Thin accretion disk

Miljenko Čemeljić

Nicolaus Copernicus Astronomical Center of the Polish Academy of Sciences,

Warsaw, Poland

&

Academia Sinica Institute of Astronomy and Astrophysics, Taipei, Taiwan, Visiting Scholar

General outline

- **PART I:** Introduction to accretion
  - Spherical (Bondi) accretion
  - Accretion disk; viscosity
  - Python code for Bondi accretion

- **PART II:** Steady disk solutions
  - Perturbative solutions for the disk
  - Shakura-Sunyaev disk
  - Python code for Shakura-Sunyaev disk

- **PART III:** General solutions
  - Urpin's and Regev's vertically averaged solutions
  - Kluźniak-Kita 3D global solution for thin disk and magnetic generalization
  - Python code for thin disk; python tool DUSTER

- Check webpage https://web.tiara.sinica.edu.tw/~miki/miki_thinacrdisk.html
  Required packages: Python (matplotlib, NumPy)

Outline, Part I: Introduction to accretion

- Spherical (Bondi) accretion

- Accretion disk; viscosity; Roche equipotential surfaces

- Python code for Bondi accretion

Accretion process – general introduction

Accretion is a process of mass collection onto a (usually rotating) central body, where a particle or a fluid element moving at some orbit transfers part of its energy and angular momentum to its surrounding.

More general, it can be described as inward motion of matter because of the gravitational force-this definition we need to include the simplest case which we will first study, the spherical (Bondi) accretion.

Gravity was the first force to be studied extensively, in mechanics, our first applied science. In astronomy it was for long the only computed force, by Newton. It was The Force of celestial motions, until we started computing the machinery of stellar power. Then it showed gravity is not enough-electromagnetic interactions (chemistry) were also helpless, Sun burning coal would expire practically on our eyes.

Nuclear forces explained this, but Gravity made a comeback when we understood sun-like stars are weaklings in comparison to a more powerful energetics of stellar compact objects, black holes and galactic nuclei.

How we arrived to accretion

Historically, accretion was first considered as a relevant process in the close binaries: after collecting a wealth of spectrographic data by Struve and collaborators, it became obvious that simple models of stellar stability are insufficient to explain the spectral features-theoretical curves were woo smoothed by the simplifications, and observational curves were not smooth at all!

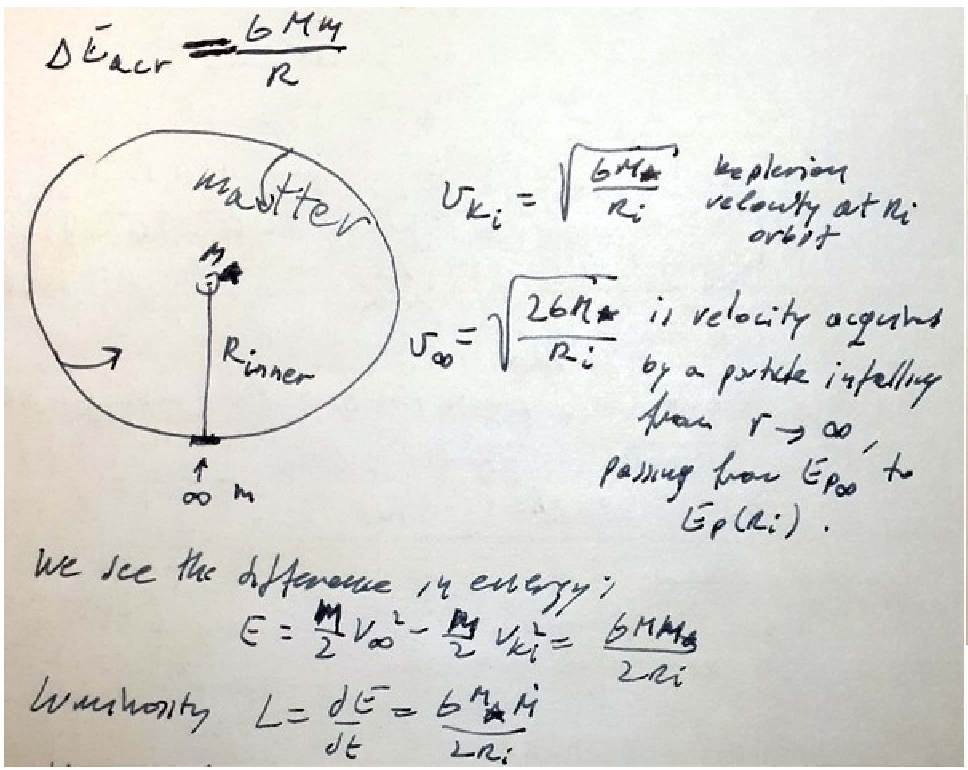
Introducing more physical processes into astrophysics of stars was called “Struve revolution” by Popper in 1970:

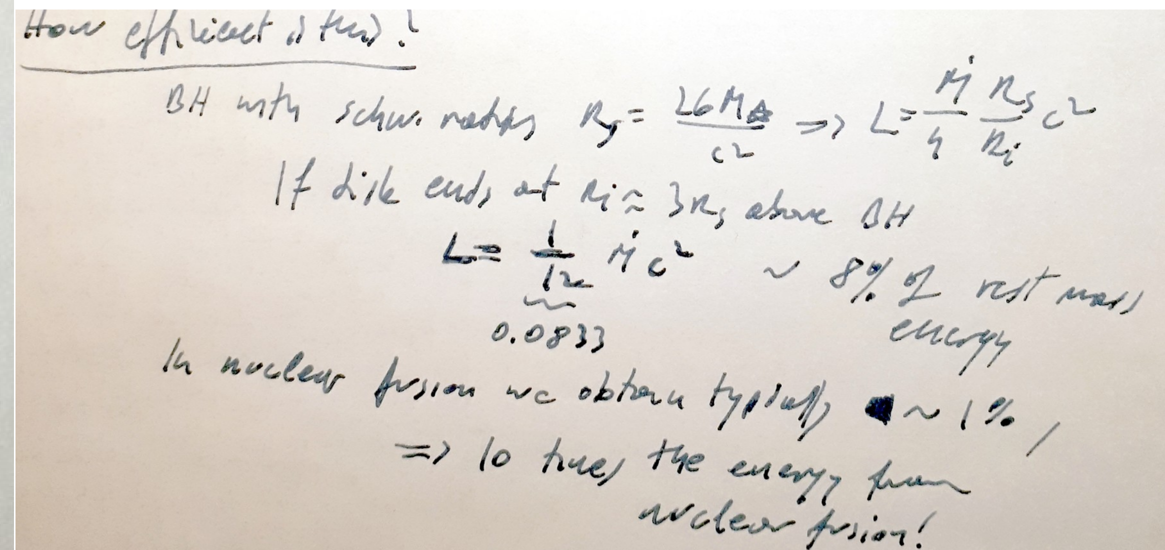
To explain the spectral features, hence the energy and angular momentum evolution in the “peculiar stars”-which mostly showed to be close binaries, astronomers around the middle of XX century had to include the streams of matter, gas rings, Roche’s equipotential surfaces, and finally Huang (1963) included a thick disk. Gradually, with increase in quality of data, similar concepts were introduced in the objects on other scales, like active galactic nuclei (AGNs), quasars, and centers of clusters of galaxies. Astrophysics of accretion could start!

Energetics of accretion

When I said that with accretion gravity made a comeback, I should say that it made it with a boom! - accretion is by far the most efficient way of extracting energy out of the matter we know: it yields about 10 times more than nuclear fusion!

We can show it in a back-of-the-envelope calculation for the luminosity of the disk acquired by the infall of material from the large distance onto a central object:





Energetics of accretion

- In the literature, one can find various numbers for the efficiency of accretion, I list some below, but what is meant by “efficiency”?

  It is the power available at given **mass accretion rate $\dot{M}$** onto an object of radius R: **$P=\dot{M}\,GM/R$.** This power is usually dissipated (through radiation) away, otherwise there will be no accretion-e.g. heat would push the matter away.

- Efficiency of pair annihilation: **$\eta=1$**

- Efficiency of Nuclear (H) fusion: $\eta \sim 10^{-4}$

- Accretion efficiency η : Earth: 10^{-9}, Sun: 10^{-6},

  White Dwarf: 10^{-4}, Neutron star: 10^{-1}

- Accretion efficiency **$\eta=GM/(Rc^2)$** onto black hole (no hard surface) depends on the details of accretion flow and spin of BH:

- **$0.057< \eta <0.42$** for thin accretion disk

Eddington limit

What is the maximum luminosity at which matter still can be accreted? (This means that gravitational force on a chunk of fluid still just exceeds the radiation pressure)

- Simplest case is radial accretion onto a mass point M. If medium is fully ionised gas of electrons and protons, and we assume Compton scattering with the simplest radiation pressure $F_{rad} = +\frac{\sigma_T}{c}\frac{L}{4\pi r^2}$ and $F_g = -m_p \frac{GM}{r^2}$

  from $F_{rad}=F_g$ we get that $L_{\rm Edd} = \frac{4\pi G M m_p c}{\sigma_T}$

- Some Eddington luminosities: solar mass NS $L_{\rm Edd} = 1.3 \times 10^{38} \left(\frac{M}{M_\odot}\right) \, {\rm erg/s}$

  supermassive BH $L_{\rm Edd} = 1.3 \times 10^{46} \left(\frac{M}{10^8 M_\odot}\right) \, {\rm erg/s}$

- Eddington mass accretion rate: $\dot{m}_{\rm Edd} = \frac{L_{\rm Edd}}{\eta c^2}$

- Usually it is said that the accretion is not possible if $L>L_{Edd}$ but there are cases when it is not true, and they are very interesting cases of supernovae and non-spherical accretion cases in disks and jets.

Spherical accretion-fast forward

H. Bondi (1952): an analytic solution for the spherically symmetric, steady-state accretion flow of an infinite gas cloud onto a point mass, in the Newtonian approach.

Such model was later extended, to be applied from the study of star formation to cosmology.

- Bondi considered adiabatic ($\mathbf{p \sim \rho^{\gamma}}$) accretion of gas. Far from central mass, gas elements move in dependence of their thermal energy only, so that with gas temperature $\mathbf{T_{inf}}$ with sound speed $\mathbf{c_s}$ we can say that at some critical distance from the central mass r_{cr} the escape velocity is equal to the speed velocity: $\mathbf{r_{cr}=2GM/c_s^2}$
For $\mathbf{r<r_{cr}}$ material falls freely onto the central mass, and for the density above the radius $\mathbf{r_{cr}}$, $\mathbf{\rho_{inf}}$ we can write the infalling mass $\mathbf{\dot{M}=4\pi G^2M^2\rho_{inf}/c_s^2}$
Hydrostatic equilibrium gives $\mathbf{\rho \sim r^{-3/2}}$ (with $\gamma=5/3$) and temperature $\mathbf{T \sim 1/r\hat{}[3(\gamma-1)/2]}$. Infalling gas reaches speed of sound at distance $\mathbf{r_s}$ from the center. We find $\mathbf{r_s=GM/c_s^2(5-3\gamma)/4}$.

Spherical accretion, more in detail

I follow Torok et al. (2020).

- We consider polytropic gas, for which $p=K\rho^{\gamma}$ where γ is a polytropic index, K is a constant.
- Gas is moving in the gravitational potential of mass M: $\Phi(r) = -\frac{GM}{r}$

Far from central mass, gas elements are at rest, $u^r=0$, asymptotic values are ρ_∞ and p_∞. From the continuity of mass $\vec{\nabla} \cdot (\rho \vec{v}) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \rho u^r\right) = 0$

and from radial component of the momentum, Euler eq., we have the momentum conservation.

- We can integrate $r^2 \rho u^r = \mathrm{const} \equiv -\frac{\dot{M}}{4\pi}$

where the constant $\dot{M}=-4\pi r^2 \rho u^r$ is the speed of mass loading onto the central object through the sphere of radius r.

Spherical accretion, derivation

Radial component of the momentum eq.:

$$u^r \frac{\mathrm{d}u^r}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{GM}{r^2} = -a^2 \frac{\mathrm{d}\ln\rho}{\mathrm{d}r} - \frac{GM}{r^2}$$

describes motion of the fluid element under gravity and pressure. From the logarithm of radial derivative we obtain:

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} = -\frac{2}{r} - \frac{1}{u^r}\frac{\mathrm{d}u^r}{\mathrm{d}r}$$

so that we can write $\dfrac{\mathrm{d}(u^r)^2}{\mathrm{d}r} = \dfrac{\frac{4a^2}{r} - \frac{2GM}{r^2}}{1 - \frac{a^2}{(u^r)^2}}$

- Next we define the dimensionless variables (with $r_* \equiv \dfrac{GM}{a_\infty^2}$)

$$y \equiv \left(\frac{-u^r}{a_\infty}\right)^2, \quad \tilde{a} \equiv \frac{a}{a_\infty}, \quad x \equiv \frac{r}{r_*}, \quad \dot{m} \equiv \frac{M}{4\pi r_*^2 \rho_\infty a_\infty}$$

we obtain ordinary differential equation, Bondi equation, with one additional expression:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{4\tilde{a}^2}{x} - \frac{2}{x^2}}{1 - \frac{\tilde{a}^2}{y}}, \quad \tilde{a}^2 = \left(\frac{\dot{m}}{x^2 y^{1/2}}\right)^{\gamma-1}$$

Spherical accretion, derivation

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- The boundary conditions at x → ∞ follow from the requirement of zero radial velocity at a large distance y → 0, with the sound speed obtaining asymptotic values ã → 1. The second condition gives asymptotic behavior

$$y \approx \left(\frac{\dot{m}}{x^2}\right)^2 \quad (x \to \infty)$$

There is a critical point, where the flow velocity equals the local speed of sound $y=\tilde{a}^2$ – since the denominator at the right side in Bondi eq. is 0, it will be regular only with the above part also equal to 0. This is called a sonic point, x_s, and we have

$$\frac{4\tilde{a}^2}{x_\mathrm{s}} - \frac{2}{x_\mathrm{s}^2} = 0 \quad \Leftrightarrow \quad y = \tilde{a}^2$$

This gives another condition on Bondi eq.: the integration curve y(x) is not necessarily in agreement with x → ∞ . Both conditions are met only for some values of ṁ .

Spherical accretion, analytical derivation

- Solution with the additional regularity condition is an interesting numerical problem. To know if we got it correct, we first check for the analytical solution-which we can do because in the spherical accretion case we can integrate the momentum equation.
- Dimensionless momentum eq.is
$$y^{1/2}\frac{\mathrm{d}y^{1/2}}{\mathrm{d}x} + \frac{1}{\rho a_\infty^2}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{1}{x^2} = 0.$$
We integrate the 1st and 3rd terms easily, and the 2nd is
$$\frac{1}{a_\infty^2}\int \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}\mathrm{d}x = \frac{1}{a_\infty^2}\int \frac{\mathrm{d}p}{\rho} = \frac{\gamma K}{a_\infty^2}\int \rho^{\gamma-2}\mathrm{d}\rho = \frac{\gamma K \rho^{\gamma-1}}{(\gamma-1)a_\infty^2} = \frac{\tilde{a}^2}{\gamma-1}$$
so we have
$$\frac{y}{2} + \frac{\tilde{a}^2}{\gamma-1} - \frac{1}{x} = \mathrm{const} \equiv \Psi$$
the dimensionless Bernoulli equation. The integration constant Ψ denotes the different integration curves, which are solutions of y(x) Bernoulli's equation. The one that meets our boundary condition must satisfy $y \to 0$ and $\tilde{a} \to 1$ for $x \to \infty$, so it is given by $\Psi \equiv \Psi_0 = 1/(\gamma - 1)$

- Which value of Ψ corresponds to a solution that passes continuously through a critical point? From the condition of regularity we know that
$$y_\mathrm{s} = \tilde{a}_\mathrm{s}^2, \quad x_\mathrm{s} = \frac{1}{2\tilde{a}_\mathrm{s}^2}$$

Spherical accretion, analytical derivation

- The equation for the conservation of mass also allows us to express the speed of sound at a sonic point using accretion speef $\tilde{a}_s = (4\dot{m})^{\frac{\gamma-1}{5-3\gamma}}$
- Inserting to Bernoulli equation, we obtain:

$$\Psi \equiv \Psi_s = \frac{\tilde{a}_s^2}{2} + \frac{\tilde{a}_s^2}{\gamma-1} - 2\tilde{a}_s^2 = \frac{5-3\gamma}{2\gamma-2}\tilde{a}_s^2 = \frac{5-3\gamma}{2\gamma-2}(4\dot{m})^{\frac{2\gamma-2}{5-3\gamma}}$$

- Since we require the solution to pass continuously through the sonic point, while giving the correct behavior to large radii, it must hold $\Psi_s = \Psi_0$. This gives us a simple analytical expression of the sonic position point x_s and local speed of sound $\tilde{a}_s$ at the sonic point: $x_s = \frac{5-3\gamma}{4}, \quad \tilde{a}_s^2 = \frac{2}{5-3\gamma}$

  Such speed of sound corresponds to the accretion rate

$$\dot{m}_0 = \frac{1}{4}\left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2\gamma-2}}$$

  which is the characteristic value in the problem.

Spherical accretion, analytical derivation

- We obtained solutions at great distances from a compact object (when $x \to \infty$). What behavior can be expected when $x \to 0$?

  Frobenius' theorem is applicable, which gives necessary and sufficient conditions for finding a maximal set of independent solutions of an overdetermined system of first-order homogeneous linear partial differential equations: " F is integrable if and only if for every p in U the stalk Fp is generated by r exact differential forms."
- Geometrically, the theorem states that an integrable module of 1-forms of rank r is the same thing as a codimension-r foliation.
- Wikipedia: The method usually breaks down like this:
- -We seek a Frobenius-type solution of the form of series $y=\sum_{\infty}^{k}=a_k x_k+r$. ...

  - The obtained series must be zero. ...

  - If the indicial equation has two real roots r1 and r2 such that r1−r2 is not an integer, then we have two linearly independent Frobenius-type solutions.

Spherical accretion, analytical derivation

- Assume that the solution near the point $x = 0$ captures the dependence $y \approx A / x^{\alpha}$. We choose from Bernoulli's equation the fastest growing terms that must be in equilibrium and by comparing them we determine α and A. From the continuity equation it can be seen that $\tilde{a}^2 \sim x^{(\alpha - 2)(\gamma - 1)}$.

- The first term of the Bernoulli equation (the term corresponding to the kinetic energy) grows as $x^{-\alpha}$, the second term (corresponding to the internal energy) grows as $x^{(\alpha - 2)(\gamma - 1)}$ and the last third term (corresponding to potential energy) grows as x^{-1}.

- Comparing the exponents, assuming that $1 \leq \gamma \leq 5/3$, we find that the highest value of α corresponds to the equilibrium between the kinetic and potential member and corresponds to $\alpha = 1$ and $A = 2$, so $y \sim 2/x$ $(x \rightarrow 0)$.

  It means that very close to a compact object the pressure is only a negligible perturbation to gravity. At large distances the gravitational potential and internal energy are in balance. In both cases, the neglected term tends to reduce the speed values. This means that it will be useful in the numerical solution of Bernoulli's equation to use the method of bisection.

Accretion disk

- Let us take another view on accretion: material point orbiting around a center of mass interacts with its surrounding, transferring part of its energy and angular momentum.
- Consequence of such scenario is slowly spiral-in of the mass point.

  Energy which can be extracted is equal to the bonding energy of the smallest orbit: E_{acr}=GMm/R , see the back of the envelope calculation.
- We consider rotating volume of gas with angular momentum **L** in cylindrical coordinates (R,φ,z), with **z** parallel to the axis of rotation.
- We further assume that distribution of L between the gas particles is much slower than radiation transfer and rotation=**L** of the particle with mass m remains constant, but its kinetic and internal energy are distributed to other particles by collisions, shocks and radiation.
- For the constant **L** the minimal energy is for the circular orbit=>we obtain the thin disk in which particles rotate with $v_\varphi = R\Omega(R)$,we can write, with potential Φ: $F_g = ma = mv_\varphi^2/R = -d\Phi/dR = F_{cf}$.

Accretion disk

- If there is a process counteracting the spread of particles (as viscosity), energy spreads through the system by heat, and escapes from the system by radiation.
- As a consequence, mass particle will orbit at smaller R, we can understand this as transformation of the orbital energy into radiation energy.
- With the gas surface density S(R,t) and radial velocity $v_r(R,t)$, we observe the element of gas with inner radius R and outer R+ΔR. Mass of such ring is Δm=2πR·ΔR·S, angular momentum **L=R**x**p**, where for angle of 90° between **R** and **p** we can write $L=Rmv_\varphi$, and in fact we can write, with $L=mR^2\Omega$, that the angular momentum of such ring is $2\pi R\Delta R S R^2\Omega$. Change of mass of such a ring is equal to the fluxes in and out from the neighboring rings (positive sign is away from the center): ∂/∂t(Δm)=flux(R)-flux(R+ΔR)=

$$= v_r(R,t)\cdot 2\pi R S(R,t) - v_r(R+\Delta R,t)2\pi(R+\Delta R)S(R+\Delta R,t) =$$

$$= v_r(R,t)2\pi R S(R,t) - v_r(R+\Delta R,t)2\pi R S(R+\Delta R,t) -$$

$$- v_r(R+\Delta R,t)2\pi\Delta R S(R+\Delta R,t) =$$

=[with ΔR→0, 3rd term=0 and f(x+Δx)-f(x)=Δx∂f(x)/∂x]= $-\Delta R\,\partial(2\pi R S v_r)/\partial R$

Accretion disk

- Now we can write $\partial/\partial t(\Delta m)/\Delta R=-\partial/\partial R(2\pi RSv_r)$ and since $\Delta m/\Delta R=2\pi RS$, so we stay, after divide by 2π, with $\partial/\partial t(RS)=-\partial/\partial R(RSv_r)$, $\partial R/\partial t=v_r$ and we can write further $v_rS+R\partial S/\partial t=-v_rS-R\partial/\partial R(Sv_r)$. Since $\partial/\partial R(RSv_r)=v_rS+R\partial/\partial R(Sv_r)$ we can write: $R\partial S/\partial t+\partial/\partial R(RSv_r)=-v_rS=0$, since we are interested only in the **change** of mass.

