic Stability (Oxford)
Dobler, W., Shukurov, A., & Brandenburg, A. 2002, Phys. Rev. E, 65, 036311
Dudley, M. L., & James, R. W. 1989, Proc. R. Soc. Lond. A, 425, 407
Gailitis, A., Lielausis, O., Platacis, E., Dement'ev, S., & Cifersons, A. 2001 Phys. Rev. Lett., 86, 3024
Ji, H., Goodman, J., & Kageyama, A. 2001, MNRAS, 325, L1
Jones, C. A. 1985, J. Fluid Mech., 157, 135
Kulsrud, R. M., & Anderson, S. W. 1992, ApJ, 396, 606
Laure, P., Chossat, P., & Daviaud, F. 2000, in Dynamo and dynamics, a mathematical challenge, Nato Sci. Ser. II, 26, 17
Marcus, P. S. 1984, J. Fluid Mech., 146, 45
McIvor, I. 1977, MNRAS, 178, 85
Moffatt, H. K. 1978, Magnetic field generation in electrically conducting fluids (Cambridge)
Roberts, P. H. 1964, Proc. Cam. Phil. Soc., 60, 635
Rüdiger, G., & Zhang, Y. 2001, A&A, 378, 302
Steiglitz, R., & Müller, U. 2001, Phys. Fluids, 13, 561

Nonlinear (hydromagnetic) dynamo

When the Lorentz force starts to affect fluid motion, the nonlinear regime is reached-it usually happens at the end of the exponential growth of the field in a kinematic dynamo, quenching the dynamo, to result in some equilibrium state.

- An exception is a self-driving dynamo, with a flow existing only because of the Lorentz force. This is the case in magnetorotational instability (MRI).

  The rotation law in accretion discs is Keplerian, resulting from a balance between centrifugal and gravitational forces, implying the specific angular momentum increasing with radius, rendering hydrodynamically stable discs. In the presence of a magnetic field, points that are separated in space may be coupled nonlocally, so two points in a Keplerian orbit that are being pulled together will actually move further apart from each other. This is the essence of the MRI, which, because of large Reynolds numbers, leads to turbulence. Turbulence then can lead to dynamo effect. Simulations in the presence of stratification have shown that there can be an α effect, although the sign is opposite to the expected one.

- Another exception is self-killing (suicidal) dynamo, in which the Lorentz force damps the exponential growth.

Magnetorotational instability (MRI)