- Exercise for homework is to do the same for conservation of the angular momentum of such a ring, but adding the term for transfer of angular momentum between the rings, because of viscous torques, $\Delta R\partial G/\partial R$:

$$\frac{\partial}{\partial t}(2\pi R\Delta RSR^2\Omega) = v_r(R,t)2\pi RS(R,t)R^2\Omega(R)-$$

$$-v_r(R+\Delta R,t)2\pi(R+\Delta R)S(R+\Delta R,t)\cdot(R+\Delta R)^2\Omega(R+\Delta R)+$$

$$+\frac{\partial G}{\partial R}\Delta R = [(R+\Delta R)^3 = R^3+3R^2\Delta R+3R(\Delta R)^2+(\Delta R)^3 =$$

$$= \text{linearna aproksimacija} = R^3+3R^2\Delta R] =$$

Accretion disk

$$= v_r(R,t)2\pi RS(R,t)R^2\Omega(R+\Delta R) -$$
$$-3v_r(R+\Delta R,t)2\pi\Delta RS(R+\Delta R,t)R^2\Omega(R+\Delta R) + \frac{\partial G}{\partial R}\Delta R$$

- In the linear approximation, first two terms are $-\Delta R\partial(2\pi v_r RSR^2\Omega)/\partial R$ and we have:

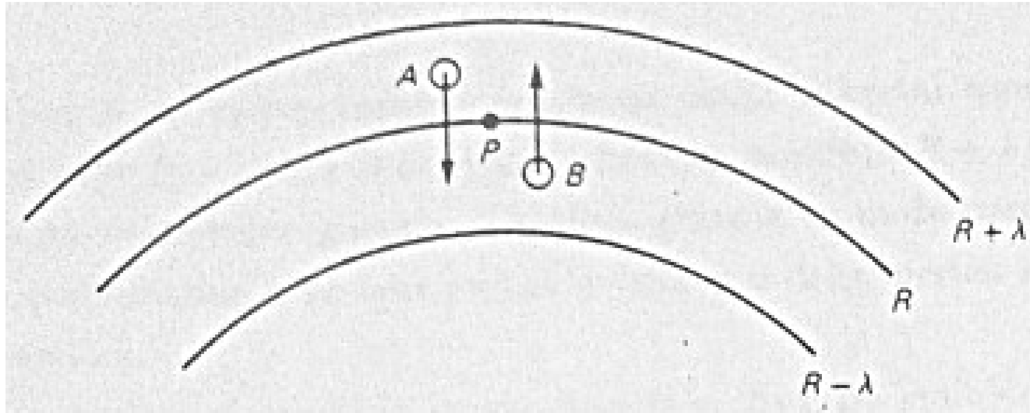
$$2\pi\Delta R\frac{\partial}{\partial t}(RSR^2\Omega) = -\Delta R\frac{\partial}{\partial R}(2\pi v_r RSR^2\Omega) - 3v_r 2\pi\Delta RSR^2\Omega + \frac{\partial G}{\partial R}\Delta R$$

$$\frac{\partial R}{\partial t}SR^2\Omega + R^2\frac{\partial}{\partial R}(SR^2\Omega) = -\frac{\partial}{\partial R}(v_r RSR^2\Omega) - 3v_r SR^2\Omega + \frac{\partial G}{\partial R}\cdot\frac{1}{2\pi}$$

- Again we discard constant 1st l.h.s and 2nd r.h.s. terms and write $\partial R/\partial t = v_r$ to get:

$$R\frac{\partial}{\partial t}(SR^2\Omega) + \frac{\partial}{\partial R}(Rv_r SR^2\Omega) = \frac{1}{2\pi}\frac{\partial G}{\partial R}.$$

Viscosity



Two neighbor rings in the disk.

- Now we will find the torque of two neighbor rings. The speed of chaotic motion in the gas is $\bar{\upsilon}$, and λ is the characteristic scale, which is also the mean free path. After exchange, element A will (in average) have torque from the position R-$\lambda/2$, and element in B from R+$\lambda/2$. Material in chaotic movement does not transfer matter (in average =0), only the steady flow can.
- Transferred mass is $\partial m/\partial t = H\rho\bar{\upsilon}$, where H is the disk height in z direction.
- For the accretion process essential is the difference in transported torques, and there is transport of torque because of chaotic motions. This is viscous torque.
- Observer in point P, rotating with $\Omega(R)$ sees fluid in R-$\lambda/2$ moving with speed

$$(R - \frac{\lambda}{2})\Omega(R - \frac{\lambda}{2}) + \Omega(R)\frac{\lambda}{2}$$

Viscosity

- This gives the average flow of angular momentum by the unit angle directed outwards:

$$\rho\bar{v}HR[(R-\frac{\lambda}{2})\Omega(R-\frac{\lambda}{2})+\Omega(R)\frac{\lambda}{2}]$$

and inwards:

$$\rho\bar{v}HR[R+\frac{\lambda}{2}\Omega(R+\frac{\lambda}{2})-\Omega(R)\frac{\lambda}{2}].$$

- Torque on the outer ring by the inner ring is equal to total outwards torque. In the first approximation we have:

$$\rho\bar{v}HR\{[(R-\frac{\lambda}{2})\Omega(R-\frac{\lambda}{2})+\Omega(R)\frac{\lambda}{2}]-[(R+\frac{\lambda}{2})\Omega(R+\frac{\lambda}{2})-\Omega(R)\frac{\lambda}{2}]\}=$$

$$=\rho\bar{v}HR\{R\Omega(R-\frac{\lambda}{2})-R\Omega(R+\frac{\lambda}{2})-\frac{\lambda}{2}\Omega(R-\frac{\lambda}{2})-\frac{\lambda}{2}\Omega(R+\frac{\lambda}{2})+2\Omega(R)\frac{\lambda}{2}\}=$$

- =[first two terms in { } give $-R\lambda\partial\Omega/\partial R$, 3rd and 4th after $\lambda/2\to 0$ give $-\lambda/2\Omega(R\ )$]$=-\lambda\rho\bar{v}HR^2\partial\Omega/\partial R$.

Viscosity, and back to the disk

- For the whole ring we multiply the obtained result with $2\pi R$, and with the surface density $\rho H=S$ [from $\rho=m/V$, $S=m/A$ we divide with H, we have $S/H=m/(AH)$, r.h.s.$=\rho$] we obtain that the torque of the outer to the inner ring
- (- inner torque to the outer ring) is $G(R)=2\pi R\nu S R^{2c}\Omega/\partial R$, where $\nu=\lambda\bar{\upsilon}$ is the kinematic viscosity coefficient.
- We had $G=G(R,t)$, and with $R\partial\Omega/\partial R=A$ we have $G(R)=2\pi R\nu SAR$, where νSA is a viscous force per unit angle.

- Now we can insert the obtained viscosity in the disk equations. We insert G into the equation we obtained from angular momentum conservation:

$$R\frac{\partial}{\partial t}(SR^2\Omega)+\frac{\partial}{\partial R}(Rv_r SR^2\Omega)=\frac{1}{2\pi}\frac{\partial G}{\partial R}.$$

- We divide with R and together with $R\partial S/\partial t+\partial/\partial R(RSv_r)=0$ we can eliminate v_r.

Viscosity, and back to the disk

- The first equation, after division with R, we can rewrite as:

$$S\frac{\partial}{\partial t}(R^2\Omega) + R^2\Omega\frac{\partial S}{\partial t} + \frac{1}{R}R^2\Omega\frac{\partial}{\partial R}(SRv_r) +$$
$$+ \frac{1}{R}SRv_r\frac{\partial}{\partial R}(R^2\Omega) = \frac{1}{R}\frac{\partial}{\partial R}(\nu SR^3\frac{\partial\Omega}{\partial R}).$$

- We write $R\partial S/\partial t+\partial/\partial R(RSv_r)=0$ as $\partial(Rsv_r)/\partial R=-R\partial S/\partial t$ and we have

$$S\frac{\partial}{\partial t}(R^2\Omega) + R^2\Omega\frac{\partial S}{\partial t} - R^2\Omega\frac{\partial S}{\partial t} + Sv_r\frac{\partial}{\partial R}(R^2\Omega) =$$
$$= \frac{1}{R}\frac{\partial}{\partial R}(\nu SR^3\frac{\partial\Omega}{\partial R}).$$

- In the first approximation **L** is a constant vector, and since $R^2\Omega$ is proportional to its length, we can discard the 1st term.

Viscosity, and back to the disk eqs.

- We obtain: $$v_r = \frac{\frac{1}{R}\frac{\partial}{\partial R}(\nu S R^3 \frac{\partial \Omega}{\partial R})}{S\frac{\partial}{\partial R}(R^2\Omega)}$$
- Inserting it into $R\partial S/\partial t+\partial/\partial R(RSv_r)=0$ we have: $$\frac{\partial S}{\partial t} = -\frac{1}{R}\frac{\partial}{\partial R}[RS\frac{\frac{1}{R}\frac{\partial}{\partial R}(\nu S R^3 \frac{\partial \Omega}{\partial R})}{S\frac{\partial}{\partial R}(R^2\Omega)}] = = \frac{1}{R}\frac{\partial}{\partial R}\{\frac{\frac{\partial}{\partial R}[\nu S R^3(-\frac{\partial \Omega}{\partial R})^2]}{\frac{\partial}{\partial R}(R^2\Omega)}\}.$$
- With $F_g=F_{cf}$, potential of point mass M is $mv^2/R=GMm/R^2$ and $v_\varphi=\Omega R$, **G=gravity const.**, Keplerian $\Omega=(GM/R^3)^{1/2}$ and $d\Omega/dR=-3/2(GM/R^5)^{1/2}$ and →
  This is the diffusion equation for the surface density S: mass diffuses inwards, angular momentum outwards.
- Diffusion timescale is $t_{visc}=R^2/\nu$.

$$\frac{\partial}{\partial R}(R^2\Omega) = \frac{\partial}{\partial R}(R^2\sqrt{GM}R^{-\frac{3}{2}}) = = \frac{\partial}{\partial R}(\sqrt{GM}R^{\frac{1}{2}}) = \frac{1}{2}\sqrt{GM}R^{-\frac{1}{2}}$$

$$\frac{\partial S}{\partial t} = \frac{1}{R}\frac{\partial}{\partial R}\{\frac{\frac{\partial}{\partial R}[\nu S R^3 \frac{3}{2}\sqrt{GM}R^{\frac{5}{2}}]}{\frac{1}{2}\sqrt{GM}R^{-\frac{1}{2}}}\} = = \frac{1}{R}\frac{\partial}{\partial R}\{\frac{\frac{\partial}{\partial R}[\nu S \frac{3}{2}\sqrt{GM}R^{\frac{1}{2}}]}{\frac{1}{2}\sqrt{GM}R^{-\frac{1}{2}}}\} = = \frac{3}{R}\frac{\partial}{\partial R}[\sqrt{R}\frac{\partial}{\partial R}(\nu S\sqrt{R})]$$

Viscosity, general discussion

- We obtained the solution for S. In general, ν depends on local conditions in the disk, and $\nu=\nu(S,R,t)$ so that we obtained nonlinear diffusion eq. for S.
- If ν depends on R only, eq. is linear in S, even for the power of R (it was clear already in 1920-ies, Jeffreys 1924, Weizsacker in 1948).

- Most of the mass moves towards the center, losing energy and torque. A tail of matter moves towards larger R to conserve the angular momentum. Matter from the initial ring arrives to the center, and total angular momentum is transported to large radii by a very small mass, compared to the disk mass. The disk slowly spreads outwards.
- In 1973 Shakura & Sunyaev gave a solution, parameterizing viscosity as $\nu=\alpha c_s H$, where $\alpha<1$ is a coefficient describing "turbulent viscosity".

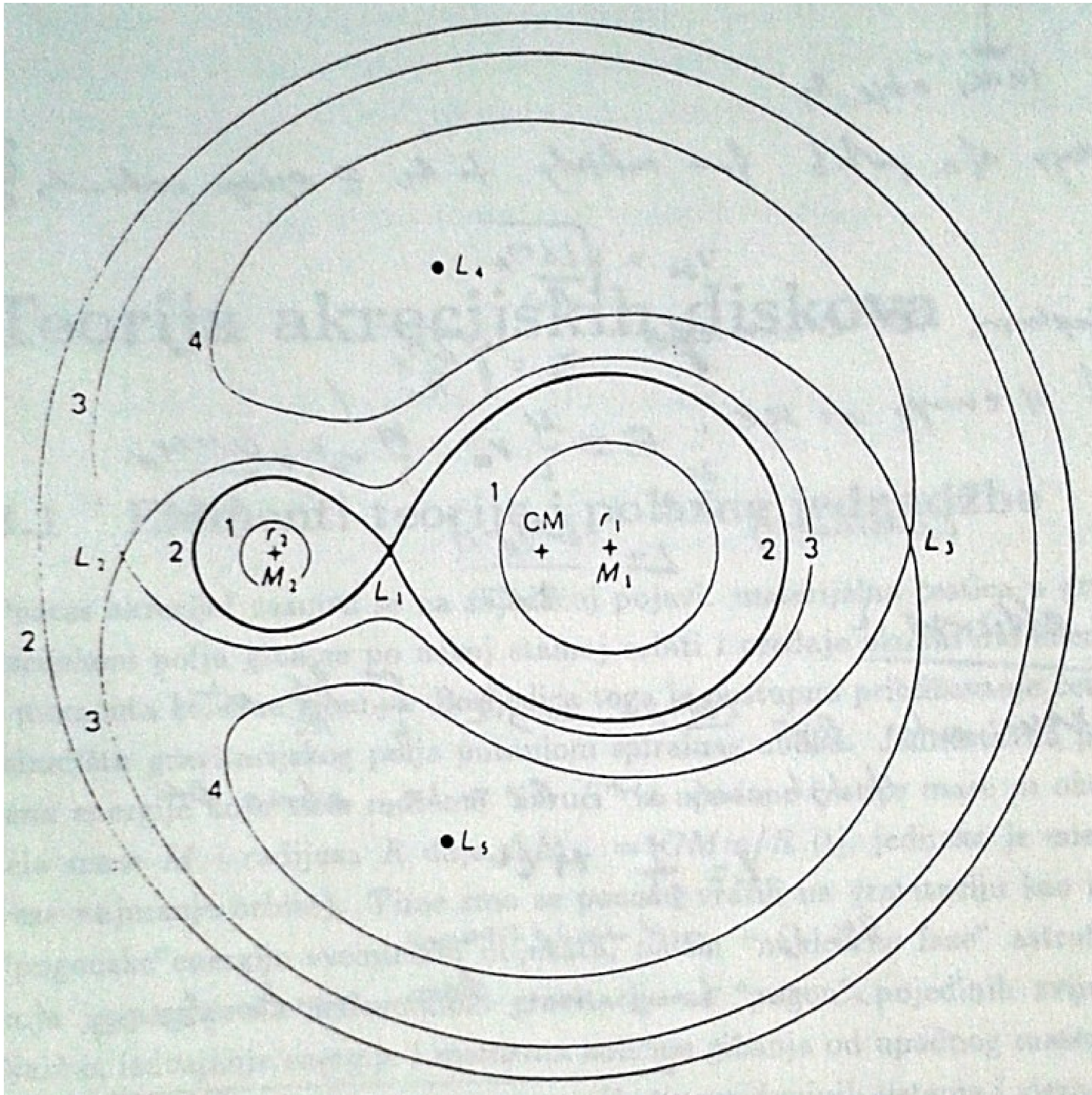
  Usually $\nu\sim LV$, where L is characteristic scale, and v characteristic velocity of the turbulent eddies-so we assumed $L\sim H$ of the disk, and $V\sim c_s$ (turbulence is usually assumed to be subsonic).

  In astrophysics we are usually dealing with large Reynolds numbers Re, defined through $Re=LV/\nu$, simply because of large L.

Viscosity, general discussion

- Re measures ratio of inertial to viscous forces, so in the disc we usually have proportionality with v_φ^2/R
- For Re<<1 viscous forces are dominating, and with Re>>1 they are unimportant. In accretion discs usually $Re>>10^{11}$ and we can not get much lower.
- Clue of the problem is exactly in so large Re: from experiments we know that fluids have some critical value Re_c, at which the velocity becomes chaotic, so we have turbulence. Typical $Re_c=10^3$, so we can conclude that disc material is turbulent.
- There were many works on turbulence, but we still do not have the full understanding of the mechanism in accretion disks. Currently accepted paradigm is the one by Balbus & Hawley (1992), where magneto-rotational (MRI) turbulence is invoked.
- Mathematically, viscous process is a diffusion process (of matter and angular momentum), this is the basics for our description.

Roche equipotential surfaces



- A small, but hopefully useful distraction: Roche surfaces usually emerge when we are dealing with close binaries.
- French matematician Edouard Albert Roche (1820-1883) discussed the problem of equipotential surfaces in the context of stability of the orbits of small planetary satellites in 1849.
- Physical model is of the small test mass m moving in the gravitational fields of two massive points. There is no gravitational back-reaction from m to M1 and M2.

Roche equipotential surfaces

- Because of tidal forces, with close binaries we usually can work in this approximation.
- Lagrange in 1772 showed the existence of 5 points (Lagrange points L_1-L_5), where forces from M1 and M2 are balanced.
- If we solve the equations in (x,y,z) put along the line connecting M1 and M2, perpendicular to it in the plane of the paper and vertical out of the paper, we obtain, with μ and 1-μ being the masses expressed in total mass parts μ=M1+M2 and distances r_1 and r_2 in the units of the masses M1 and M2 distance:

$$v^2 = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C$$

- Constant C is defined by the initial conditions.
- The velocity v=0 defines a surface at which velocities are >0 or <0, real or imaginary, and C defines from which side of this surface the mass m is moving.

Roche equipotential surfaces

- Jacobi found that v^2=V-C, where

$$V = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}.$$

- It shows that we can use the same approach for close binaries, even when there are flows between them, or they are filling their respective “Roche lobes”, so that the solution

  gives us the equipotential

  surfaces, U is the potential:

$$-U = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

- We have v^2=-2U-C

Python code, jupyter notebook

- We will solve the Bondi spherical accretion problem using Python.
- We will use the relaxation method, for coordinate network with logarithmic distribution.
- To ease the work, we use the **jupyter notebook**, but we also work with Python scripts alone.
- jupyter notebook is useful tool, because one can directly follow the pieces of python code with the explaining text.
- Few hints for solving the problems: Depending on your python installation,you might need to upgrade existing or install missing packages. Browsing for the answers using the error messages shown on the execution gives you hints on how to solve the problem.
- Few examples, to remind you of commands for this. I find it simplest to use pip (for python3, which we use here, use pip3-update it on your machine by typing: python3 -m pip install --upgrade pip).
- I also needed to install dvipng package for creating png files, in ubuntu: sudo apt-get install dvipng
- You might encounter problem with jax, install it, it needs jaxlib, install it too
- If something does not work with python3.5, try 3.8 or else, python is a delicate animal and sometimes you have to beg it to get what you want!

Python code for Bondi accretion in analytical solution

- First we exercise use of python on showing our analytical solution.
- Run the routine bondi-relaxation.ipynb which you got in slack in jupyter:
  jupyter-notebook bondi-relaxation.ipynb
  this will launch the notebook in your default browser.
- If there are any problems with jupyter notebook, extract the bare python code from jupyter notebook version and try to run it in the terminal. Some tweaks might be needed, try with other versions of python.
- After you solve the problems in bare script and install/activate all the needed routines, jupyter version should also work.

Python code for Bondi accretion solver

- After installing everything, we are ready to run the solver itself.
- Run the solver parts of the routine bondi-relaxation.ipynb.
- After you get it running in jupyter, extract the needed part into a separate file (there is such a function in jupyter), and run from the terminal as a python script.

PyAstronomy & Roche equipotential surfaces

- If you need some routine for astrophysical use, it is very probable someone else also needed it. There is a pile of astro-routines from last decades, but not in python, this is relatively new fashion, before for long time a standard package was IDL, a quite expensive proprietary software.

- There is **Open source initiative version of IDL, called GDL**, which has most-but not all-functionality of IDL. For simpler graphics or most of computations it usually works, give it a try-”G” in GDL is from Gnu…, so it is distributed with major linuxen, or can easily be installed.

- If you are lucky, your needed code might be already translated to python, and available in Github or similar. One such library is **PyAstronomy**-currently at 0.18 beta edition, but growing, so best is you browse for it and find the current wersion, it will for certain grow steadily.

- I show an example of a routine from PyAstronomy, computation od Lagrange points and Roche equipotential surfaces. Note that, depending on software used for viewing the results, you can read the positions in wished points by just moving the cursor on your screen. The data about mass ratio etc is printed out in the terminal.

Summary of the Part I

- General introduction to accretion
- Energetics of accretion
- Eddington limit
- Spherical (Bondi) accretion

- Accretion disk
- Viscosity

- Python code for Bondi accretion
- PyAstronomy package

Outline, Part II: Steady disk solutions

- Perturbative solutions for the disk

- Shakura-Sunyaev disk

- Python code for Shakura-Sunyaev disk

General equations

- We will go through (sometimes painful) detail into the accretion disk equations.
- The obtained solution is still a starting point for explanation of the birth of stars and larger structures.
- Matter which we consider, when undergoing accretion, is gaseous, which means that interaction is by the collisions, not short distance forces. We use, as we did before, λ for the mean free path of the particles, ῡ for the mean velocity (velocities are measured in the comoving coordinates, and distributed following a Maxwell-Boltzmann distribution, which is dependent on the temperature, T), ρ for the mass density of gas. When observing the gas at scales L>>λ, we can consider it as a continuous fluid, with density, velocity and temperature defined in every point of the flow. The equations to describe such fluid are the equations of conservation of mass, momentum and energy.
- Conservation of mass: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$

Conservation of momentum follows from the force acting on a fluid element:

$$-\oint_S P d\vec{n} = (\text{Gauss} - \text{Ostrogradski}) = -\int_V \nabla P dV$$

P is pressure, and the direction of ort vector **n** is outwards from the volume.

General equations

- Force acting on the unit volume element of the gas is (-∇P), and its equation of motion we obtain from the 2nd Newton, multiplying it with the unit volume mass=density ρ and acceleration, so we can write:

$$\rho \frac{d\vec{v}}{dt} = -\nabla P$$

- Acceleration is also with respect to the comoving coordinates, not in the background rest system, so we have two parts in the velocity change in this equation: one is the change of velocity in the given point of space at a time interval dt: ∂**v**/∂tdt and another is the difference in velocities at two points of space, distanced **r**, through which the fluid flows during dt, what we can write as d**r**∇**v**, so we can write all together:

$$d\vec{v} = \frac{\partial \vec{v}}{\partial t} dt + d\vec{r} \nabla \cdot \vec{v} / \frac{1}{dt}$$

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \nabla \cdot \vec{v}$$

- When we insert it to the above equation of motion, we obtain

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \nabla \cdot \vec{v} = -\nabla P$$

- General equation of motion should add the source term for the external forces acting on the system, we obtain the Euler equation:

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \nabla \cdot \vec{v} = -\nabla P + \vec{f} \quad \text{(E)}$$

- If we insert **f**=ρ**g** for a gas in gravity field (**g** is the gravity acceleration), **f** could contain contributions from viscosity, external magnetic field etc.

General equations

- Momentum of the fluid element is ρ**v**, conservation of the momentum is:

$$\frac{\partial}{\partial t}\rho\vec{v} = 0 = \frac{\partial\rho}{\partial t}\vec{v} + \rho\frac{\partial\vec{v}}{\partial t}$$

- For a stationary flow ∂ρ/∂t=0, so we also the last derivative is zero, and we have

$$\rho\vec{v}\nabla\cdot\vec{v} + \nabla P - \rho\vec{g} = 0 \qquad \text{(A)}$$

- From $m\vec{g} = -GmM\vec{r_0}/r^2$ is $\vec{g} = -GM\vec{r_0}/r^2$.

- For the accretion onto spherical object of mass M, we choose spherical coordinates (r,ϑ,φ), radial component of the equation (A) is (G=grav. const)

$$\rho v_r \frac{1}{r^2}[\frac{\partial}{\partial r}(r^2 v_r)] + \frac{\partial P}{\partial r} + \rho\frac{GM}{r^2} = 0 \;/\frac{1}{\rho}$$

- where []=$2rv_r + r^2\partial v_r/\partial r$ so we have

$$\frac{2v_r^2}{r} + v_r\frac{\partial v_r}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{GM}{r^2} = 0 \qquad \text{(B)}$$

General equations

- From the continuity equation we have, for the stationary case with $\partial P/\partial t=0$ that $\partial\rho/\partial t+\nabla(\rho v_r)=0$. For any vector A radial part is $\nabla\cdot\vec{A}=\frac{1}{r^2}[\frac{\mathrm{d}}{\mathrm{d}r}(r^2A_r)]$, so we have $\rho\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2v_r)=0\ /\int$ which means that r^2v_r=const.

Since $(-\rho v_r)$ is inflow mass flux, this const must be related to mass flux,

- i.e. the accretion rate $\dot{M}=4\pi r\rho(-v_r)$, since $r^2\cdot$(inflow flux)=const=$\dot{M}/4\pi$,

for the whole sphere is $4\pi r^2\dot{M}\cdot$(inflow flux).

Now we insert $r^2v_r= -\dot{M}/4\pi\rho$ into eq.(B) from the previous slide to obtain $v_r=-\dot{M}/4\pi\rho r^2$, which in the limit $r\rightarrow 0$ gives $v_r=0$ and for the stationary spherical accretion we stay with

$$v_r\frac{dv_r}{dr}+\frac{1}{\rho}\frac{dP}{dr}+\frac{GM}{r^2}=0$$

General equations

- Energy conservation:

The gas element energy is a sum of kinetic term $1/2\rho v^2$ (by unit volume) and internal (thermal) energy $\epsilon\rho$ (ϵ is specific energy-by mass unit, dependent on temperature T). From the equipartition of energy we know that each degree of fredom has average energy of 1/2kT, so for mono-atomic gas we have only 3 translational directions and we can write $\epsilon = 3/2kT$.

- Energy conservation equation we write similar to mass conservation, plus adding source terms, depending on physics we include in our model, now instead of ρ we conserve the kinetic and internal energy, and in the
- spatial derivative we will have work done by the pressure, P**v**:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\epsilon\right) + \nabla \cdot \left[\left(\frac{1}{2}\rho v^2 + \rho\epsilon + P\right)\vec{v}\right] - \vec{f}\vec{v} = 0$$

and for a stationary case:

$$\nabla \cdot \left[\left(\frac{1}{2}\rho v^2 + \rho\epsilon + P\right)\vec{v}\right] = \vec{f}\vec{v}.$$

On the r.h.s. we can add the losses (so, - sign!) by radiation, heat etc. as the source terms inside $-\nabla(\ )$.

Perturbative solutions for the disk

- Now we move to the perturbation method-we compute the perturbation in relation to the hydrostatic balance. We obtained

$$\nabla \cdot (\rho \vec{v}) = 0$$

$$\rho(\vec{v} \cdot \nabla)\vec{v} = -\nabla P + \vec{f}$$

$$\nabla \cdot [(\frac{1}{2}\rho v^2 + \rho\epsilon + P)\vec{v}] = \vec{f}\vec{v}.$$

and with **v**=0 in the hydrostatic case we stay only with

$$\nabla P = \mathbf{f}$$

- For the ideal gas, which we can assume everywhere except degenerate gas in some dense objects or near the centres of the normal stars, we have

$$P = \frac{\rho k T}{\mu m_H}$$

- with $m_H \sim m_{proton}$ is the hydrogen atom mass, and μ is the average molecular mass in units of m_H, so that for completely ionized hydrogen it is $\mu=0.5$ and for neutral hydrogen $\mu=1$.

Perturbative solutions for the disk

- Now we assume a **small** shift in the density and pressure (ρ', P') from the initial balance values (ρ_0, P_0): $\rho=\rho_0+\rho'$, $P=P_0+P'$, $\mathbf{v}=\mathbf{v}'$
- Depending on the processes, perturbations can be isothermal or adiabatic.
- For adiabatic changes with $\gamma=5/3$ and isothermal with $\gamma=1$ we can write P/ρ^{γ} = const=k, so we can write $P_0+P' = k(\rho_0+\rho')^{\gamma}$ [**try γ=1.05 in simuls!**]
- Linearizing the mass continuity eq.:
- with $\nabla(\rho')v \rightarrow 0$ in the first approx.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = \frac{\partial}{\partial t}(\rho_0 + \rho') + \nabla \cdot [(\rho_0 + \rho') \cdot \vec{v}'] =$$
$$= \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{v}' = 0$$

- We do the same with Euler eq. to obtain:

$$(\rho_0 + \rho')\frac{\partial \vec{v}'}{\partial t} + (\rho_0 + \rho')\vec{v}' \cdot \nabla \cdot \vec{v}' = -\nabla(P_0 + P') + \vec{f}$$
$$\rho_0 \frac{\partial \vec{v}'}{\partial t} + \rho_0 \vec{v}' \cdot \nabla \cdot \vec{v}' = -\nabla P_0 - \nabla P' + \vec{f}$$

- Since $\nabla P_0=\mathbf{f}$, and products of second and higher orders are neglected, we obtain $\rho_0 \partial \mathbf{v}'/dt = -\nabla P'$.
- We obtained two eqs.:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{v}' = 0 \qquad \text{(C)}$$
$$\frac{\partial \vec{v}'}{\partial t} + \frac{1}{\rho_0}\nabla P' = 0$$

Perturbative solutions for the disk

- From $P_0+P' = k(\rho_0+\rho')^\gamma$ we see that P is a function of ρ only, so that we can write $\nabla P' = (\partial P/\partial\rho)_0 \nabla\rho'$, to the first order, where with a subscript $_0$ we assigned that we evaluate the derivation for the equilibrium state.
- The second of the eqs. (C) we can write now as:

$$\frac{\partial \vec{v}'}{\partial t} + \frac{1}{\rho_0}(dP/d\rho)_0 \nabla \rho' = 0$$

- We act on it with ∇:

$$\nabla \cdot \frac{\partial \vec{v}'}{\partial t} + [\frac{1}{\rho_0}(\partial P/\partial\rho)_0 \nabla^2 \rho'] = 0$$

- We act on the 1st eq.in (C) with ∂/∂t:

$$\frac{\partial^2 \rho'}{\partial t^2} + (\rho_0 \nabla \frac{\partial \vec{v}'}{\partial t}) = 0$$

- We subtract the two eqs.to obtain:

$$\frac{\partial^2 \rho'}{\partial t^2} = (\partial P/\partial\rho)_0 \nabla^2 \rho'$$

the wave equation! With ↑ as the sound speed, c_s^2, we can write

$$\frac{\partial^2 \rho'}{\partial t^2} = c_s^2 \nabla^2 \rho'$$

Perturbative solutions for the disk

- For P' and **v**' we obtain the equivalent equations, so we conclude that small perturbations around the *hydrostatic equilibrium positions spread with the speed of sound*.
- Depending on the kind of perturbation, we have two possibilities,

adiabatic

$$c_s^{ad} = \sqrt{\frac{5P}{3\rho}} = \sqrt{\frac{5kT}{3\mu m_H}} \propto \rho^{\frac{1}{3}}$$

or isothermal:

$$c_s^{izot} = \sqrt{\frac{P}{\rho}} = \sqrt{\frac{kT}{\mu m_H}}$$

Stationary thin disk

- Back in "Viscosity, and back to the disk eqs", slide 25, we obtained the equation: (**G is gravity const.** here!)

$$\frac{\partial S}{\partial t} = \frac{1}{R}\frac{\partial}{\partial R}\left\{\frac{\frac{\partial}{\partial R}\left[\nu S R^3 \frac{3}{2}\sqrt{GM} R^{\frac{5}{2}}\right]}{\frac{1}{2}\sqrt{GM} R^{-\frac{1}{2}}}\right\} =$$

$$= \frac{1}{R}\frac{\partial}{\partial R}\left\{\frac{\frac{\partial}{\partial R}\left[\nu S \frac{3}{2}\sqrt{GM} R^{\frac{1}{2}}\right]}{\frac{1}{2}\sqrt{GM} R^{-\frac{1}{2}}}\right\} =$$

$$= \frac{3}{R}\frac{\partial}{\partial R}\left[\sqrt{R}\frac{\partial}{\partial R}(\nu S \sqrt{R})\right]$$

- To continue, we needed the viscosity. That the disk would be "stationary" and that viscosity would work, we need that the mass accretion rate $\dot{M}$ would be slow enough. Then we can set $\partial/\partial t=0$ and from the mass conservation we can write $\dot{M}=2\pi RS(-v_r)$ and from the angular momentum conservation (middle of slide 23) we have $RSV_rR^2\Omega=(G(R,t)+C)/(2\pi)$, with C=const. related to the angular momentum rate of the accreted matter. Star must rotate slower than the breakup rotation at the equator, so when approaching closer to the star, there is a region in the disk where the disk corotates with the star. Even closer to the star, our approximation breaks-here starts the discussion and departure from the simple estimates.

Stationary thin disk

- In slide 23 we had G=G(R,t), and with $R\partial\Omega/\partial R=A$ it was $G(R)=2\pi R\nu SAR$, where νSA is a viscous force per unit angle. After integration:

$$-\nu S\frac{\partial\Omega}{\partial R} = S(-v_r\Omega) + \frac{C}{2\pi R^3} \quad \text{(D)}$$

- Inside a ring at R_A+b, the rotation of the disk approaches Keplerian, reaches **$\partial\Omega/\partial R=0$**, and increases until it reaches $R\sim R_A$. We can write $\Omega(R_A+b) = \sqrt{\frac{GM}{R_A^3}}[1+\mathcal{O}(\frac{b}{R_A})]$ (**G is gravity constant now**!).

- Closer than R_A The thin disk approx. is not valid animore. To find C we insert $R=R_A+b$ and evaluate $C=2\pi R_A^3 S v_r \Omega(R_A+b)|_{R\_A+b}$ (now Ω() , not multiplying!), which gives, after inserting mass accretion rate $\dot{M}$ and $\Omega(R_A+b)$, $C=-\dot{M}(GMR_A)^{1/2}$ that $\nu S = \frac{\dot{M}}{3\pi}[1-\sqrt{\frac{R_A}{R}}]$, exact to order $O(b/R_A)$. We insert it to eq. D to obtain

- Loss of energy because of viscosity is $D(R) = \frac{g}{4\pi}\frac{\partial\Omega}{\partial R} = \frac{1}{2}(\nu S)(R\frac{\partial\Omega}{\partial R})^2$ which is $D(R)=g/(4\pi)\partial\Omega/\partial R$ per unit disk surface. Inserted back to (D) it it gives that D(R) is independent of viscosity: $D(R) = \frac{3GM\dot{M}}{8\pi R^3}[1-\sqrt{\frac{R_A}{R}}]$

Stationary thin disk

- Now we can estimate the luminosity of a disk between R_1 and R_2 (2 is for 2 disk sides: $L(R_1,R_2) = 2\int_{R_1}^{R_2} D(R)2\pi R\,\mathrm{d}R$, $L(R_1,R_2) = \frac{3GM\dot{M}}{2}\int_{R_1}^{R_2}\left[1-\sqrt{\frac{R_A}{R}}\right]\frac{\mathrm{d}R}{R^2}$

- If we substitute $x=R/R_A$, we obtain $L(R_1,R_2) = \frac{3GM\dot{M}}{2}\left\{\frac{1}{R_1}[1-\frac{2}{3}\sqrt{\frac{R_A}{R_1}}]-\frac{1}{R_2}\left[1-\frac{2}{3}\sqrt{\frac{R_A}{R_2}}\right]\right\}$
- For $R=R_1$ and $R_2 \to \infty$ we obtain the complete disk luminosity (**G is gravity const**):

$$L_{\text{disk}} = \frac{GM\dot{M}}{2R_A} = \frac{1}{2}L_{\text{akrecije}}$$

- where we defined $L_{\text{akrecije}} = \frac{\Delta E_{\text{akr}}}{\Delta t} = \frac{GM\dot{M}}{R_A}.$

- This means that half of the energy is radiated from the disk, and half is released very close to the central star, which takes the same amount like the whole disk! (which has a much, much larger surface).

Stationary thin disk

- This was for radial direction, is it all consistent with the vertical direction? In the vertical, z direction, there is mainly no flow, we have the hydrostatic equilibrium: $\frac{1}{\rho}\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left[\frac{GM}{\sqrt{R^2+z^2}}\right]$ which we get from the vertical component of

  Euler eq. (eq.E on slide 38) neglecting all the terms with velocities.
  For the thin disk z<<R we have $\frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{GMz}{R^3}$

- Since H||z we can write ∂P/∂z~P/H and z~H, and condition for a thin disk becomes H<<R. For P∝ρc_s^2 we have $H \simeq c_s\sqrt{\frac{R}{GM}}R^4$ which means that it has to be c_s<<(GM/R)$^{1/2}$; additional condition for a thin disk: local Keplerian speed must be highly supersonic. Only with this satisfied, the approximation of thin disk can be used. - This is a strong condition for the inner workings of a disk, and tells us that the local orbiting speed will be close to the Keplerian speed.

Stationary thin disk

- The radial component of the Euler eq.is: $v_r \frac{\partial v_r}{\partial R} - \frac{v_\varphi^2}{R} + \frac{1}{\rho}\frac{\partial P}{\partial R} + \frac{GM}{R^2} = 0$
  If we neglect the pressure term, because
- of $c_s << (GM/R)^{1/2}$, we have $\rho^{-1}\partial P/\partial R \sim c_s^2/R$ in comparison to a larger gravitational term GM/R^2, with $\dot{M} = 4\pi r\rho(-v_r)$ which we know from before, and $\nu S = \frac{\dot{M}}{3\pi}[1 - \sqrt{\frac{R_A}{R}}]$ from slide 47, we have $v_r = -\frac{3\nu}{2R}(1 - \sqrt{\frac{R_A}{R}})^{-1}$
- Now we are slowly shifting to the Shakura & Sunyaev (1973) main assumption: for any reasonable viscosity, the radial velocity v_r is highly **sub**sonic, while orbital velocity is highly supersonic and approximately Keplerian: with $\nu \propto c_s H$ we have $v_r \propto \nu/R \sim c_s H/R << c_s$
- Now we have all the equations for the disk structure.

Lynden-Bell on steady model disk

Galactic Nuclei as Collapsed Old Quasars

by

D. LYNDEN-BELL

Royal Greenwich Observatory,
Herstmonceux Castle, Sussex

Powerful emissions from the centres of nearby galaxies may represent dead quasars.

RYLE gives good evidence[1] that quasars evolve into powerful radio sources with two well separated radio components, one on each side of the dead or dying quasar. The energies involved in the total radio outbursts are calculated to be of the order of 10^{61} erg, and the optical variability of some quasars indicates that the outbursts probably originate in a volume no larger than the solar system. Now 10^{61} erg have a mass of 10^{40} g or nearly 10^7 Suns. If this were to come from the conversion of hydrogen into helium, it can only represent the nuclear binding energy, which is 3/400 of the mass of hydrogen involved. Hence 10^9 solar masses would be needed within a volume the size of the solar system, which we take to be 10^{15} cm (10 light h). But the gravitational binding energy of 10^9 solar masses within 10^{15} cm is GM^2/r which is 10^{62} erg. Thus we are wrong to neglect gravity as an equal if not a dominant source of energy. This was suggested by Fowler and Hoyle[2], who at once asked whether the red-shifts can also have a gravitational origin. Greenstein and Schmidt[3], however, earlier showed that this is unlikely because the differential red-shift would wash out the lines. Attempts to avoid this difficulty have looked unconvincing, so I shall adopt the cosmological origin for quasar red-shifts. Even with this hypothesis the numbers of quasar-like objects are very large, or rather they were so in the past. I shall assume that the quasars were common for an initial epoch lasting 10^9 yr, but that each one only remained bright for 10^6 yr, and take Sandage's estimate (quoted in

which we shall call the Schwarzschild throat. We would be wrong to conclude that such massive objects in space-time should be unobservable, however. It is my thesis that we have been observing them indirectly for many years.

Effects of Collapsed Masses

As Schwarzschild throats are considerable centres of gravitation, we expect to find matter concentrated toward them. We therefore expect that the throats are to be found at the centres of massive aggregates of stars, and the centres of the nuclei of galaxies are the obvious choice. My first prediction is that when the light from the nucleus of a galaxy is predominantly starlight, the mass-to-light ratio of the nucleus should be anomalously large.

We may expect the collapsed bodies to have a broad spectrum of masses. True dead quasars may have 10^{10} or 10^{11} $M_\odot$ while normal galaxies like ours may have only 10^7–10^8 $M_\odot$ down their throats. A simple calculation shows that the last stable circular orbit has a diameter of $12\ GM/c^2 = 12m$ so we shall call the sphere of this diameter the Schwarzschild mouth. Simple calculations on circular orbits yield the following results, where M_7 is the mass of the collapsed body in units of 10^7 $M_\odot$, so that M_7 ranges from 1 to 10^4.

Circular velocity

$$V_c = [GM/(r-2m)]^{1/2} \text{ where } r > 3m \qquad (1)$$

Usually cited before Shakura & Sunyaev disk is Lynden-Bell (1969) discussion of the origin of emission from galactic nuclei-”old quasars”, Schwarzschild mouth was still the term for the event horizon.

Shakura & Sunyaev viscous alpha disk

- Now we are in a better position to discuss the SS73 paper, which is one of the most cited papers on accretion disks (10657 at noon Sunday 27th 02 2022)
- It got a reprint in A&A in 2009, and a review by Andrew King, which best describes its importance. We will first follow this short review.

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Accretion: the gold mine opens

A. King

Theoretical Astrophysics Group, University of Leicester, Leicester LE1 7RH, UK
e-mail: ark@astro.le.ac.uk

In the early 1970s, Yakov Zeldovich suggested that two young Moscow astronomers, Nikolai Shakura and Rashid Sunyaev, should combine their efforts to understand how accretion could power bright X-ray sources.

The result was a paper now amongst the most cited in all astronomy. Another highly influential paper on accretion disks (Pringle 1981) calls it "seminal", and so it is. Yet it is not by any means the first paper on disks. The references in the paper make it clear that the authors knew of a large literature on the subject going back to the early 1960s. Indeed Pringle (1981) shows that knowledge of the principles and even the equations of disk theory dates back still earlier (to the 1920s and 1940s, respectively).

The reason for the paper's vast and deserved success lies partly in its approach to the problem it attacked, and partly in a technical innovation. Shakura and Sunyaev's title: "Black Holes in Binary Systems. Observational Appearance" makes their aim

momentum and so drives accretion, thus tapping the gravitational energy of infall. There is a direct connection between viscosity and accretion rate, and in a steady disk one can infer the surface dissipation rate purely in terms of the latter. Observations made it clear that none of the obvious candidate mechanisms – certainly not the standard "molecular" viscosity for example – was adequate to drive accretion at the rates required.

Authors therefore resorted to various parametrizations of the unknown process, usually thought of as some kind of turbulent effect. Shakura and Sunyaev adopted the now famous alpha-prescription, in modern notation usually written

$$\nu = \alpha c_s H.$$

Here ν is the kinematic viscosity, c_s the sound speed, and H the local disk scaleheight, i.e. the disk semithickness.

Shakura & Sunyaev viscous alpha disk

- With the thin disk approximation, we can compute the structure of the disk. In practice, we are solving the 1D with only a radial dependence, as we decoupled it from the vertical, z-dependence, which is essentially written as a hydrostatic equilibrium and energy transport.
- In the radial direction, the disk structure enters only in the local energy dissipation rate D(R).
- From the hydrostatic eq. $\frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{GMz}{R^3}$ for isothermal structure we obtain the solution $\rho(R,z) = \rho_c(R)\, e^{-\frac{z^2}{2H^2}}$ where ρ_c stands for the density at z=0.