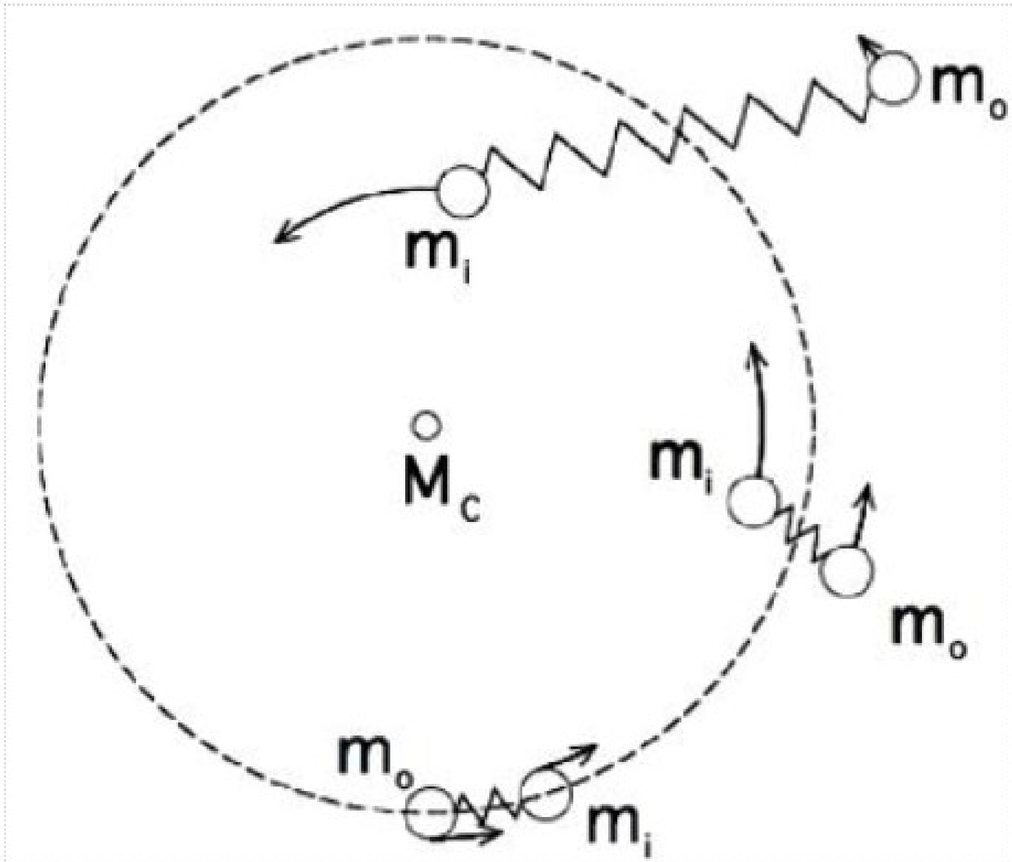


Figure 1: The magnetorotational instability. Magnetic fields in a disk bind fluid elements precisely as though they were masses in orbit connected by a spring. The inner element m_i orbits faster than the outer element m_o, even though the former has less angular momentum. Thus, the spring causes a net transfer of angular momentum from m_i to m_o. This transfer is unstable, as described in the text. The inner mass continues to sink, whereas the outer mass rises farther outward. (Courtesy of H. Ji)

An element of magnetized fluid experiences the Lorentz force **J**x**B**, which can be disruptive, causing instability, even with very weak B. For angular velocity of rotation decreasing with radial distance, the motion is unstable, with a destabilizing force proportional to the displacement. This is the origin of MRI (Balbus & Hawley, 1991, but actually first studied by Chandrasekhar, 1953; and 1961 and Velikhov, 1959).

- **m$_i$** experiences a retarding torque, loses angular momentum, and must fall to an orbit of smaller radius, corresponding to a smaller angular momentum. **m$_o$** experiences a positive torque, acquires more angular momentum, and moves to a higher orbit. The “spring” stretches even more, the torques become larger, which results in the unstable motion.

Magnetorotational instability (MRI)

We consider a fluid element of mass circling about the central mass with Ω, which is a function of the distance R from the rotation axis. We assume that the orbital radius is $r=R_0$. The centripetal force required to keep the mass in orbit is $-R\Omega^2(R)$, the minus sign indicates a direction toward the center. If this force is gravity from a point mass M at the center, then the centripetal force is just $-GM/R^2$, where G is Newton's constant.

For small departures from the circular motion of the orbiting mass element caused by some perturbing force. We transform variables into a rotating frame moving with the orbiting mass element at angular velocity $\Omega(R_0)=\Omega_0$, with origin located at the unperturbed, orbiting location of the mass element. Working in a rotating frame, we need to add to the equations of motion a Coriolis force $-2\mathbf{\Omega_0}\times\mathbf{v}$ plus a centrifugal force $R\Omega_2{}^2$. The velocity $\mathbf{v}$ is the velocity as measured in the rotating frame. We restrict our analysis to a small neighborhood near R_0 , say $R_0+\mathbf{x}$, with x much smaller than R_0 . Then the sum of the centrifugal and centripetal forces is

$$R\left[\Omega_0^2-\Omega^2(R_0+z)\right]\approx -xR\frac{d\Omega^2}{dR}$$ to the 1st (linear) order in z.

The x and y components of eqs. of motion, for a small departure from $R=R_0$:

$$\ddot{x}-2\Omega_0\dot{y}=-xR\frac{d\Omega^2}{dR}+f_x$$
$$\ddot{y}+2\Omega_0\dot{x}=f_y$$

, with f_x and f_y assigning the forces per unit mass in both directions.

Without these external forces, we have solutions with the time dependence $e^{i\omega t}$, with $\omega^2=4\Omega_0^2+R\frac{d\Omega^2}{dR}\equiv\varkappa^2$, the epicyclic frequency, which we can rewrite as $\frac{1}{R^3}\frac{dR^4\Omega^2}{dR}$, it is proportional to radial derivative of angular momentum per unit mass, or specific angular momentum $\rightarrow$ which must increase outward for having stable epicyclic oscillations [if not, we would have instability, with exponential growth of oscillations] (Chandrasekhar, 1961, Rayleigh criterion for stability.

If there is an external restoring force $f_x=-kx$, $f_y=-ky$ ($k=$ const) solutions for displacements with time dependence $e^{i\omega t}$ give

$$\omega^4-(2k+\varkappa^2)\omega^2+k\left(k+R\frac{d\Omega^2}{dR}\right)=0$$

If k is sufficiently small, a decreasing outward angular velocity would produce negative ω^2, with $\pm$ imaginary values for ω, which give exponential growth of very small displacements, and not oscillations. A large k will produce oscillations.