- The central density of the disk we can approximate as $\rho = S/H$, $H = \rho c_s / v_\varphi$. $c_s^2 = P/\rho$, where P is a sum of gas and radiation pressure $P = \frac{\rho k T_c}{\mu m_p} + \frac{4\sigma}{3c} T_c^4$ with an assumption $T(T,z) \sim T_c(R,0)$. The central temperature T_c is determined by the relation between the vertical energy flux and the energy dissipation because of viscosity.
- Locally, using the thin disk approximation, we now have the vertical temperature gradient, so that for z=const surface we have radiated energy flux ($\varkappa_R$ is the Rosseland mean opacity): $F(z) = -\frac{16\sigma T^3}{3\kappa_R \rho}\frac{\partial T}{\partial z}$

Shakura & Sunyaev viscous alpha disk

- We assumed the optically thick disk: $\tau=\rho H\varkappa_R=S\varkappa_R>>1$, so that the radiation is locally very close to the black body radiation. In the case with $\tau\leq 1$ radiation could directly exit the disk, and the equation for F(z) from the bottom of previous slide would not be valid any more.
- For the energetic balance must be F(H)-F(0)=D(R), so that $F(z) \sim \frac{4\sigma}{3\tau}T^4(z)$ which, with $T_c^4>>T^4(H)$ gives $\frac{4\sigma}{3\tau}T_c^4 = D(R)$.
- For the full set of eqs.we need the $\varkappa_R=\varkappa_R(\rho,T_c)$ relation, and expression for ν and its relation to S and $\dot{M}$. This all amounts to 8 equations for **ρ, S, H, T_c, c_s, P, τ, ν** in dependence of R, M and $\dot{M}$,with some parameter in the viscosity, which are describing the thin disk model:

$$(1)\quad \rho = \frac{S}{H}$$

$$(2)\quad H = \frac{c_s R^{+\frac{3}{2}}}{\sqrt{GM}}$$

$$(3)\quad c_s^2 = \frac{P}{\rho}$$

$$(4)\quad P = \frac{\rho k T_c}{\mu m_p} + \frac{4\sigma}{3c}T_c^4$$

$$(5)\quad \frac{4\sigma T_c^4}{3\tau} = \frac{3GM\dot{M}}{8\pi R^3}[1-\sqrt{\frac{R_A}{R}}]$$

$$(6)\quad \tau = S\kappa_R(\rho,T_c) = \tau(S,\rho,T_c)$$

$$(7)\quad \nu S = \frac{\dot{M}}{3\pi}[1-\sqrt{\frac{R_A}{R}}]$$

$$(8)\quad \nu = \nu(\rho,T_c,S,\alpha,\ldots)$$

Shakura & Sunyaev viscous alpha disk

- With "alpha viscosity" parameterization $\nu=\alpha c_s H$ Shakura & Sunyaev ('73) gave the first solution. They used the Kramers' law (**$6.6\cdot 10^{22}$ wrongly sometimes!!**) $\varkappa_R=5\cdot 10^{24}\rho T_c^{-7/2}$ cm^2/g and neglected the radiation pressure in eq.(4). Now the system of 8 eqs.can be solved. I give steps (from "Accretion power…"):
- First we simplify $f^4=1-(R_A/R)^{1/2}$ and write whole r.h.s. of eq.5 as equal to D. Now with eq.6, $\varkappa_R$ and eq.2 we can write eq.5 as

$4\sigma T^4/(3\tau)=D=4\sigma T^{15/2}/(3*5\cdot 10^{24}\rho S)=[\rho=S/H]=4\sigma H T^{15/2}/(15\cdot 10^{24}S^2)=$
$=\{H=c_s R^{3/2}/(GM)^{1/2}$ and from eqs. 3 and 4 (without rad pressure term) write
$H=R^{3/2}T^{1/2}[k_B/(GM\mu m_p)]^{1/2}\}=4\sigma R^{3/2}T^8[k_B/(GM\mu m_p)]^{1/2}/(15\cdot 10^{24}S^2)$, and from that obtain $T^8=15\cdot 10^{24}S^2 D[k_B/(GM\mu m_p)]^{1/2}/(4\sigma R^{3/2})$. We insert D back as the r.h.s of eq.5 and use eqs.7 and 8 (where finally $\nu=\alpha c_s H$ comes into game), to write the solution-I give the detailed derivation of solutions on the next slide, this is usually not shown in literature; then we can write v_r from the equation in the slide 50.

Shakura & Sunyaev viscous alpha disk

5) with 6, $\varkappa_R = 5\cdot10^{24}\rho T^{-7/2}$ and 2) gives

$$\frac{4\sigma T^4}{3\Sigma\, 5\cdot10^{24}\rho T^{-7/2}} = \frac{3GM\dot M}{8\pi R^3}\left(1-\sqrt{\frac{R_A}{R}}\right)$$

$$\rho = \frac{\Sigma}{H} = \frac{\Sigma}{c_s}\sqrt{\frac{GM}{R^3}} = \Sigma\sqrt{\frac{GM}{R^3}\frac{\mu m_H}{kT}}$$

$$c_s = \sqrt{\frac{P}{\rho}} = \sqrt{\frac{kT}{\mu m_H}}$$

$$\frac{4\sigma T^4}{3\cdot\Sigma\cdot5\cdot10^{24}\,\Sigma\sqrt{\frac{GM}{R^3}\frac{\mu m_H}{kT}}\,T^{-7/2}} = \frac{3GM\dot M}{8\pi R^3}\left(1-\sqrt{\frac{R_A}{R}}\right)$$

$$T^8 = \frac{45\cdot10^{24}}{32\pi\sigma}\Sigma^2\left(\frac{GM}{R^3}\right)^{3/2}\dot M\sqrt{\frac{\mu m_H}{k}}\left(1-\sqrt{\frac{R_A}{R}}\right)$$

8) in 7) $\alpha c_s H\Sigma = \frac{\dot M}{3\pi}\left(1-\sqrt{\frac{R_A}{R}}\right)$, since $\frac{\alpha c_s^2}{\sqrt{\frac{GM}{R^3}}}\Sigma = \frac{\dot M}{3\pi}\left(1-\sqrt{\frac{R_A}{R}}\right)$

and (3,4) $c_s^2 = \frac{P}{\rho} = \frac{kT}{\mu m_H}$, gives $\alpha\frac{kT}{\mu m_H}\frac{\Sigma}{\sqrt{\frac{GM}{R^3}}} = \frac{\dot M}{3\pi}\left(1-\sqrt{\frac{R_A}{R}}\right)$

Now we put it into T8: $T = \frac{\mu m_H}{k\Sigma\alpha}\frac{\dot M}{3\pi}\left(1-\sqrt{\frac{R_A}{R}}\right)$

$$\left[\frac{\mu m_H}{k\Sigma\alpha}\frac{\dot M}{3\pi}\left(1-\sqrt{\frac{R_A}{R}}\right)\right]^8 = \frac{45\cdot10^{24}}{32\pi\sigma}\Sigma^2\left(\frac{GM}{R^3}\right)^{3/2}\dot M\sqrt{\frac{\mu m_H}{k}}\left(1-\sqrt{\frac{R_A}{R}}\right)$$

$$\Sigma^{10} = \left(\frac{\mu m_H}{k}\right)^{15/2}\dot M^7\frac{32\pi\sigma}{45\cdot10^{24}}\frac{1}{(3\pi\alpha)^8}\left(1-\sqrt{\frac{R_A}{R}}\right)^7\left(\frac{R^3}{GM}\right)^{3/2}$$

$$\Sigma = \left(\frac{\mu m_H}{k}\right)^{3/4}\left(\frac{32\pi\sigma}{45\cdot10^{24}}\right)^{1/10}\frac{\alpha^{-4/5}\dot M^{7/10}}{(3\pi)^{4/5}}\left(\frac{GM}{R^3}\right)^{-3/20}\left(1-\sqrt{\frac{R_A}{R}}\right)^{7/10}$$

$$H = c_s\sqrt{\frac{R^3}{GM}} = \sqrt{\frac{P}{\rho}\frac{R^3}{GM}} = \sqrt{\frac{kT}{\mu m_H}\frac{R^3}{GM}} \quad /^{16}$$

$$H^{16} = \left(\frac{k}{\mu m_H}\frac{R^3}{GM}\right)^8 T^8 = \left(\frac{k}{\mu m_H}\frac{R^3}{GM}\right)^8\frac{45\cdot10^{24}}{32\pi\sigma}\Sigma^2\left(\frac{GM}{R^3}\right)^{3/2}\dot M\sqrt{\frac{\mu m_H}{k}}\left(1-\sqrt{\frac{R_A}{R}}\right) \quad /\sqrt{\ }$$

$$H^8 = \left(\frac{k}{\mu m_H}\frac{R^3}{GM}\right)^4\left(\frac{45\cdot10^{24}}{32\pi\sigma}\right)^{1/2}\Sigma\left(\frac{GM}{R^3}\right)^{3/4}\dot M^{1/2}\left(\frac{\mu m_H}{k}\right)^{1/4}\left(1-\sqrt{\frac{R_A}{R}}\right)^{1/2}$$

$$= \left(\frac{k}{\mu m_H}\right)^{15/4}\left(\frac{R^3}{GM}\right)^{13/4}\dot M^{1/2}\left(\frac{45\cdot10^{24}}{32\pi\sigma}\right)^{1/2}\left(1-\sqrt{\frac{R_A}{R}}\right)^{1/2}\Sigma =$$

$$= \left(\frac{k}{\mu m_H}\right)^{15/4}\left(\frac{R^3}{GM}\right)^{13/4}\dot M^{1/2}\left(\frac{45\cdot10^{24}}{32\pi\sigma}\right)^{1/2}\left(1-\sqrt{\frac{R_A}{R}}\right)^{1/2}\cdot$$

$$\cdot\left(\frac{\mu m_H}{k}\right)^{3/4}\left(\frac{32\pi\sigma}{45\cdot10^{24}}\right)^{1/10}\frac{\alpha^{-4/5}}{(3\pi)^{4/5}}\left(\frac{GM}{R^3}\right)^{-3/20}\left(1-\sqrt{\frac{R_A}{R}}\right)^{7/10}\dot M^{7/10}$$

$$= \left(\frac{k}{\mu m_H}\right)^3\left(\frac{R^3}{GM}\right)^{31/10}\dot M^{6/5}\left(\frac{32\pi\sigma}{45\cdot10^{24}}\right)^{[illegible]/10}\left(1-\sqrt{\frac{R_A}{R}}\right)^{6/5}\frac{\alpha^{-4/5}}{(3\pi)^{4/5}} \quad /\sqrt[8]{\ }$$

$$H = \left(\frac{k}{\mu m_H}\right)^{3/8}\left(\frac{R^3}{GM}\right)^{31/80}\dot M^{3/20}\left(\frac{32\pi\sigma}{45\cdot10^{24}}\right)^{1/20}\left(1-\sqrt{\frac{R_A}{R}}\right)^{3/20}\frac{\alpha^{-1/10}}{(3\pi)^{1/10}}$$

$$\rho = \frac{\Sigma}{H} = \frac{\left(\frac{\mu m_H}{k}\right)^{3/4}\left(\frac{32\pi\sigma}{45\cdot10^{24}}\right)^{1/10}\dot M^{7/10}\frac{\alpha^{-4/5}}{(3\pi)^{4/5}}\left(\frac{GM}{R^3}\right)^{-3/20}\left(1-\sqrt{\frac{R_A}{R}}\right)^{7/10}}{\left(\frac{k}{\mu m_H}\right)^{3/8}\left(\frac{32\pi\sigma}{45\cdot10^{24}}\right)^{1/20}\dot M^{3/20}\frac{\alpha^{-1/10}}{(3\pi)^{1/10}}\left([illegible]\right)^{-31/80}\left(1-\sqrt{\frac{R_A}{R}}\right)^{3/20}} =$$

$$= \left(\frac{\mu m_H}{k}\right)^{3/8}\left(\frac{32\pi\sigma}{45\cdot10^{24}}\right)^{1/20}\dot M^{11/20}\frac{\alpha^{-7/10}}{(3\pi)^{7/10}}\left(\frac{GM}{R^3}\right)^{-53/80}\left(1-\sqrt{\frac{R_A}{R}}\right)^{11/20}$$

$$T = \left(\frac{45\cdot10^{24}}{32\pi\sigma}\right)^{1/4}\Sigma^{1/4}\left(\frac{GM}{R^3}\right)^{3/16}\dot M^{1/8}\left(\frac{\mu m_H}{k}\right)^{1/16}\left(1-\sqrt{\frac{R_A}{R}}\right)^{1/8} =$$

$$= \left(\frac{45\cdot10^{24}}{32\pi\sigma}\right)^{1/8}\left(\frac{GM}{R^3}\right)^{3/16}\dot M^{1/8}\left(\frac{\mu m_H}{k}\right)^{1/16}\left(1-\sqrt{\frac{R_A}{R}}\right)^{1/8}$$

$$\cdot\left(\frac{\mu m_H}{k}\right)^{3/16}\left(\frac{32\pi\sigma}{45\cdot10^{24}}\right)^{1/40}\frac{\alpha^{-1/5}}{(3\pi)^{1/5}}\left(\frac{GM}{R^3}\right)^{-3/80}\left(1-\sqrt{\frac{R_A}{R}}\right)^{7/40}\dot M^{7/40} =$$

$$= \left(\frac{45\cdot10^{24}}{32\pi\sigma}\right)^{1/10}\left(\frac{GM}{R^3}\right)^{3/20}\left(\frac{\mu m_H}{k}\right)^{1/4}\dot M^{3/10}\frac{\alpha^{-1/5}}{(3\pi)^{1/5}}\left(1-\sqrt{\frac{R_A}{R}}\right)^{3/10}$$

$$\tau = 5\cdot10^{24}\,\Sigma\rho T^{-7/2} = 5\cdot10^{24}\left(\frac{\mu m_H}{k}\right)^{1/4}\left(\frac{32\pi\sigma}{45\cdot10^{24}}\right)^{1/2}\dot M^{1/5}\frac{\alpha^{-4/5}}{(3\pi)^{4/5}}\left(\frac{GM}{R^3}\right)^{3/80}\left(1-\sqrt{\frac{R_A}{R}}\right)^{1/5}$$

$$\nu = \alpha c_s H = \alpha H^2\sqrt{\frac{GM}{R^3}} = \frac{\alpha^{4/5}}{(3\pi)^{1/5}}\left(\frac{GM}{R^3}\right)^{-11/40}\left(\frac{k}{\mu m_H}\right)^{3/4}\dot M^{3/10}\left(\frac{32\pi\sigma}{45\cdot10^{24}}\right)^{1/10}\left(1-\sqrt{\frac{R_A}{R}}\right)^{3/10}$$

$$v_R = -\frac{3\nu}{2R}\left(1-\sqrt{\frac{R_A}{R}}\right)^{-1} = -\frac{3}{2R}\frac{\alpha^{4/5}}{(3\pi)^{1/5}}\left(\frac{GM}{R^3}\right)^{-11/40}\left(\frac{k}{\mu m_H}\right)^{3/4}\dot M^{3/10}\left(\frac{32\pi\sigma}{45\cdot10^{24}}\right)^{1/10}\left(1-\sqrt{\frac{R_A}{R}}\right)^{-7/10}$$

Shakura & Sunyaev viscous alpha disk

- So, with "alpha viscosity" parameterization $\nu = \alpha c_s H$ Shakura & Sunyaev ('73) gave the first solution. They used the Kramers' law (**$6.6 \cdot 10^{22}$ wrongly sometimes!!**) $\varkappa_R = 5 \cdot 10^{24} \rho T_c^{-7/2}$ cm²/g and neglected the radiation pressure in eq.(4) to write, in the form variations of which we usually find in the literature:

$R_{10} = R/(10^{10}\ \mathrm{cm}), M_1 = M/M_\odot$ i $\dot{M}_{16} = \dot{M}/(10^{16}\ \mathrm{g\,s^{-1}}), \mu = 0.615, f = [1 - (R_A/R)^{0.5}]^{1/4}$:

$$\begin{aligned}
S &= 5.2\alpha^{-4/5}\dot{M}_{16}^{7/10}M_1^{1/4}R_{10}^{-3/4}f^{11/5}\ \mathrm{g\ cm^{-2}},\\
H &= 1.7\cdot 10^{8}\alpha^{-1/10}\dot{M}_{16}^{3/20}M_1^{-3/8}R_{10}^{9/8}f^{3/5}\ \mathrm{cm},\\
\rho &= 3.1\cdot 10^{-8}\alpha^{-7/10}\dot{M}_{16}^{11/20}M_1^{5/8}R_{10}^{-15/8}f^{11/5}\ \mathrm{g\ cm^{-3}}\\
T_c &= 1.4\cdot 10^{4}\alpha^{-1/5}\dot{M}_{16}^{3/10}M_1^{1/4}R_{10}^{-3/4}f^{6/5}\ \mathrm{K} \qquad (3.39)\\
\tau &= 190\,\alpha^{-4/5}\dot{M}_{16}^{1/5}f^{4/5},\\
\nu &= 1.8\cdot 10^{14}\alpha^{4/5}\dot{M}_{16}^{3/10}M_1^{-1/4}R_{10}^{3/4}f^{6/5}\ \mathrm{cm^2\ s^{-1}},\\
v_r &= 2.7\cdot 10^{4}\alpha^{4/5}\dot{M}_{16}^{3/10}M_1^{-1/4}R_{10}^{-1/4}f^{-14/5}\ \mathrm{cm\ s^{-1}}.
\end{aligned}$$

- It is important that α is nowhere coming with large power, so that any error because of our not knowing it, is less.

Shakura & Sunyaev viscous alpha disk

- The Kramers' law for $\varkappa_R$ is critical, because when it is not holding any more, our approximation breaks down, but until it holds, disk can extend far in R, of the order of Roche boundary of the more massive star.
- Mass in the disk is $M_{disk} = 2\pi \int_{R_A}^{R_{vanjski}} S R dR \lesssim (10^{-10} M_\odot)\alpha^{-4/5}\dot{M}_{16}^{7/10}$, which is even in the very large disks negligible in comparison with the central object. This justifies the neglect of self-gravity of a disk, which is valid until $\rho << M/R^3$. Only for a very small α, of the order of 10^{-10}, this would not be fulfilled.
- The disk thickness in z-direction means that each element of the disc surface radiates as a blackbody with a temperature T (R) given by equating the dissipation rate D(R) per unit face area to the blackbody flux: $\sigma T^4 = D(R)$
- If we insert D from the bottom of slide 47: $$T(R) = \left\{ \frac{3GM\dot{M}}{8\pi R^3 \sigma} \left[1 - \left(\frac{R_*}{R} \right)^{1/2} \right] \right\}^{1/4}$$
- For $R \gg R_*$,

$$T = T_*(R/R_*)^{-3/4}$$

where

$$\left.\begin{aligned} T_* &= \left(\frac{3GM\dot{M}}{8\pi R_*^3 \sigma} \right)^{1/4} \\ &= 4.1 \times 10^4\, \dot{M}_{16}^{1/4} m_1^{1/4}\, R_9^{-3/4}\,\mathrm{K} \\ &= 1.3 \times 10^7\, \dot{M}_{17}^{1/4} m_1^{1/4}\, R_6^{-3/4}\,\mathrm{K}. \end{aligned}\right\}$$

$\dot{M}_{16} = \dot{M}/10^{16}$ g s^{-1}, $m_1 = M/M_\odot$, $R_9 = R_*/10^9$ cm etc. for disk around WD (R_9) & NS (R_6). **Note $R_A = R_*$ now!**

Shakura & Sunyaev viscous alpha disk

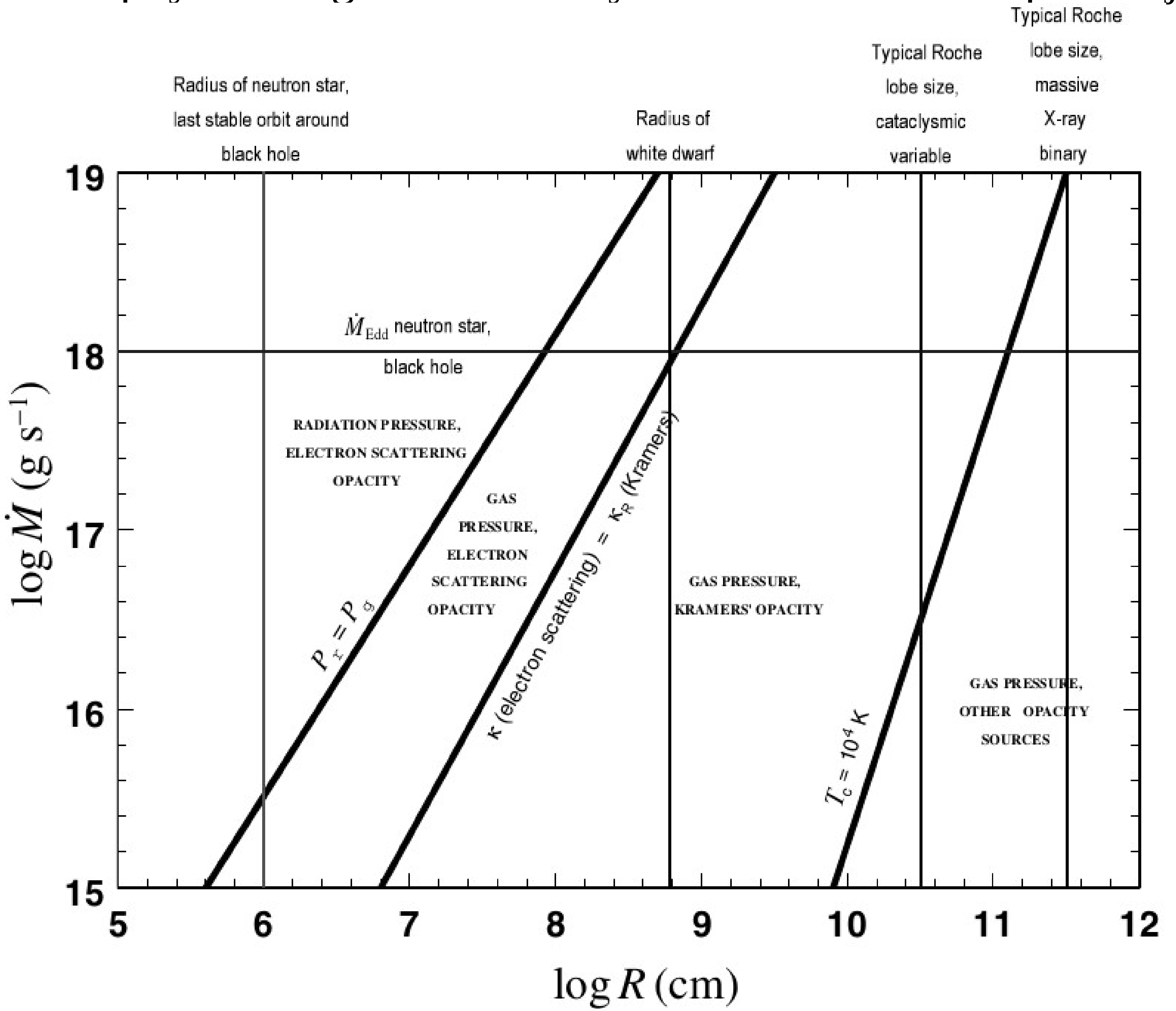
- The low power of α in the equations is good for usefulness of α as a parameter, but it also means we cannot expect to learn the typical size of α by direct comparison of steady-state disc theory with observations. This is something what is troubling disk astrophysics until today. No free lunch!
- A good thing is that for $\alpha \leq 1$ we obtained believable solutions, which are not too much off the models from observational data.

Where we expect the assumptions (Kramers' opacity and the neglect of radiation pressure) to break down? From the top of slide 54 we have

- $\varkappa_R = \tau / S = 36\, \dot{M}_{16}^{-1/2}\, m_1^{1/4}\, R_{10}^{3/4}\, f^{-2}$ independent of α. We compare with other opacity sources-the major competitive opacity is electron scattering where $\varkappa_R = \sigma_T / m_p \sim 0.4 \mathrm{cm}^2/\mathrm{g}$ with Kramers' opacity dominating for $R > 2.5\times10^7\, \dot{M}_{16}^{2/3}\, m_1^{1/3} f^{8/3}$ cm. This is smaller than the radius of a white dwarf for any reasonable $\dot{M}$, so for the accretion discs in cataclysmic variables we expect Kramers' opacity to dominate in most of the disc.

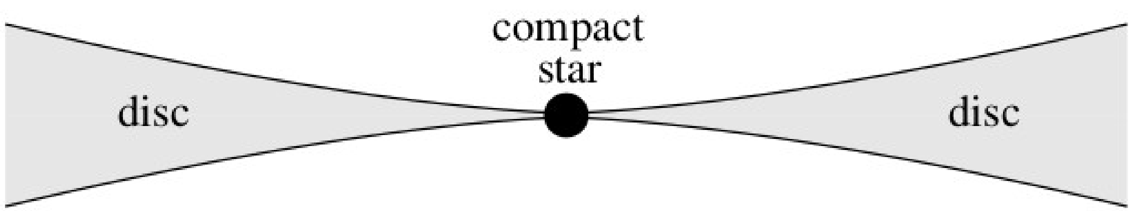
Shakura & Sunyaev viscous alpha disk

- In reasonable range we can rely on the results shown in the figure below for the physical regimes in steady α-discs around compact objects:

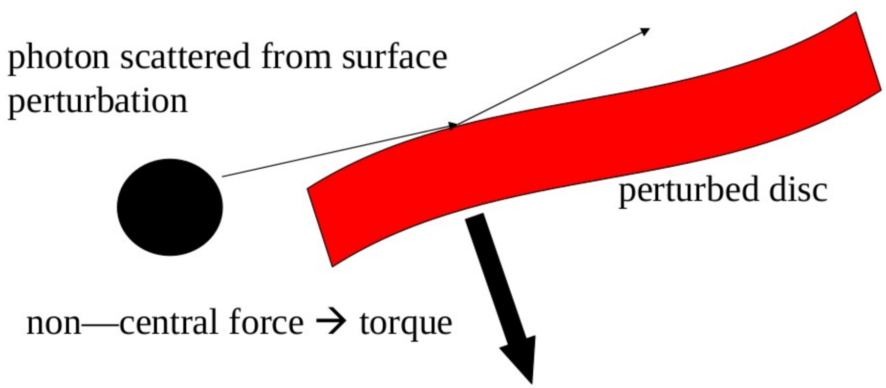


Shakura & Sunyaev viscous alpha disk

- If the disk is concave, then the central, hot regions, could irradiate the more distant, colder parts of the disk with hard radiation, and the picture complicates-this would show in observations.



- Similar is for a warped disk, complications in the picture:



SS73 concave disc

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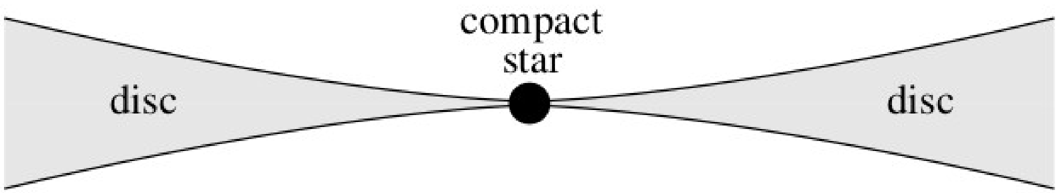
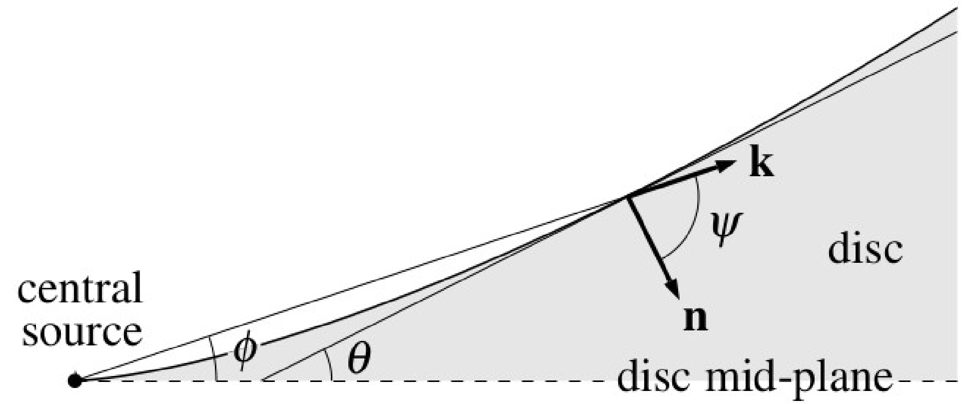




k indicates the direction of propagation of the radiation incident on the disc and **n** is the local inward-directed normal vector.

- In low-mass X-ray binaries the disk is probably heated by irradiation by the central accretion source. If the accretor is a luminous star, we can have a similar effect.
- If the central source can be regarded as a point, its total radiative flux at disc radius R is $L_{pt}/(4\pi R^2)$, source with L_{pt} total luminosity. The flux crossing the disc surface at this point is $F = \frac{L_{pt}}{4\pi R^2}(1-\beta)\cos\psi$

  β is the albedo, the effective fraction of incident radiation scattered from the surface without absorption, and ψ is the angle between the local inward-directed disc normal and the direction of the incident radiation.

We can write:

$$\psi = \frac{\pi}{2} - \theta + \phi,$$

where

$$\tan\theta = \frac{dH}{dR},$$

and

$$\tan\phi = \frac{H}{R},$$

SS73 concave disc

- Since dH/dR, H/R are both <<1 for a thin disc

$$\cos\psi = \sin(\theta - \phi) = \cos\theta\cos\phi[\tan\theta - \tan\phi] \simeq \frac{dH}{dR} - \frac{H}{R}$$

- With effective temperature T_{pt} resulting from irradiation by the point source and F from the previous slide,

$$T_{\rm pt}^4 = \frac{L_{\rm pt}}{4\pi R^2\sigma}\left(\frac{H}{R}\right)\left[\frac{d\ln H}{d\ln R} - 1\right](1-\beta),$$

or

$$\left(\frac{T_{\rm pt}}{T_e}\right)^4 = \frac{H}{R}\left(\frac{R_*}{R}\right)^2\left[\frac{d\ln H}{d\ln R} - 1\right](1-\beta)$$

T_e is the effective temperature of the central source, defined by $L_{pt} = 4\pi R_*^2 \sigma T_e^4$, with the characteristic source dimension R_*. With H varying as $R^{9/8}$, for the non-irradiated disk in the solutions for the disk, in a disc deriving **all** its luminosity from irradiation by a point source, one can show that $H \propto R^{9/7}$, and the factor in square brackets in solution lies between 1/8 and 2/7, which we name g, (to add another g). The ratio H/R is roughly constant in a disc, so T_{pt} falls off as $R^{-1/2}$. For a large enough disc T_{pt} dominates the disk effective temperature, which goes as $R^{-3/4}$. We obtain:

$$\left(\frac{T_{\rm pt}}{T_{\rm eff}}\right)^4 = \frac{2}{3}\frac{RL_{\rm pt}}{GM\dot{M}}\frac{H}{R}g(1-\beta)\,.$$

- If the central luminosity L_{pt} results from accretion, as in low-mass X-ray binaries, we have $L_{pt} = GM\dot{M}/R_*$ where $R_* = 10$ km for a neutron star, and a similar value for a black hole. Then

$$\left(\frac{T_{\rm pt}}{T_{\rm eff}}\right)^4 = \frac{2}{3}\frac{R}{R_*}\frac{H}{R}g(1-\beta).$$

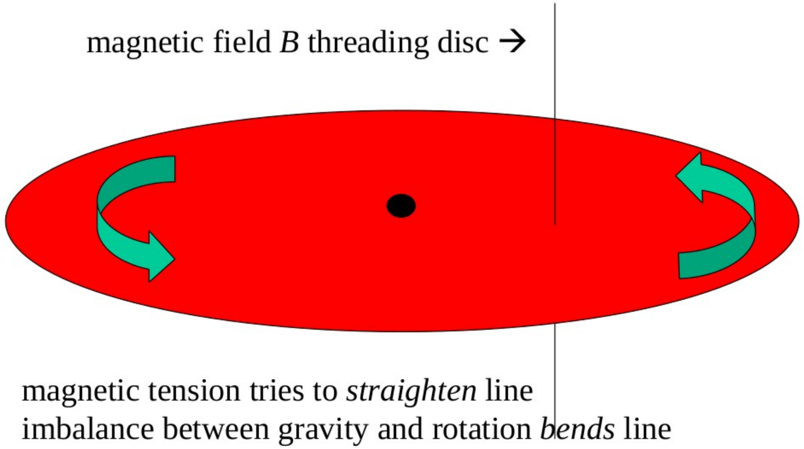
- This means that even if the combination $(H/R)g(1 - \beta)$ can be $<10^{-3}$, the central source will dominate for a disc with a large enough ratio R/R_*. In low-mass X-ray binaries $R \sim 10^6$cm and outer disk radius is $\sim 10^{10}$cm, so $R/R_* \sim 10^4$ and there will be a large range of surface temperatures in the disk.

Working of MRI in a disk

- Alpha viscosity does not give us predictive power.
- Since ∂/∂R($R^2\Omega$)=0 [Rayleigh criterion, stability against axisymmetric perturbations] and ∂Ω/∂R<0 . Most potential mechanisms are sensitive to the angular momentum gradient, so they work in such
- a way that they are bringing angular momentum INWARDS. We need a mechanism sensitive to Ω.
- If not alpha viscosity, then what? How the MRI works?
- Balbus-Hawley (magnetorotational, MRI) instability (1992).

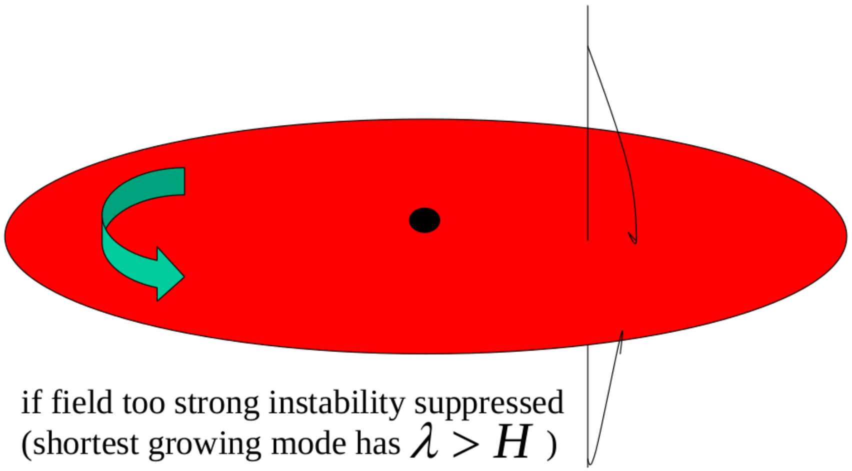
If we imagine a straight magnetic field B line threading a rotating disc, magnetic tension tries to straighten line, there is imbalance between gravity and rotation which bends the line (**figures in this and next 2 slides are from A. King's lecture I found online**).

magnetic field *B* threading disc →



magnetic tension tries to *straighten* line
imbalance between gravity and rotation *bends* line

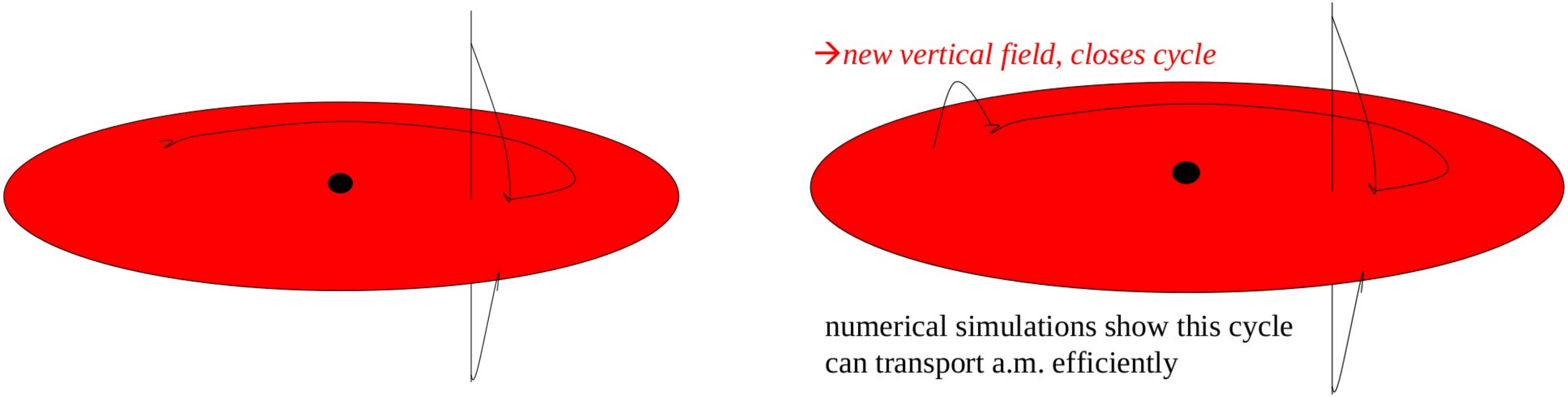
Vertical fieldline perturbed outwards, rotates faster than surroundings, so centrifugal force > gravity → *kink increases*
Line connects fast-moving (inner) matter with slower (outer) matter, and speeds latter up: ***outward a.m. transport***



if field too strong instability suppressed
(shortest growing mode has $\lambda > H$)

Working of MRI in a disk

- Vertical fieldline perturbed outwards, rotates faster than surroundings, so centrifugal force > gravity, so that kink increases. Line connects fast-moving (inner) matter with slower (outer) matter, and speeds latter up: outwards a.m. transport!

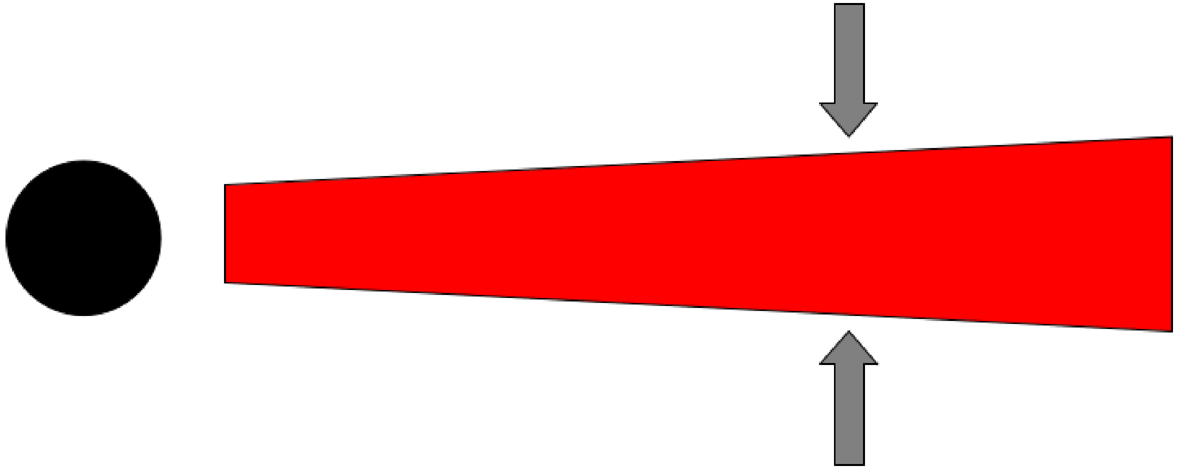


- For a too large mag. field, instability is supressed. Distorted fieldline stretched azimuthally by differential rotation, strength grows, pressure balance between flux tube and surroundings requires $B^2/8\pi+P_{gas,in}=P_{gas,out}$, so that gas pressure (and density) are lower inside tube; buoyant (Parker) instability works, and Flux tube rises above the disk, creating another vertical field, which closes the cycle, which can transport the angular momentum – this was shown to work in numerical simulations.

Self-gravity of disk

- Another effect which will change the picture is when the disk becomes larger:

main difference: *size of AGN disc* set by *self—gravity*



vertical component of gravity from central mass is $\sim GMH / R^3$

cf that from self—gravity of disc $\sim G\rho H^3 / H^2 \sim G\rho H$

Thus self—gravity takes over where $\rho \sim M / R^3$, or

$$M_{disc} \sim R^2 H\rho \sim \frac{H}{R} M$$

disc breaks up into stars outside this

Python code for Shakura & Sunyaev viscous alpha disk

For the beginning, some **ideological** points:

- Physical interpretation of observations is usually given, even in the cases with publicly funded experiments and observatories gathering the data, by individuals with proprietary and private modeling software. Such a practice produces results difficult to reproduce or verify. As with the recent initiatives, which is slowly becoming a policy, that publications should be freely available, similar initiatives are promoted for software and tools.
- It will take time, but we will get there, because it is in public interest, and it is also not very justified that some - and not at all so numerous, because of historical reasons - publishers would earn from the publicly paid research.

- As for Python, and in general tools, in line with green approach I gave a presentation on this back in 2020, I will repeat here some interesting points, might be of use.

The Ecological Impact of High-performance Computing in Astrophysics

Simon Portegies Zwart

[1] *Leiden Observatory, Leiden University, PO Box 9513, 2300 RA, Leiden, The Netherlands* [1]

[1] Non-anonymous Dutch scientists.

Computer use in astronomy continues to increase, and so also its impact on the environment. To minimize the effects, astronomers should avoid interpreted scripting languages such as Python, and favor the optimal use of energy-efficient workstations.

ArXiv:2009.11295; Nature Astronomy vol.4, 819 (2020)

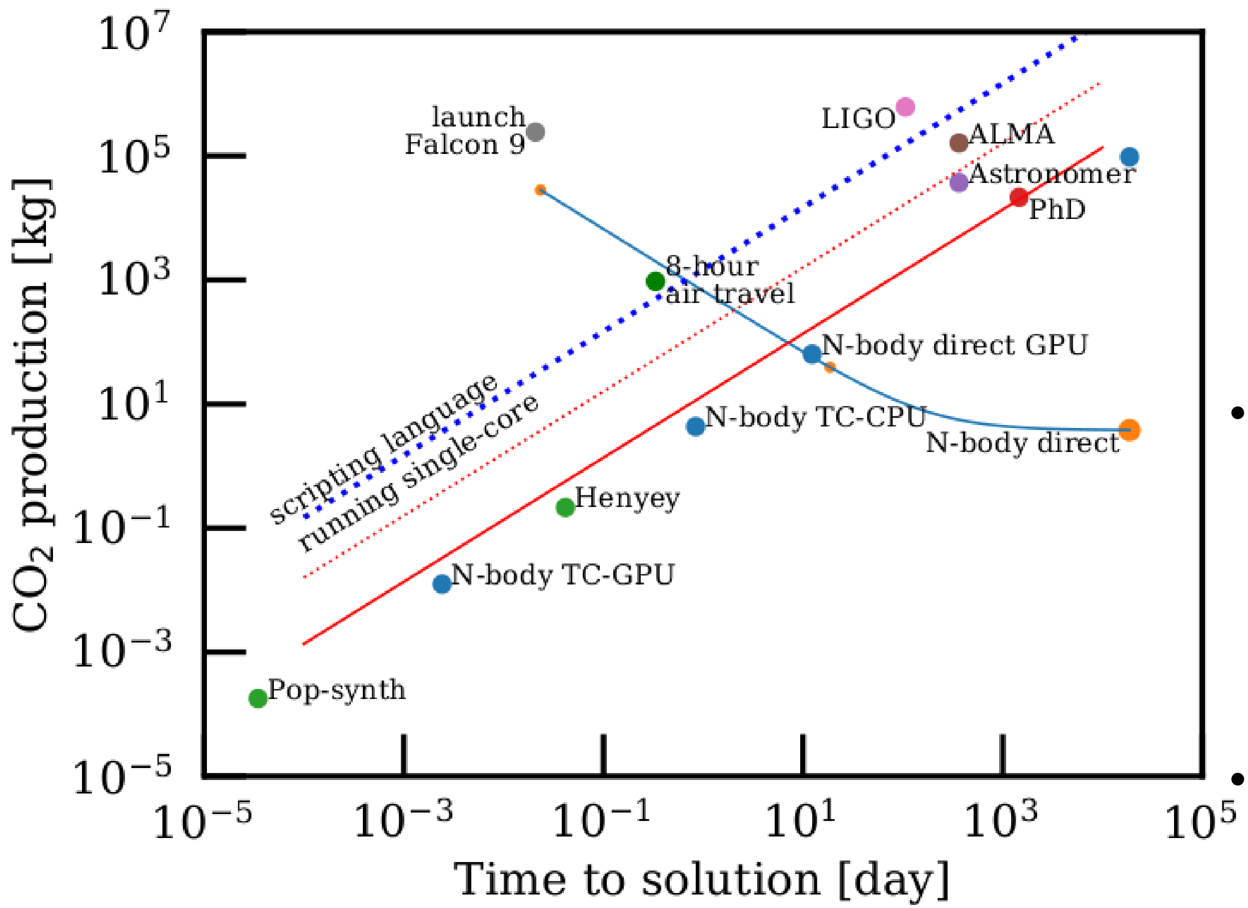


Figure 1: **CO^2 emission (in kg) as a function of the time to solution (in days) for a variety of popular computational techniques employed in astrophysics, and other activities common among astronomers [3][4]. The solid red curve gives the current individual world-average production, whereas the dotted curves give the maximum country average. The LIGO carbon production is taken over its first 106-day run (using ~ 180 kW) [5], and for ALMA a 1-year average [6]. A Falcon 9 launch lasts about 32 minutes during which $\sim 110\,000$ liters of highly refined kerosene is burned. The tree-code running on GPU is performed using $N = 2^{20}$ particles. The direct N-body code on CPU (right-most blue bullet) was run with $N = 2^{13}$ [7], and the other codes with $N = 2^{16}$. All performance results were scaled to $N = 2^{20}$ particles. The calculations were performed for 10 N-body time units [8]. The energy consumption was computed using the scaling relations of [9] and a conversion of KWh to Co_2 of 0.283 kWh/kg. The blue dotted curve shows the estimated carbon emission when these calculations would have been implemented in Python running on a single core. The solid blue curve to the right, starting with the orange bullet shows how the performance and carbon production changes while increasing the number of compute cores from 1 to 10^6 (out of a total of $7\,299\,072$, left-most orange point) using the performance model by [10].**

Carbon footprint of astronomy and computing

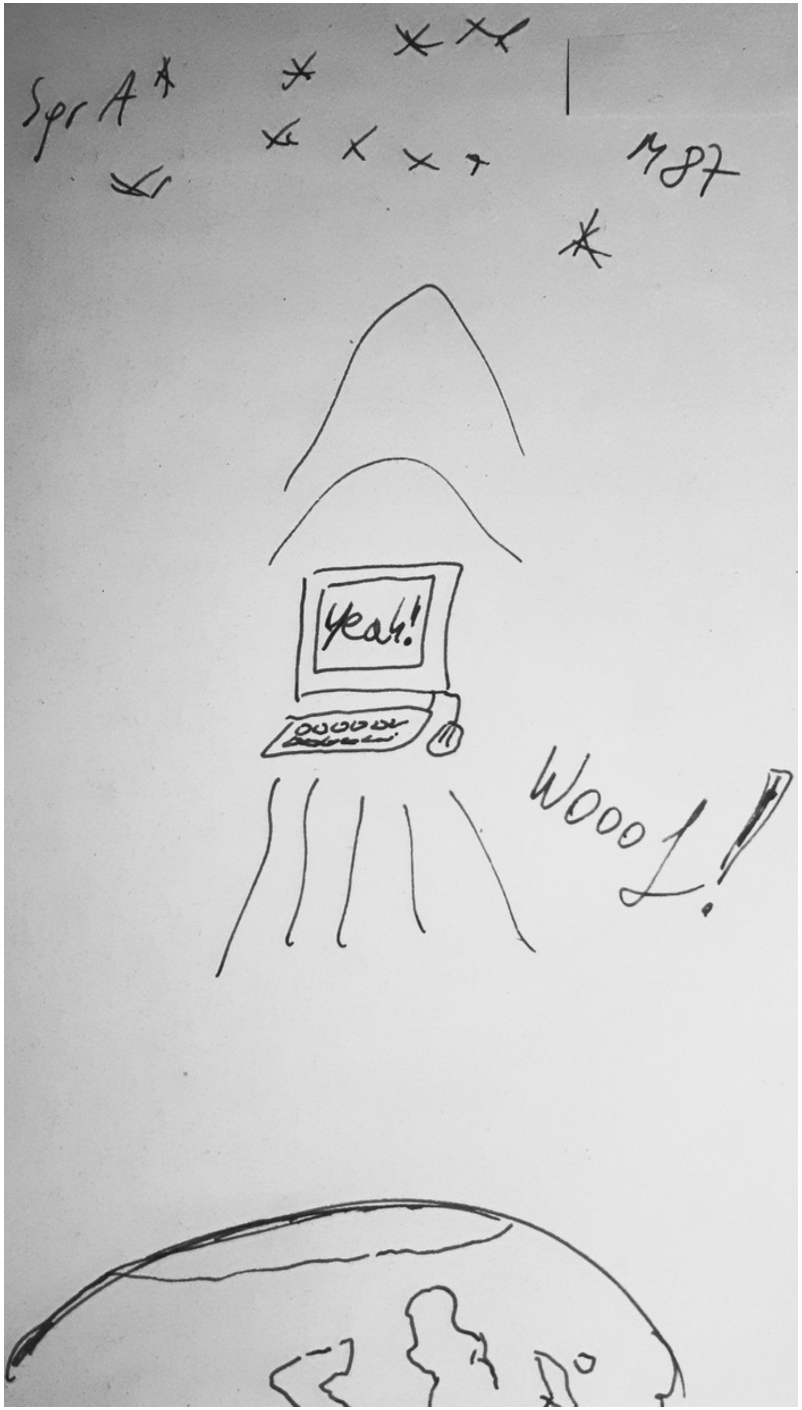
- Comparison of the average Human production of CO_2 (red line) with other activities, such as telescope operation, the emission of an average astronomer, and finishing a (four year) PhD.
- The emission of carbon while running a workstation is comparable to the world's per-capita average.