Magnetorotational instability (MRI)

Magnetic fields act on a fluid in motion as a spring with constant k, acting against the Lorentz force on free charges. Charges are locally rearranged so to produce an internal electric field $\vec{\mathcal{E}} = -\vec{v}\times\vec{B}$, counterbalancing the $\vec{F}_L = \vec{\mathcal{E}} + \vec{v}\times\vec{B}$ on charges. This induced $\vec{\mathcal{E}}$ produces further changes in $\vec{B}$ by Faraday:

$$\frac{\partial \vec{B}}{\partial t} = -\nabla\times\vec{\mathcal{E}} \quad \text{or} \quad \frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{v}\times\vec{B}) \quad \text{[ideal MHD]}$$

, or, written via displacement $\vec{\xi} = \vec{v}\,\delta t$ of the fluid in δt: $\delta\vec{B} = \nabla\times(\vec{\xi}\times\vec{B})$, for the 'frozen-in' field in the fluid and small $\delta\vec{B}$, with $\nabla\cdot\vec{\xi} = 0$, $\delta\vec{B} = (\vec{B}\cdot\nabla)\vec{\xi}$, changes in $\vec{B}$ happen only if there is a shearing displacement along the field line. If $\vec{B}$ is uniform in vertical direction z, and ξ changes as e^{ikz} $\delta\vec{B} = ik B\vec{\xi}$. Force per unit volume on conducting fluid is $\vec{J}\times\vec{B}$, since by Biot-Savart $\mu_0\vec{J} = \nabla\times\vec{B}$, we write $\frac{1}{\mu_0}(\nabla\times\vec{B})\times\vec{B} = -\nabla\left(\frac{B^2}{2\mu_0}\right) + \left(\frac{1}{\mu_0}\right)(\vec{B}\cdot\nabla)\vec{B}$

Magnetorotational instability (MRI)-from scholarpedia

The first term on the right is analogous to a pressure gradient. In our problem it may be neglected because it exerts no force in the plane of the disk, perpendicular to z . The second term acts like a magnetic tension force, analogous to a taut string. For a small disturbance $\delta\boldsymbol{B}$, it exerts an acceleration given by

$$\left(\frac{1}{\mu_0\rho}\right)(\boldsymbol{B}\cdot\boldsymbol{\nabla})\delta\boldsymbol{B} = \left(\frac{ikB\delta\boldsymbol{B}}{\mu_0\rho}\right) = -\frac{k^2B^2}{\mu_0\rho}(\xi) \qquad (11)$$

Thus, a magnetic tension force gives rise to a return force which is directly proportional to the displacement. This means that the oscillation frequency ω for small displacements in the plane of rotation of a disk with a uniform magnetic field in the vertical direction satisfies an equation ("dispersion relation") exactly analogous to equation (5), with $K = k^2B^2/\mu_0\rho$:

$$\omega^4 + [2(k^2B^2/\mu_0\rho) + \kappa^2]\omega^2 + (k^2B^2/\mu_0\rho)[(k^2B^2/\mu_0\rho) + Rd\Omega^2/dR] = 0 \qquad (12)$$

As before, if $d\Omega^2/dR < 0$, there is an exponentially growing root of this equation for wavenumbers k satisfying $(k^2B^2/\mu_0\rho) < -Rd\Omega^2/dR$. This corresponds to the MRI.

Notice that the magnetic field appears in equation (12) only as the product kB . Thus, even if B is very small, for very large wavenumbers k this magnetic tension can be important. This is why the MRI is so sensitive to even very weak magnetic fields: their effect is amplified by multiplication by k . Moreover, it can be shown that MRI is present regardless of the magnetic field geometry, as long as the field is not too strong.

In astrophysics, one is generally interested in the case for which the disk is supported by rotation against the gravitational attraction of a central mass. A balance between the Newtonian gravitational force and the radial centripetal force immediately gives

$$\Omega^2 = \frac{GM}{R^3} \qquad (13)$$

where G is the Newtonian gravitational constant, M is the central mass, and R is radial location in the disk. Since $Rd\Omega^2/dR = -3\Omega^2 < 0$, this so called *Keplerian disk* is unstable to the MRI. Without a weak magnetic field, the flow would be stable.

For a Keplerian disk, the maximum growth rate is $\gamma = 3\Omega/4$, which occurs at a wavenumber satisfying $(k^2B^2/\mu_0\rho) = 15\Omega^2/16$. γ is very rapid, corresponding to an amplification factor of more than 100 per rotation period.

Summary

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