Carbon footprint of computing

- The relation between the time-to-solution and the carbon footprint of the calculations is not linear. When running a single core, a supercomputer-used to capacity-produces less carbon than a workstation. More cores result in better performance, at the cost of producing more carbon.

- Similar **performance as a single GPU is reached when running 1000 cores, but** when the number of cores is further increased, the performance continues to grow at an enormous cost in carbon production.

- When **running a million cores, the emission by supercomputer** by far exceeds air travel and **approaches the carbon footprint of launching a rocket into space**.



Ecological impact of computing language

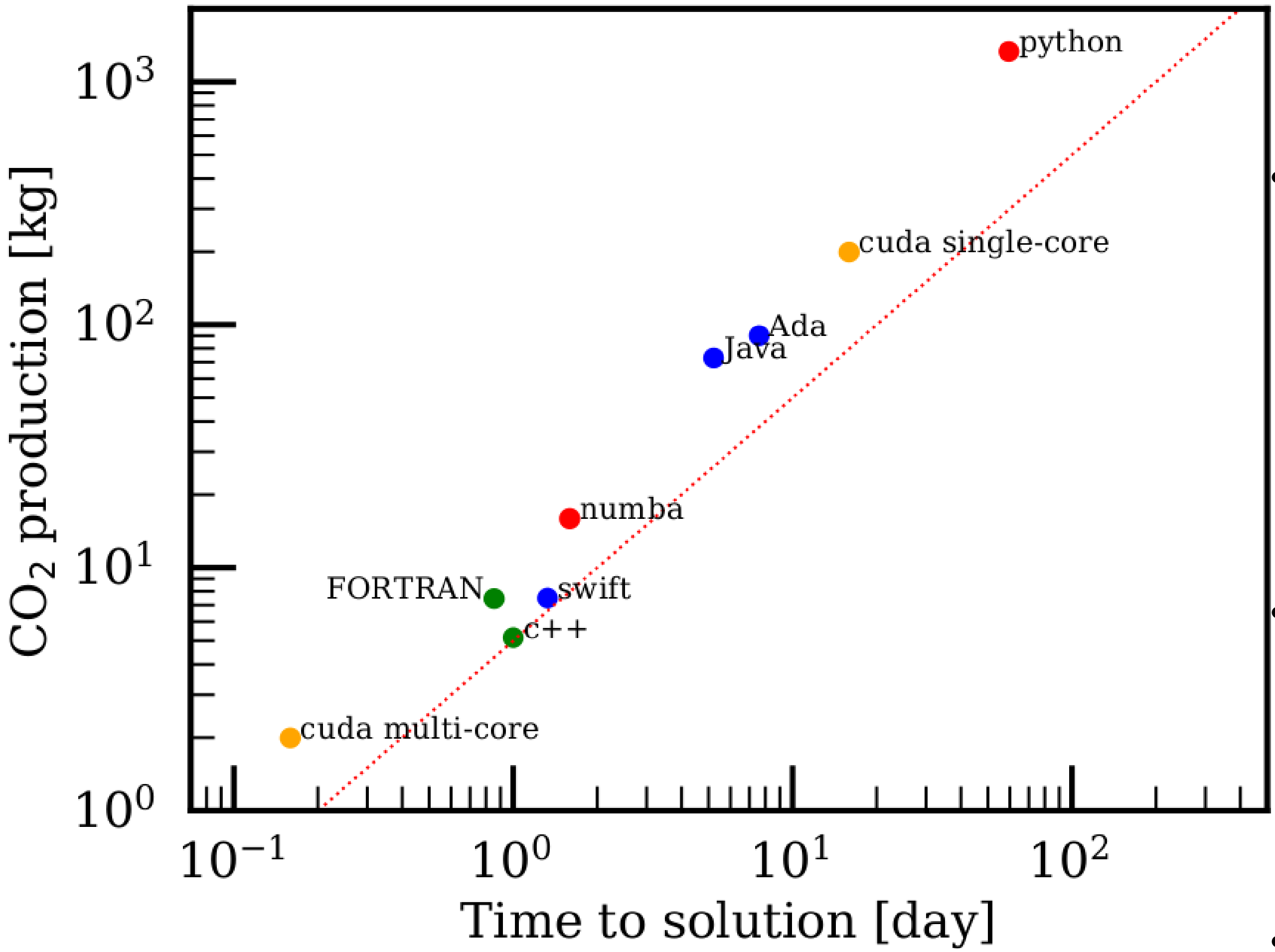


Figure 3: **Here we used the direct N-body code from [23] to measure execution speed and the relative energy efficiency for each programming language from table 3 of [22]. The dotted red curve gives a linear relation between the time-to-solution and carbon footprint (~ 5 kg CO_2/day). The calculations were performed on a 2.7GHz Intel Xeon E-2176M CPU and NVIDIA Tesla P100 GPU.**

- Results were obtained with the assumption that astrophysicists invest in full code optimization that uses the hardware optimally.
- In practice, most effort is generally invested into solving the research question; designing, writing, and running the code is not the primary concern, if the result is obtained reasonably fast. This is why **inefficient** (and **slow**) scripting languages as Python flourish.
- According to the Astronomical Source Code Library, ~ 43% of the code is written in Python, 7 % Java, IDL and Mathematica. Only 18%, 17% and 16% of codes are written in Fortran, C and C++ respectively.
- Python and Java are also less efficient in terms of energy per operation than compiled languages, which explains the offset away from the dotted curve.
- Among 27 tested languages, only Perl and Lua are slower than Python-**popularity of Python should be confronted with the ecological consequences**.

How to improve?

- Runtime performance of Python can be improved **using numba or NumPy libraries**, which offer pre-compiled code for common operations-it leads to an enormous increase in speed and reduced carbon emission. However, these libraries are rarely adopted for reducing carbon emission or runtime with more than an order of magnitude.
- NumPy, for example, is mostly used for its advanced array handling and support functions. Using these will reduce runtime and, therefore, also carbon emission, but **optimization** is generally stopped as soon as the calculation runs within an unconsciously determined reasonable amount of time, such as the coffee-refill time-scale or a holiday weekend. We even teach Python to students, but without realizing the ecological impact.
- The carbon footprint of computational astrophysics can be reduced substantially by **running on GPUs,** but the development time of such code requires major investments in time and expertise.
- As an alternative, one could run concurrently **using multiple cores, rather than a single thread**. It is even better to **port the code to a supercomputer and share the resources**.
- Best for the environment is to **abandon Python** for a more environmentally friendly (compiled) programming language.
- Even better is to **use other interesting strongly-typed languages** with characteristics similar to Python, such as Alice, Julia, Rust, and Swift. They offer the flexibility of Python but with the performance of compiled C++.
- Educators may want to **reconsider teaching Python** to University students. There are plenty environmentally friendly alternatives.

Python package agnpy

- Back to our topic. As an intro and example, we install and run the agnpy, python package modelling the radiative processes of relativistic particles accelerated in the jets of AGNs. We try to learn from it, eventually use some of its content.
- The package is fresh, arXiv paper was out in January of this year, information about the installation and the first example I took from the poster from Jets conference in last year.

agnpy: an open-source, do it yourself, approach to (jetted) AGN modelling

C. Nigro [1] J. Sitarek [2] P. Gliwny [2] D. Sanchez [3]

[1] Institut de Física d'Altes Energies (IFAE), Barcelona [cosimo.nigro@ifae.es]
[2] University of Lodz
[3] Laboratoire d'Annecy de Physique des Particules (LAPP), Annecy

νF_ν

AGNpy

Introduction

agnpy [1] is a python package modelling the radiative processes of relativistic particles accelerated in the jets of Active Galactic Nuclei (AGN). It additionally includes classes representing the AGN thermal and line emitters and computes the γ-γ absorption produced in their photon fields. The package is built on numpy [2] and astropy [3] and is affiliated with the latter project. [**GitHub**, **Docs**]

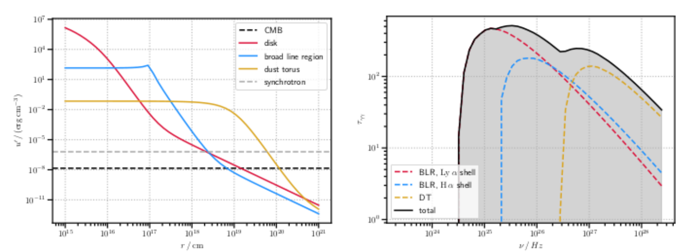
Package Modules

Emission Regions

agnpy.emission_regions describes the region responsible for particle acceleration and radiation. It contains an $e^\pm$ energy distribution parametrised with an analytical function.

Evaluate Energy Density and Absorption of the Line and Thermal Emitters

agnpy allows to evaluate the energy density, $u\,/\,(\mathrm{erg\,cm^{-3}})$, of the li and thermal emitters as a function of the distance from the jet axis or their γ-γ opacity, $\tau_{\gamma\gamma}$, as a function of the escaping γ energy.



Python package agnpy

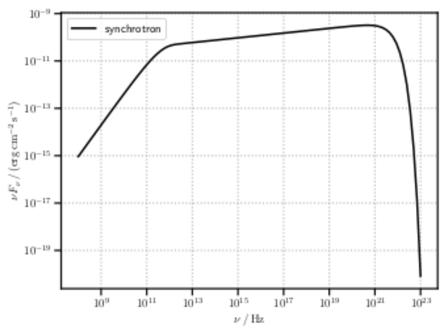
They can be used as target for EC or γ-γ absorption; their broad-band emission can also be evaluated.

Examples

Compute the SED for a Given Radiative Process

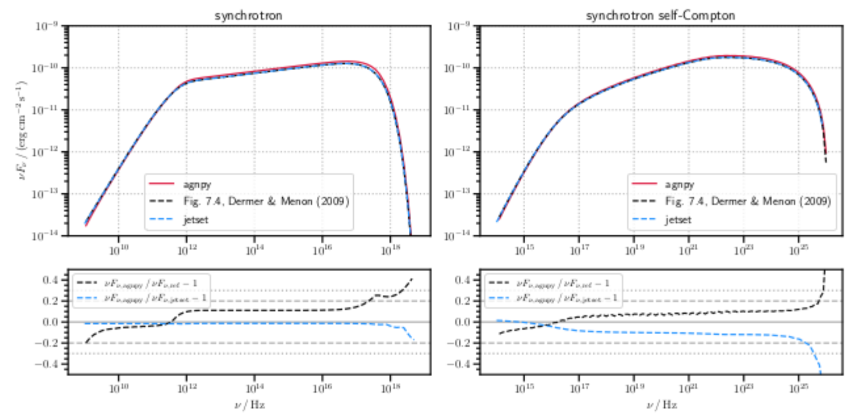
With few lines of python the user can evaluate the SED for a specific radiative process

```
import numpy as np
import astropy.units as u
from agnpy.emission_regions import Blob
from agnpy.synchrotron import Synchrotron
from agnpy.utils.plot import plot_sed
import matplotlib.pyplot as plt
# define the emission region and the radiative process
blob = Blob()
synch = Synchrotron(blob)
# compute the SED over an array of frequencies
nu = np.logspace(8, 23) * u.Hz
sed = synch.sed_flux(nu)
# plot it
plot_sed(nu, sed, label="synchrotron")
plt.show()
```



Validation

agnpy is thoroughly validated against bibliographic references and against other modelling codes relying on the same physical assumptions. An agreement between 10% and 30% with other sources is achieved. Below we compare agnpy's synchrotron and SSC spectra against the ones in [4] and the ones produced with jetset.



References

[1] Nigro C. et al.; *agnpy: modelling Active Galactic Nuclei radiative processes with python*; https://doi.org/10.5281/zenodo.4055175.

[2] Harris, C.R. et al.; *Array programming with NumPy*; Nature 585, 357–362 (2020).

[3] Astropy Collaboration; *Astropy: Building an Open-science Project and Status of the v2.0 Core Package*; AJ, 156, id.123 (2018).

[4] Dermer C., Menon. G.; *High Energy Radiation from Black Holes*; Princeton (2009).

[5] Finke J.; *External Compton Scattering in Blazar Jets and the Location of the Gamma-Ray Emitting Region*; ApJ, 830:94 (2016).

Python package agnpy

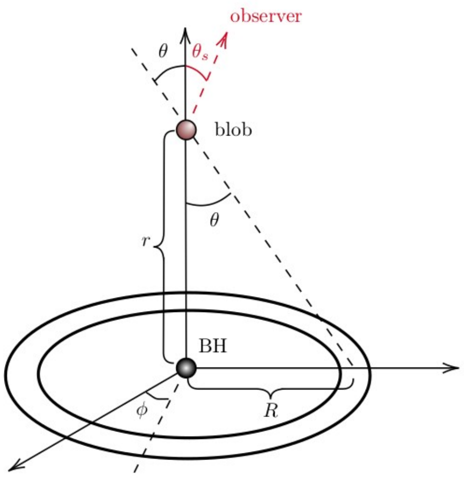


Fig. A.1: Geometry used for the energy density and external Compton scattering calculations. A disc-like emitter with radial coordinate R is represented as an example.

Appendix A.2.3: Shakura-Sunyaev accretion disc

Let us consider an accretion disc with inner and outer radii $R_{\rm in}$ and $R_{\rm out}$, respectively. From the sketch in Fig. A.1 we see that the cosine of the photon incoming angle $\mu = \cos\theta$ and the coordinate along the disc radius R are related via

$$\mu = \left(1 + \frac{R^2}{r^2}\right)^{-1/2}, \; R = r\sqrt{\mu^{-2} - 1}. \tag{A.13}$$

The minimum and maximum incoming angles of photons from the disc impinging on the emission region are therefore

$$\mu_{\rm min} = \left(1 + \frac{R_{\rm out}^2}{r^2}\right)^{-1/2}, \; \mu_{\rm max} = \left(1 + \frac{R_{\rm in}^2}{r^2}\right)^{-1/2}. \tag{A.14}$$

The specific spectral energy density of photons coming from a Shakura & Sunyaev (1973) accretion disc can be parametrised, following Dermer & Schlickeiser (2002) and Dermer et al. (2009), as:

$$\underline{u}(\epsilon, \Omega; r) = \frac{3GM\dot{m}}{(4\pi)^2 c R^3}\,\varphi(R)\,\delta(\epsilon - \epsilon_0(R)) \tag{A.15}$$

where $\varphi(R)$ represents the variation of energy flux along the radius

$$\varphi(R) = 1 - \sqrt{\frac{R_{\rm in}}{R}} \tag{A.16}$$

and $\epsilon_0(R)$ is the monochromatic approximation for the mean photon energy emitted from the disc at radius R

$$\epsilon_0(R) = 2.7 \times 10^{-4} \left(\frac{l_{\rm Edd}}{M_8 \eta}\right)^{1/4} \left(\frac{R}{R_{\rm g}}\right)^{-3/4}. \tag{A.17}$$

In the above equations, G is the gravitational constant, $M = M_8 \times 10^8\,M_\odot$ is the mass of the black hole in solar mass units, $\dot{m}\,[{\rm g\,s^{-1}}]$ the black hole mass accretion rate, η the fraction of gravitational energy converted to radiant energy ($L_{\rm disc} = \eta \dot{m} c^2$), $l_{\rm Edd}$ the ratio of the disc luminosity to the Eddington luminosity ($L_{\rm Edd} = 1.26 \times 10^{46}\,M_8\,{\rm erg\,s^{-1}}$), and $R_{\rm g} = GM/c^2$ is the gravitational radius of the BH. When using Eq. (A.16) and Eq. (A.17), we make explicit the fact that the photons emitted at a given disc radius R will have different incidence angles depending on the blob position r, by replacing $\varphi(R) \to \varphi(\mu; r)$ and $\epsilon_0(R) \to \epsilon_0(\mu; r)$.

The integral energy density in the galactic frame is

$$u(r) = \frac{3}{8\pi c}\frac{GM\dot{m}}{r^3}\int_{\mu_{\rm min}}^{\mu_{\rm max}} {\rm d}\mu\,\frac{\varphi(\mu; r)}{(\mu^{-2} - 1)^{3/2}}. \tag{A.18}$$

In the blob frame the energy density is

$$u'(r) = \frac{3}{8\pi c}\frac{GM\dot{m}}{r^3} \int_{\mu_{\rm min}}^{\mu_{\rm max}} {\rm d}\mu \frac{\varphi(\mu; r)}{\Gamma^6 (1 - \mathcal{B}\mu)^2 (1 + \mathcal{B}\mu')^4 (\mu^{-2} - 1)^{-3/2}}. \tag{A.19}$$

Using NumPy, SciPy, and astropy, agnpy is still "kosher" and provides another element to the modular astrophysical software system envisioned in Portegies Zwart (2018). But, how much of SS73 disk is there in agnpy? There is rather an update of their result-good for science, but not for our goal of using SS73. But maybe we could simply make it and incorporate into agnpy, they claim it is easy to do. Let's check it, can we do it in 2 hrs? We wish to plot this updated result aside the original one.

Python code for Shakura & Sunyaev viscous alpha disk

Next we prepare our own Python code to plot the results for Shakura & Sunyaev thin disk:

-load the necessary routines and prepare the definition of quantities.

-type-in the equations, constants and write the interactive input interface.

-make the visualisation part of the code.

Summary of the Part II

- We introduced the simplest equations for disk description

- Detailed perturbative solutions, step by step.

- Shakura & Sunyaev ‘73 solution
- If not alpha viscosity, then what? How the MRI works?
- Self-gravity of disk

- Python code for Shakura & Sunyaev viscous alpha disk
  -some ideological points about open access
  -agnpy package
  -our own python routine
  -merging of our routine for original SS73 with agnpy

Outline, Part III: General solutions for thin disk

- z-averaged solutions: Urpin (1984) ; Regev (1983) solution

- Kluźniak-Kita global solution for a thin HD disk and its magnetic generalization

- Umurhan (2006) generalized γ solutions

- Python code for thin disk; python tool DUSTER

Urpin solution

Hydrodynamic flows in accretion disks

V. A. Urpin

Ioffe Physics and Technology Institute, USSR Academy of Sciences, Leningrad

(Submitted December 14, 1982)

Astron. Zh. **61**, 84–90 (January–February 1984)

Calculations of the velocity field in an accretion disk show that matter may flow in the disk not only toward but also away from the central object. The rates of flow are determined, and the geometry of hydrodynamic motions in nonstationary disks is discussed.

I was mentioning before the z-averaged solutions. Assumptions in Urpin are all as we already know:

We shall adopt the turbulent-viscosity model.[6] According to this model the influence of small-scale turbulent motions upon the regular motions is tantamount simply to a renormalization of the viscosity coefficient. The equation of motion will then take the form

$$\rho\frac{\partial \mathbf{V}}{\partial t}+\rho(\mathbf{V}\cdot\nabla)\mathbf{V}=-\nabla p+\rho\nabla\psi+\nabla\hat{\pi},$$
$$(\hat{\pi})_{ik}=\rho\mathfrak{H}\left(\frac{\partial V_i}{\partial x_k}+\frac{\partial V_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\nabla\mathbf{V}\right). \quad (1)$$

I noticed here we forgot to discuss on viscous tensor, of which the above eq.is a component:

The viscosity was defined by the

$$\boldsymbol{\tau}=\eta_{\mathrm{v}}\left[(\nabla\boldsymbol{v})+(\nabla\boldsymbol{v})^T-\frac{2}{3}(\nabla\cdot\boldsymbol{v})\boldsymbol{I}\right],$$

with the dynamic viscosity $\eta_{\mathrm{v}}=\rho\nu_{\mathrm{v}}$

$$\frac{\partial\rho}{\partial t}+\nabla(\rho\mathbf{V})=0. \quad (2)$$

It has been suggested that the motions along the r, z directions may take place at subsonic velocities, while the rotational velocity V_φ may exceed the sound speed c_s. Estimates indicate (Refs. 1, 2) that these conditions will often be satisfied in real objects. Moreover, we shall consider only objects in which the accretion rate varies so slowly that the inequality

$$t_a>P_{\mathrm{rot}}=2\pi r/V_\varphi \quad (3)$$

holds for all r; here t_a is the time scale for change in the accretion rate. In this event the disk will be axisymmetric.

With these assumptions the z component of Eq. (1) reduces to the simple hydrostatic-equilibrium condition

$$\frac{1}{\rho}\frac{\partial p}{\partial z}=-\frac{GMz}{r^3},$$

where M is the mass of the central object (we neglect the self-gravitation of the disk). The radial component of Eq. (1) will take the form

$$V_r\frac{\partial V_r}{\partial r}+V_z\frac{\partial V_r}{\partial z}-\frac{V_\varphi^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{\partial\psi}{\partial r}+\frac{1}{\rho}(\nabla\hat{\pi})_r.$$

Urpin solution

Since $V_r \sim \mathfrak{H}/r$ and $V_z \sim (z_0/r)V_r$ (z_0 is the characteristic disk thickness; see, for example, Shakura and Syunyaev[1]), one can readily show that the first two terms on the left are much smaller than $(z_0/r)^4 V_\varphi^2/r$. We may therefore neglect them. The last term on the right is easily estimated:

$$\frac{1}{\rho}(\nabla\hat{\pi})_r \lesssim \frac{\mathfrak{H} V_\varphi}{r} \sim \left(\frac{v_t}{c_s}\right)\left(\frac{l_t}{z_0}\right)\frac{V_\varphi^2}{r^2}\frac{z_0^2}{r} \sim$$
$$\sim \left(\frac{v_t}{c_s}\right)\left(\frac{l_t}{z_0}\right)\left(\frac{z_0}{r}\right)^2\frac{V_\varphi^2}{r} \ll \left(\frac{z_0}{r}\right)^2\frac{V_\varphi^2}{r};$$

it too may be neglected. Then

$$\frac{V_\varphi^2}{r} \approx -\frac{\partial\psi}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r}.$$

Later we will need an expression for V_φ accurate to terms of order $(z/r)^2$. Since $\psi = GM/R$, where $R = (r^2 + z^2)^{1/2}$ is the distance from the compact object, we readily find

$$V_\varphi \approx \left(\frac{GM}{r}\right)^{1/2}\left\{1 - \frac{3}{4}\frac{z^2}{r^2} + \frac{r^2}{2GM\rho}\frac{\partial p}{\partial r}\right\}. \quad (4)$$

If the condition (3) holds, the φ component of the equation of motion will become

$$\frac{V_r}{r}\frac{\partial}{\partial r}(rV_\varphi) + V_z\frac{\partial V_\varphi}{\partial z} = \frac{1}{\rho}(\nabla\hat{\pi})_\varphi.$$

In the disk $V_z \sim (z_0/r)V_r$, and furthermore $\partial V_\varphi/\partial r \sim (z/r)\partial V_\varphi/\partial r$. We may therefore neglect the second term on the left. Then

$$V_r = \frac{2r}{\rho}\left(\frac{r}{GM}\right)^{1/2}(\nabla\hat{\pi})_\varphi. \quad (5)$$

According to the definition of the tensor $\hat{\pi}$ we may write

$$\nabla\hat{\pi} = [\nabla(\rho\mathfrak{H})\nabla]\mathbf{V} + \rho\mathfrak{H}\nabla^2\mathbf{V} + \rho\mathfrak{H}\nabla(\nabla\mathbf{V}) + \nabla[\mathbf{V}\nabla(\rho\mathfrak{H})]$$
$$-(\mathbf{V}\nabla)\nabla(\rho\mathfrak{H}) - \frac{2}{3}\nabla(\rho\mathfrak{H}\nabla\mathbf{V}).$$

Urpin solution

Since we have assumed the disk to be axisymmetric with $\partial/\partial\varphi = 0$,

$$(\nabla\hat{\pi})_\varphi = \frac{\partial V_\varphi}{\partial z}\frac{\partial(\rho\mathfrak{H})}{\partial z} + \frac{\partial(\rho\mathfrak{H})}{\partial r}\left(\frac{\partial V_\varphi}{\partial r} - \frac{V_\varphi}{r}\right) + \rho\mathfrak{H}\left[\frac{\partial^2 V_\varphi}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{\partial V_\varphi}{\partial r} + \frac{V_\varphi}{r}\right)\right].$$

Substituting this expression into Eq. (5), we obtain

$$V_r \approx \frac{2r}{\rho}\left(\frac{r}{GM}\right)^{1/2}\left\{\frac{\partial V_\varphi}{\partial z}\frac{\partial(\rho\mathfrak{H})}{\partial z} + \frac{\partial(\rho\mathfrak{H})}{\partial r}\left[\frac{\partial V_\varphi}{\partial r} - \frac{V_\varphi}{r}\right] + \rho\mathfrak{H}\left[\frac{\partial^2 V_\varphi}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{\partial V_\varphi}{\partial r} + \frac{V_\varphi}{r}\right)\right]\right\}$$
$$\approx \frac{2r}{\rho}\left\{\left(\frac{r}{GM}\right)^{1/2}\left[\frac{\partial V_\varphi}{\partial z}\frac{\partial(\rho\mathfrak{H})}{\partial z} + \rho\mathfrak{H}\frac{\partial^2 V_\varphi}{\partial z^2}\right] - \frac{3}{2r}\frac{\partial(\rho\mathfrak{H})}{\partial r} - \frac{3}{4r^2}\rho\mathfrak{H}\right\}. \quad (6)$$

It is evident from Eq. (6) that to determine the radial component of the velocity correctly we must know V_φ to terms of order $(z/r)^2$, since $\partial^2 V_\varphi/\partial z^2 \sim \partial^2 V_\varphi/\partial r^2 \sim V_\varphi/r^2$. Our expression (4) provides this accuracy.

The vertical velocity component V_z can be determined from the equation of continuity:

$$\frac{\partial}{\partial z}(\rho V_z) = -\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) - \frac{\partial\rho}{\partial t}. \quad (7)$$

Next he discusses the three zones of the disk, neglecting the closest one to the star:

firm that the law (8) holds for zones B, C. In zone A, Shakura and Syunyaev[1] found that ρ is independent of height, but if one allows for turbulent heat transfer, this conclusion will no longer be valid.[7] The density variation in zone A then can also be described to fair accuracy by Eq. (8). However, in the accretion disks around neutron stars and white dwarfs zone A evidently is often absent; we therefore shall consider in detail only the flows in zones B, C.

I have obtained the following expressions for the scale z_0 in these two zones[7]:

$$\text{zone C:}\quad z_0 = 10^4\alpha^{-1/10} m^{9/10} \dot{m}^{3/20} \Re^{9/8}\ \text{cm},$$
$$\text{zone B:}\quad z_0 = 2{,}1\cdot 10^4\,\alpha^{-1/10} m^{9/10} \dot{m}^{1/5} \Re^{21/20}\ \text{cm}. \quad (9)$$

Here $m = M/M_\odot$; $\dot{m} = \dot{M}/(1.9\cdot 10^{18}\ m\cdot g/sec)$, with $\dot{M}$ the accretion rate; $\Re = rc^2/6GM = r/(8.89\cdot 10^5\ m\cdot cm)$; and the parameter $\alpha = \mathfrak{H}/z_0 c_s$ measures the level to which turbulence has developed in the disk $[c_S = (kT_c/m_p)^{1/2}]$.

Along with these expressions we shall also need the condition that the mass flux be conserved. We can obtain it from the equation of continuity, together with Eqs. (6), (8):

$$\dot{M} \approx \text{const} = -\int_{-\infty}^{+\infty} 2\pi r\rho V_r\, dz = 3\pi\sqrt{\pi}\,\rho_c \mathfrak{H} z_0, \quad (10)$$

where ρ_c is the density in the central disk plane.

Urpin solution

Now we compute v_r: we have from eq.4, because of vertical hydrostatic equilibrium

$$\frac{\partial V_\varphi}{\partial z}=\frac{1}{2}\left(\frac{GM}{r^3}\right)^{1/2}\left(-\frac{z}{\rho}\frac{\partial\rho}{\partial r}-\frac{r^3}{C_r M\rho^2}\frac{\partial p}{\partial r}\frac{\partial\rho}{\partial z}\right). \quad (11)$$

We introduce the convenient dimensionless parameters $\xi = z/z_0$, $f = T/T_c$, $\Psi = \rho/\rho_c$. With these definitions we obtain

$$\frac{\partial\ln\rho}{\partial r}=\frac{\partial\ln\rho_c}{\partial r}-\xi\frac{\dot\Psi}{\Psi}\frac{\partial\ln z_0}{\partial r},$$
$$\frac{1}{\rho}\frac{\partial p}{\partial r}=\frac{2kT_c}{m_p}f\left\{\frac{\partial\ln\rho_c T_c}{\partial r}-\xi\left(\frac{\dot\Psi}{\Psi}+\frac{\dot f}{f}\right)\frac{\partial\ln z_0}{\partial r}\right\}. \quad (12)$$

(the dot signifies differentiation with respect to ξ).

The r-dependence of the parameters ρ_c, T_c, z_0, $\mathfrak{H}$ has been investigated by Shakura and Syunyaev (Ref. 1) and the author.[7] For these parameters the following relations hold:

zone C: $\frac{\partial\ln\rho_c}{\partial\ln r}=-\frac{15}{8}$, $\frac{\partial\ln T_c}{\partial\ln r}=-\frac{3}{4}$, $\frac{\partial\ln z_0}{\partial\ln r}=\frac{9}{8}$, $\frac{\partial\ln\mathfrak{H}}{\partial\ln r}=\frac{3}{4}$;

zone B: $\frac{\partial\ln\rho_c}{\partial\ln r}=-\frac{33}{20}$, $\frac{\partial\ln T_c}{\partial\ln r}=-\frac{9}{10}$, $\frac{\partial\ln z_0}{\partial\ln r}=\frac{21}{20}$, $\frac{\partial\ln\mathfrak{H}}{\partial\ln r}=\frac{3}{5}$. (13)

As mentioned above, $\Psi\sim\exp(-\xi^2)$ in zones B, C. Substituting the expressions (12), (13) into Eq. (11) and noting that $\dot\Psi/\Psi\approx-2\xi$, we obtain

$$\text{zone C:}\quad \frac{\partial V_\varphi}{\partial z}=\frac{3z_0}{16r^2}\left(\frac{GM}{r}\right)^{1/2}\xi(5-7f-6\xi^2+6f\xi^2-3\xi\dot f),$$
$$\text{zone B:}\quad \frac{\partial V_\varphi}{\partial z}=\frac{3z_0}{40r^2}\left(\frac{GM}{r}\right)^{1/2}\xi(11-17f-14\xi^2+14f\xi^2-7\xi\dot f).$$

Using these expressions, it is not hard to evaluate V_r. In zone C,

$$V_r=\frac{3\mathfrak{H}}{r}\left\{\frac{5}{4}-\frac{7}{8}f-\frac{23}{4}\xi^2+4f\xi^2+\frac{3}{2}\xi^4-\frac{3}{2}f\xi^4-\frac{13}{8}\xi\dot f+\frac{3}{2}\xi^3\dot f-\frac{3}{8}\xi^2\ddot f\right\}, \quad (14)$$

while in zone B,

$$V_r=\frac{3\mathfrak{H}}{20r}\{22-17f-106\xi^2+76f\xi^2+28\xi^4-28f\xi^4-31\xi\dot f+28\xi^3\dot f-7\xi^2\ddot f\}. \quad (15)$$

Notice that when $\xi=0$ the radial component of the velocity is positive, and equal to $9\mathfrak{H}/8r$ in zone C and $3\mathfrak{H}/4r$ in zone B. Thus in the vicinity of the central plane, matter is flowing away from the compact object.

Urpin solution

If ξ is not too large, we may use for the function f the asymptotic expressions[7]:

zone C: $f \approx 1 - 0.25\xi^2 + 0.0048\xi^4 - \ldots,$

zone B: $f \approx 1 - 0.28\xi^2 + 0.0054\xi^4 - \ldots.$

With these we can easily obtain analytic expressions for V_r at reasonably small heights ξ:

zone C: $V_r = \frac{9\mathfrak{H}}{8r}(1 - 1.4\xi^2 - 4.8\xi^4 - \ldots),$

zone B: $V_r = \frac{3\mathfrak{H}}{4r}(1 - 0.8\xi^2 - 7.6\xi^4 - \ldots).$

Comparison with a numerical calculation of V_r from Eqs. (14), (15) indicates that the asymptotic expressions (16) reproduce the V_r profile almost exactly if $\xi \leq 0.6$; they work quite well for $0.6 \leq \xi \leq 1$ [accurate to $\approx$(20-30)%], and are qualitatively valid for $1 < \xi < 1.5$.

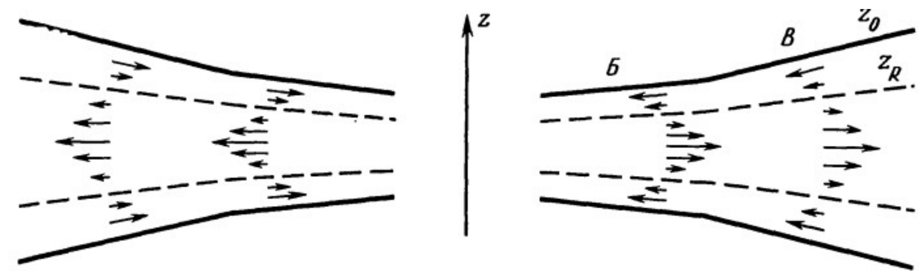
Having V_r, it is not hard to compute from Eq. (7) the vertical velocity component:

$$\text{zone C:} \quad V_z = \frac{9\mathfrak{H}}{8r}\frac{\partial z_0}{\partial r}\xi(1 - 1.4\xi^2 - 4.8\xi^4 - \ldots),$$

$$\text{zone B:} \quad V_z = \frac{3\mathfrak{H}}{4r}\frac{\partial z_0}{\partial r}\xi(1 - 0.8\xi^2 - 7.6\xi^4 - \ldots); \qquad (17)$$

V_z changes sign at the same height z_{cr} as V_r. When $z < z_{cr}$, V_z is directed away from the central plane; when $z > z_{cr}$, towards.

Next he discusses the three zones of the disk, neglecting the closest one to the star:



On substituting into Eqs. (16), (17) the appropriate values[7] of $\mathfrak{H}$, we obtain the following expressions for the velocity components in the two zones:

zone C: $V_r \approx 9\cdot10^5\alpha^{4/5}m^{-1/5}\dot{m}^{3/10}\mathfrak{R}^{-1/4}(1 - 1.4\xi^2 - 4.8\xi^4 - \ldots),$
$V_z \approx 1{,}1\cdot10^4\alpha^{7/10}m^{-3/10}\dot{m}^{9/20}\mathfrak{R}^{-1/8}\xi(1 - 1.4\xi^2 - 4{,}8\xi^4 - \ldots),$

zone B: $V_r \approx 2{,}6\cdot10^6\alpha^{4/5}m^{-1/5}\dot{m}^{3/5}\mathfrak{R}^{-2/5}(1 - 0.8\xi^2 - 7.6\xi^4 - \ldots),$
$V_z \approx 6{,}4\cdot10^4\alpha^{7/10}m^{-3/10}\dot{m}^{3/5}\mathfrak{R}^{-7/20}\xi(1 - 0.8\xi^2 - 7.6\xi^4 - \ldots).$

Figure 2 schematically illustrates the geometry of the regular motions in the disk.

Now let us estimate the rates at which mass flows in different directions in the disk. Denoting by $\dot{M}_+$, $\dot{M}_-$ the flux toward and away from the compact object, we will have $\dot{M}_+ - \dot{M}_- = \dot{M} \equiv \text{const}(r)$. Equations (8), (10), (16) then readily yield

$$\dot{M}_- = \int_0^{z_{cr}} 4\pi r\rho V_r\, dz \approx \begin{cases} 0{,}35\dot{M} & \text{in zone C}, \\ 0{,}25\dot{M} & \text{in zone B}. \end{cases}$$

Thus the mass flow toward the center exceeds the opposite flow by a factor of $\approx$4 in zone C and 5 in zone B.

Some of the matter flowing into the disk will change direction immediately adjacent to the compact object; that is where most of the outward flow will originate. Our similarity solution is not valid in this region.

Regev solution

The disk-star boundary layer and its effect on the accretion disk structure

O. Regev

Department of Physics, Technion – Israel Institute of Technology, Haifa 32000, Israel

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Summary. The method of matched asymptotic expansions is proposed for a self consistent calculation of an accretion disk and the disk-accreting star boundary layer. A model of a thin, optically thick, region (c) accretion disk and boundary layer is calculated with the help of the method. The disk in this model is hotter and denser than in the corresponding Shakura and Sunyaev (1973) model.

The significance of the model is discussed and the method is proposed for other cases of interest.

soft X-rays, emerging from the top and bottom
Pringle and Savonije (1979) propose strong shoc
thin emitting region in the boundary layer to e
emission of dwarf novae. Finally, Tylenda (
possibility of hard X-ray emission in an optic
boundary layer without strong shocks. In a rece
al. (1982) point out that observations show that
significantly smaller than predicted by a theoreti
model.

- In Regev's solution by expansion on a small parameter of H/R is proposed. On this, later similar solutions are developed.
- We will contrast his solution with the later ones, to better understand the development.

Regev solution

- Boundary layer between the inner disk radius and stellar surface is important-there Ω_K of the infalling material changes to Ω_* in a very thin layer, compared to the disk extension in radius.
- As we obtained at slide 48, up to a half of the accretion luminosity is generated in this thin layer.
- In SS73, Pringle81 and similar, $d\Omega/dr=0$ is used at $r=r_*$ boundary.
- Solutions by Regev, applied to a disk around a white dwarf relate to region (c) in SS73.
- He searches for steady, axisymmetric solutions, with $\partial/\partial t=\partial/\partial\varphi=0$, viscosity is present (only r-φ component of the viscous stress tensor is present, the rest is neglected), the disk is optically thick, with radiation transfer treated in the diffusion approximation.

Regev solution

$$u\frac{\partial u}{\partial r}+v\frac{\partial u}{\partial z}-\Omega^2 r=-\frac{1}{\varrho}\frac{\partial p}{\partial r}-\frac{GM_*}{r^2}\left(1+\frac{z^2}{r^2}\right)^{-3/2},$$

$$u\frac{\partial v}{\partial r}+v\frac{\partial v}{\partial z}=-\frac{1}{\varrho}\frac{\partial P}{\partial z}-\frac{GM_* z}{r^3}\left(1+\frac{z^2}{r^2}\right)^{-3/2},$$

$$\frac{u}{r^2}\frac{\partial}{\partial r}(r^2\Omega)+v\frac{\partial \Omega}{\partial z}=\frac{1}{\varrho r^3}\frac{\partial}{\partial r}\left(v\varrho r^3\frac{\partial \Omega}{\partial r}\right),$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\varrho u)+\frac{\partial}{\partial z}(\varrho v)=0,$$

$$\varrho C_v\left(u\frac{\partial T}{\partial r}+v\frac{\partial T}{\partial z}\right)=-\chi P\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial v}{\partial z}\right]$$

$$+v\varrho r^2\left(\frac{\partial \Omega}{\partial r}\right)^2-\frac{1}{r}\frac{\partial}{\partial r}(rF^r)-\frac{\partial}{\partial z}F^z.$$

We are already familiar with the equations: momentum eq.in r and z directions (cylindrical coords), angular momentum (in r), mass continuity and energy eqs. Radiative energy fluxes in r and z directions are, with the assumptions from the bottom of previous slide:

$$F^r=-\frac{4ac}{3}\frac{T^3}{\kappa\varrho}\frac{\partial T}{\partial r},\quad F^z=-\frac{4ac}{3}\frac{T^3}{\kappa\varrho}\frac{\partial T}{\partial z}.$$

We need to supply the equation of state for P and opacity $\varkappa_R$.

The constant mass flux through the disk is another requirement, which is a constraint for the solution:

$$-2\pi r\int_{-\infty}^{\infty}\varrho u dz=\dot{M}\ \text{(constant)}$$

Regev solution

We will not follow Regev solution in detail but just outline it, because we will repeat it related to KK solution:

In order to proceed it is imperative to write the equations in a non-dimensional form. To this end various quantities are scaled by their typical values (denoted by a tilde $\sim$). Thus ϱ, T, v, κ are in units of $\tilde{\varrho}$, $\tilde{T}$, $\tilde{v}$, $\tilde{\kappa}$, respectively. Velocities are expressed in the units of a typical sound velocity $\tilde{v}_s$, pressure in units of $\tilde{\varrho}\tilde{v}_s^2$ and Ω in units of the Keplerian angular velocity at the surface $\Omega_{k*} \equiv (GM_*/R_*^3)^{1/2}$. Two length scales are introduced – r is scaled by R_* and z by $\tilde{H}$, a typical value for the disk half-thickness.

Before writing down the non-dimensional equations it is advantageous to state the next two assumptions (in the framework of which this entire work is done).

(v) The circular flow is highly supersonic or, equivalently, the disk is (geometrically) thin $-z/r \ll 1$ in the disk.

(vi) The viscosity in the bulk of the disk is given by an "alpha model", i.e. $\nu = \alpha v_s H$.

The above assumptions are customary and are used by SS (see also Pringle, 1981). Assumption (v) introduces a small parameter into the problem $\varepsilon \equiv (\tilde{v}_s/\Omega_{k*} R_*) \ll 1$, with the help of which a typical vertical scale of the disk $\tilde{H}$ is expressed $-\tilde{H} = \varepsilon R_*$.

The scaled non-dimensional equations are:

$$\varepsilon^2 u \frac{\partial u}{\partial r} + \varepsilon v \frac{\partial u}{\partial z} - \Omega^2 r = -\varepsilon^2 \frac{1}{\varrho}\frac{\partial P}{\partial r} - \frac{1}{r^2} + \varepsilon^2 \frac{3}{2}\frac{z^2}{r^4}, \tag{7}$$

$$\varepsilon u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial z} = -\frac{1}{\varrho}\frac{\partial P}{\partial z} - \frac{z}{r^3} + \varepsilon^2 \frac{3}{2}\frac{z^3}{r^5}, \tag{8}$$

$$\varepsilon \frac{1}{r}\frac{\partial}{\partial r}(r\varrho u) + \frac{\partial}{\partial z}(\varrho v) = 0, \tag{9}$$

$$\varepsilon \frac{u}{r^2}\frac{\partial}{\partial r}(r^2\Omega) + v\frac{\partial \Omega}{\partial z} = \varepsilon^2 \alpha \frac{1}{\varrho r^3}\frac{\partial}{\partial r}\left(v\varrho r^3 \frac{\partial \Omega}{\partial r}\right), \tag{10}$$

$$\beta_g\left(\varepsilon u \frac{\partial T}{\partial r} + v\frac{\partial T}{\partial z}\right)\varrho C_v = -\chi P\left[\varepsilon \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial z}\right] + \alpha v \varrho r^2 \left(\frac{\partial \Omega}{\partial r}\right)^2 - \eta\left[\varepsilon^2 \frac{1}{r}\frac{\partial}{\partial r}(rF^r) + \frac{\partial}{\partial z}F^z\right]. \tag{11}$$

C_v is expressed in units of $\mathscr{R}/\mu_A$ with $\mathscr{R}$ the gas constant and μ_A the mean molecular weight. The new nondimensional constants here are:

$$\beta_g \equiv \frac{\tilde{P}_{\text{gas}}}{\tilde{\varrho}\tilde{v}_s^2}; \quad \eta \equiv \beta_r \lambda,$$

where

$$\beta_r \equiv \frac{\tilde{P}_{\text{rad}}}{\tilde{\varrho}\tilde{v}_s^2} \quad \text{and} \quad \lambda \equiv \frac{4c}{\tilde{v}_s}\frac{1}{\tilde{\tau}_z}.$$

$\tilde{P}_{\text{gas}}$ and $\tilde{P}_{\text{rad}}$ are the gas and radiation pressure respectively for typical conditions, c is the velocity of light and $\tilde{\tau}_z$ is the optical depth in the vertical direction in a typical point of the disk $\tilde{\tau}_z \equiv \tilde{\kappa}\tilde{\varrho}\tilde{H}$. The physical meaning of these constants is self evident and their numerical value depends on the scaling quantities which reflect a particular regime of interest.

The scaled form of Eq. (6) is:

$$r \int_{-\infty}^{\infty} \varrho u dz = -\varepsilon \mu \dot{M}, \tag{12}$$

where $\dot{M}$ is in units of $\tilde{\dot{M}}$ and $\mu \equiv \dot{M}/(2\pi \tilde{H}^2 \tilde{\varrho}\tilde{v}_s)$. The fluxes F^r and F^z are given as in Eq. (5), but with the constant term $(4ac/3)$ now missing.

Regev solution

Equations 7-12 depend on the small parameter ε and other nondimensional constants, order of which is important for finding an approximate solution. The "method of matched asymptotic expansions" is used [Regev refers to it as "Bender & Orszag (1978, chapter 9) solution for differential eqs.which exibit a boundary layer structure", I usually refer to it as doing Taylor expansion in a small parameter ε, as this is to what we resort at the end.]. Scaling is done to the typical values as in the table:

Scaling variables	Non-dimensional constants
$R_* = 9\ 10^8$ cm	$\varepsilon = \frac{\tilde{H}}{R_*} = 1.37\ 10^{-2}$
$M_* = 1\ M_\odot$	
$\Omega_{*k} = \left(\frac{GM_*}{R_*^3}\right)^{1/2} = 0.427$ rad s^{-1}	$\beta_g = \frac{\tilde{P}_{gas}}{\tilde{\varrho}\tilde{v}_s^2} = \frac{1}{\gamma} = \frac{3}{5}$
$\tilde{T} = 10^5$ °K	
$\tilde{V}_s = \left(\frac{5}{3}\frac{\mathcal{R}}{\mu_A}\tilde{T}\right)^{1/2} = 5.265\ 10^6$ cm s^{-1}	$\beta_r = \frac{\frac{1}{3}a\tilde{T}^4}{\tilde{\varrho}\tilde{v}_s^2} = 1.819\ 10^{-3}$
$\tilde{H} = \frac{\tilde{v}_s}{\Omega_{*k}} = 1.233\ 10^7$ cm	$\lambda = \frac{4c}{\tilde{v}_s} \cdot \frac{1}{\tilde{\kappa}\tilde{\varrho}\tilde{H}} = 3.525\ 10^2$
$\tilde{\varrho} = 5\ 10^{-6}$ g cm^{-3}	$\eta = \beta_r \lambda = 0.641$
$\tilde{P} = \tilde{\varrho}\tilde{v}_s^2 = 1.386\ 10^8$ dyne cm^{-2}	$\alpha = 1$
$\tilde{\kappa} = 0.11\,\tilde{n}\tilde{T}^{-3.5} = 1.048$ cm^2 g^{-1}	$\chi = \left(\frac{\partial \ln P}{\partial \ln T}\right)_e = 1$
$\tilde{\dot{M}} = 2\ 10^{16}$ g s^{-1} $= 3.15\ 10^{-10}\ M_\odot$ yr^{-1}	$C_v = \frac{3}{2}$
	$\mu = \frac{\tilde{\dot{M}}}{2\pi\tilde{H}^2\tilde{\varrho}\tilde{v}_s} = 0.795$

The ideal gas eq.is assumed, the radiative term neglected, opacity for free-free transitions is assumed, in the main solution and also in the boundary layer. Viscosity in the boundary layer is assumed as $\nu_{BL} = K v_{turb} \Lambda$, with

K and V_{turb} constant and Λ being a characteristic length scale in the boundary layer. In the nondimensional units it is

$$\nu_{BL} = \frac{K}{\alpha}\,\frac{v_{turb}}{\tilde{v}_s}\,\frac{\Lambda}{\tilde{H}}$$

Now comes the "matched" part: solution of eq.7 in previous slide in the leading order is proportional with $r^{-3/2}$, this can not be valid at the boundary. An "outer" expansion at $r \neq 1$, $\varepsilon \to 0$ is constructed and matched to the **inner expansion, valid in the disk**.

Regev solution

1. The outer solution

Let $\Omega=\Omega_0+\varepsilon\Omega_1+\dots$; $u=u_0+\varepsilon u_1+\dots$; $v=v_0+\varepsilon v_1+\dots$ with the other variables expanded in ε in a similar way. In lowest order we obtain from Eq. (7) $\Omega_0^2=\frac{1}{r^3}$, and thus:

$$\Omega_0=r^{-3/2} \quad \text{(Keplerian).} \tag{14}$$

Equation (8) gives

$$\frac{1}{\varrho_0}\frac{\partial P_0}{\partial z}=-\frac{z}{r^3}. \tag{15}$$

v_0 has been set to be equal to zero for the following reason. From Eq. (9) it follows that $(\partial/\partial z)(\varrho_0 v_0)=0$ and thus $\varrho_0 v_0=f(r)$. At $z=0$ $v_0=0$ (symmetry), thus $f(r)=0$ and since $\varrho_0\neq 0$, $v_0=0$ everywhere.

The lowest order term in Eq. (10) is trivial $v_0(\partial\Omega_0/\partial z)=0$, and from Eq. (11) we get

$$\alpha v_0\varrho_0 r^2\left(\frac{\partial\Omega_0}{\partial r}\right)^2=\eta\frac{\partial}{\partial z}F_0^z. \tag{16}$$

Order ε of Eq. (10) gives: $u_0(\partial/\partial r)(r^2\Omega_0)=0$, which implies $u_0=0$. Using this in the order ε of Eq. (9) gives $(\partial/\partial z)(\varrho_0 v_1)=0$ and thus also $v_1=0$. Thus, the ε^2 order of Eq. (10) implies

$$ru_1\varrho_0\frac{\partial}{\partial r}(r^2\Omega_0)=\alpha\frac{\partial}{\partial r}\left(v_0\varrho_0 r^3\frac{\partial\Omega_0}{\partial r}\right). \tag{17}$$

- Similar is done for the inner region, in the case when star is fast rotating, near the breakup velocity, he assumes $\Omega_*=\varepsilon^{1/2}\Omega_{K*}$. This is different from our solution later, when we will relax this constraint, so we do not follow it further in detail here, just an outline:

$$\Sigma_0(r)=\int_{-\infty}^{\infty}\varrho_0(r,z)dz=2\bar\varrho_0(r)H(r) \tag{18}$$

with $H(r)$, the half thickness of the disk to be given from (15) by

$$H(r)=\frac{v_{s0}}{\Omega_0}=T_0^{1/2}r^{3/2}. \tag{19}$$

Also

$$v_0=v_{s0}H(r)=T_0 r^{3/2}. \tag{20}$$

Integrating now Eqs. (16) and (17) over z

$$\alpha v_0\Sigma_0 r^2\left(\frac{\partial\Omega_0}{\partial r}\right)^2=2\eta F_0^z, \tag{21}$$

$$-\mu\dot M\frac{\partial}{\partial r}(r^2\Omega_0)=\alpha\frac{\partial}{\partial r}\left(v_0\Sigma_0 r^3\frac{\partial\Omega_0}{\partial r}\right), \tag{22}$$

where condition (13) has been used in (21).

Proceeding as in SS we put $F_0^z=\frac{1}{2}T_0^4/(\kappa_0\Sigma_0)$ with $\kappa_0=\bar\varrho_0 T_0^{-3.5}$ and thus $F_0^z=T_0^{7.5}H(r)/\Sigma_0^2=T_0^8\Sigma_0^{-2}r^{3/2}$. Using (14) and (20) Eq. (21) takes the form:

$$T_0^7=\left(\frac{9\alpha}{8\eta}\right)\Sigma_0^3 r^{-3}. \tag{23}$$

Integration of (22) gives

$$r^2\Omega_0-C=-\left(\frac{\alpha}{\mu\dot M}\right)T_0\Sigma_0 r^{9/2}\frac{\partial\Omega_0}{\partial r}, \tag{24}$$

where C is a constant to be determined by the inner boundary condition. SS (and others–see Pringle, 1981) put $C=1$ arguing that $(\partial\Omega_0/\partial r)=0$ at approximately $r=1$ where $\Omega_0=1$. This is entirely inconsistent with (14). In the framework of our treatment C is left undetermined at this stage and will be found from the matching with the inner solution. This is the important difference between this and the SS solution, giving rise to a significant correction (as will be seen below). Substituting Ω_0 and v_0 in (24) solving it together with (23) for T_0 and Σ_0 one gets

$$T_0=A\alpha^{-1/5}\dot M^{3/10}r^{-3/4}(1-Cr^{-1/2})^{3/10}, \tag{25}$$

$$\Sigma_0=B\alpha^{-4/5}\dot M^{7/10}r^{-3/4}(1-Cr^{-1/2})^{7/10}, \tag{26}$$

where A and B are known constants:

$$A\equiv(\mu^3/3\eta)^{1/10};\qquad B\equiv A(8\eta/9)^{1/3}.$$

Equations (25) and (26) are equivalent to the SS formulae for region (c) of the disk except for the constant C. Other variables can be found also; for instance the inflow velocity:

$$u_1=-\mu\dot M r^{-1}\Sigma_0^{-1} \qquad \text{etc.} \tag{27}$$

Regev solution

It is convenient to use here P_0 as a variable. In our units $P_0 = 1/\gamma\varrho_0 T_0$ (γ is the adiabatic exponent $= 5/3$ in our case) and thus $\varrho_0 = \gamma P_0 / T_0$.

From the thin disk approximation $H = T_0^{1/2}/\Omega_0$ and $\Sigma_0 = 2\varrho_0 H = 2\gamma P_0 T_0^{-1/2}\Omega_0^{-1}$. Using these relations the following set of equations is obtained:

$$\frac{\partial P_0}{\partial R} = \gamma \frac{P_0}{T_0}(\Omega_0^2 - 1), \qquad (34)$$

$$\frac{\partial \Omega_0}{\partial R} = \left(\frac{\mu}{2\gamma}\right) \dot{M}\Omega_0(1-\Omega_0)\, T_0^{1/2} P_0^{-1}, \qquad (35)$$

$$\frac{\partial Q_0}{\partial R} = \frac{\mu^2}{4\eta\gamma} \dot{M}^2 \Omega_0^2 (1-\Omega_0)^2 P_0^{-1}, \qquad (36)$$

$$\frac{\partial T_0}{\partial R} = -\gamma\kappa_0 \frac{P_0}{T_0^4} Q_0 \qquad (37)$$

with $\kappa_0 = P_0 T_0^{-4.5}$ for free-free opacity. It should be pointed out that various quantities like P_0, ϱ_0 etc. are in fact z-average here (like

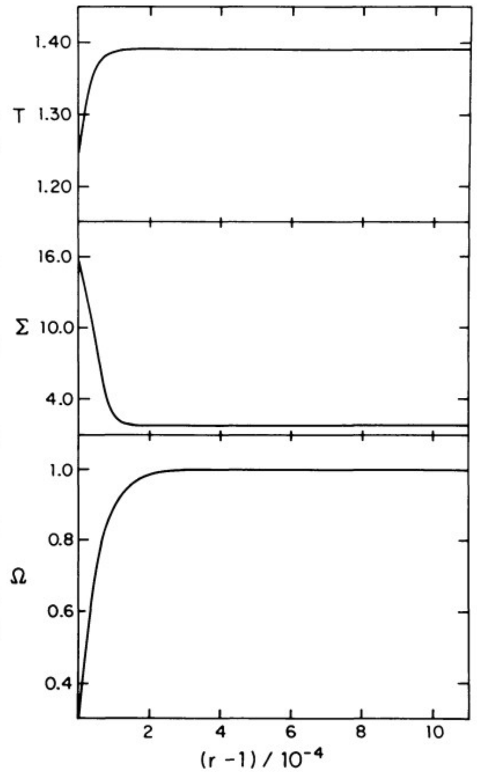

Fig. 1. The structure of the boundary layer for the model example. T – the temperature, Σ – the surface density and Ω – the angular velocity are given as a function of radius very near the star's surface. The quantities are in units used in the text (see Table)

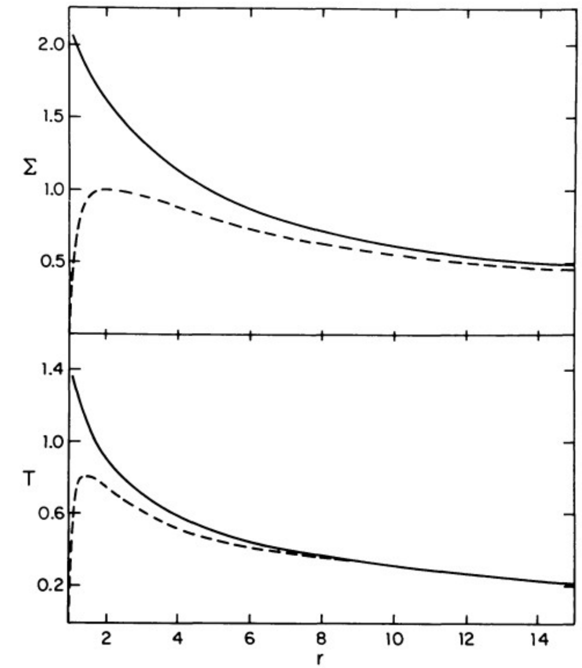

Fig. 2. The structure of the disk as obtained from the matched solution for the model example (solid curves). The dashed curves correspond to a solution with $C=1$ like that of SS (see text)

This is another z-averaged solution, only more involved than Urpin's. Those equations need to be solved numerically, in difference to KK solution, which we will derive analytically. Regev obtains curves from Figs. 1&2 for the disk structure. Since KK obtained a general solution, all this became obsolete.

Kluźniak-Kita solution

THREE-DIMENSIONAL STRUCTURE OF AN ALPHA ACCRETION DISK

Włodzimierz Kluźniak and David Kita

Physics Department, University of Wisconsin, Madison, WI 53706, USA

ABSTRACT

An analytic solution is presented to the three-dimensional problem of steady axisymmetric fluid flow through an accretion disk. The solution has been obtained through a systematic expansion in the small parameter $\epsilon = \bar{H}/\bar{R}$ (the ratio of disk thickness to its radial dimension) of the equations of viscous hydrodynamics. The equation of state was assumed to be polytropic. For all values $\alpha < 0.685$ of the viscosity parameter, we find significant backflow in the midplane of the disk occurin at all radii larger than a certain value; however, in the inner regions of the disk the fluid always flows toward the accreting object. The region of backflow is separated from the region of inflow by a surface flaring outwards from a circular locus of stagnation points situated in the midplane of the disk.

The work in KK00 paper, which exists only in arXiv version, is actually a PhD thesis of David Kita from 1995 at Madison University, USA. It is a general solution obtained similarly to Regev's, but without assumptions he used at the inner disk radius. It is a 3D, axisymmetric, purely HD solution.

- David Kita's Thesis is not available online, it is only in hard copy in the library in USA and an example in CAMK, Warsaw-but arXiv paper is actually very similar to the Thesis, all the formalism is copied in the paper.
- Paper in arXiv is with figures given at the end, I rearranged it and made a more handy version with figures positioned in their places in the text. You can download it from my orange webpage:

http://web.tiara.sinica.edu.tw/~miki/PostPrez/KK00mikiversion.pdf

Kluźniak-Kita solution

- We will go through the process of deriving asymptotic matched solutions in, again sometimes painful, detail. It is a very instructive example, and it could be of use for other similar work.
- Motivation of KK00 paper is to find the solutions which would show that the backflow, which appeared also in other solutions except Urpin and Regev, is not of a thermal origin. Urpin included thermal effects but made the simplification of zero net angular momentum flow in the disk (equivalently, his self-similar solution is valid asymptotically for large radii). KK chose the opposite route—neglect thermal effects, but include the inner boundary condition. They were able to find a global solution. They show how the backflow is fed by the inflowing fluid.
- An interesting note: Narayan & Yi (1995) went beyond the one-dimensional solutions by numerically constructing axisymmetric ADF solutions which factorize the three-dimensional equations, i.e., solutions of the type $f(r, \theta) = R(r)\Theta(\theta)$. Solutions in KK00 are **not** factorizable.

Magnetic Kluźniak-Kita disk solution (MKK)

- Not to repeat the lengthy derivations twice, we will do the magnetic version, and outline the HD solutions by setting **B**=0. *It is interesting that both are non-published work, present only in arXiv, referees did not appreciate the contributions, yet. KK00 paper has a decent following and garnered 60-some citations until now...more than many "published" papers, so it will stay in arXiv domain. The more recent magnetic generalization is still in push for publication.*

- HD solutions can be obtained since the set of HD equations is closed. For the magnetic case it is not the case, and only some general conditions can be obtained. I verify, with help of numerical simulations-which you are already familiar with-both the HD and non-ideal MHD solutions.

Magnetic Kluźniak-Kita disk solution (MKK)

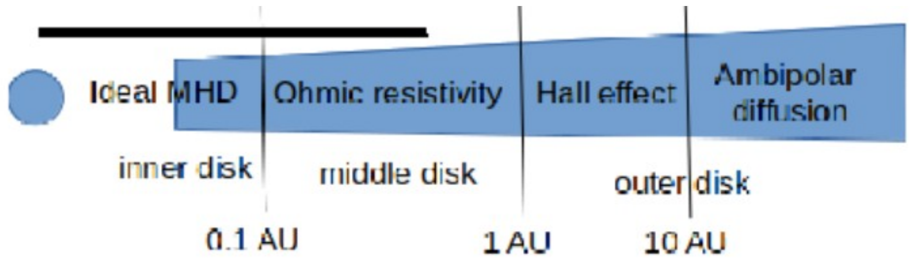

Fig. 1. Illustration of the reach of the inner, middle and outer disk regions in the case of Young Stellar Objects. In the innermost disk region the disk is in the ideal MHD regime. Further away from the star, in the middle disk region, the Ohmic resistivity adds to the viscous dissipation. In the outer disk, which we do not analyze here, other resistive terms prevail in the induction equation. Radial extension of the physical domain in our simulations is indicated with the horizontal thick black solid line.

We search for the quasi-stationary state solutions, assuming that all the heating is radiated away from the disk. This is why the dissipative viscous and resistive terms are not present in the energy equation, nor are the cooling terms. We still solve the equations in the non-ideal MHD regime, because of the viscous terms in the momentum equation, and the Ohmic resistive term in the induction equation. We are solving viscous and resistive MHD equations (in the cgs units):

$$\nabla \cdot \mathbf{B} = 0, \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (2)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v} + \eta_{\mathrm{m}} \boldsymbol{J}) = 0, \quad (3)$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \left[\rho \mathbf{v}\mathbf{v} + \left(P + \frac{\mathbf{B} \cdot \mathbf{B}}{8\pi} \right) \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} - \boldsymbol{\tau} \right] = \rho \boldsymbol{g}, \quad (4)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot \left[\left(E + P + \frac{\mathbf{B} \cdot \mathbf{B}}{8\pi} \right) \mathbf{v} - \frac{(\mathbf{v} \cdot \mathbf{B})\mathbf{B}}{4\pi} \right] = \rho \boldsymbol{g} \cdot \mathbf{v}, \quad (5)$$

where ρ, P, $\mathbf{v}$ and $\boldsymbol{B}$ are the density, pressure, velocity and magnetic field, respectively. The symbols η_{m} and $\boldsymbol{\tau}$ represent the Ohmic resistivity and the viscous stress tensor, respectively, with $\boldsymbol{\tau} = \eta \boldsymbol{T}$, where η is the dynamic viscosity and $\boldsymbol{T}$ is the strain tensor.

The gravity acceleration is $\boldsymbol{g} = -\nabla \Phi_{\mathrm{g}}$, with the gravitational potential of the star with mass $M_\star$ equal to $\Phi_{\mathrm{g}} = -GM_\star/R$. The total energy density $E = P/(\gamma - 1) + \rho(\mathbf{v} \cdot \mathbf{v})/2$, and the electric current is given by the Ampere's law $\boldsymbol{J} = \nabla \times \boldsymbol{B}/4\pi$. We assume the ideal gas with an adiabatic index $\gamma = 5/3$ and polytropic index $n = 3/2$, $(\gamma = 1 + 1/n)$.

Magnetic Kluźniak-Kita disk solution (MKK)

- To compare the magnitude of the different terms in the equations, they have to be written in normalized units. We will repeat what was done in Regev (1983): all the variables are written in the Taylor expansion, with the coefficient of expansion given by the characteristic ratio of disk height to the radius, $\varepsilon = H/R \ll 1$. For a variable X we have then $X=X_0+\varepsilon X_1+\varepsilon^2 X_2+\varepsilon^3 X_3+\ldots$ and we can compare the terms of the same order in ε for each variable.
- In the case of a viscous HD disk ($\mathbf{B} = 0$), the equations can be solved inside the disk (KK00). One can assume that the disk density decreases smoothly to zero towards the disk surface, which greatly simplifies the solution. In the case with a stellar magnetic field, the disk solution is connected with the magnetosphere of a star-disk system, through the coronal magnetic field. Reconnection and outflows complicate the solution in the magnetosphere, together with a back-reaction from the disk.
- In a magnetic case, we can obtain only the most general conditions for the disk magnetic field from the equations.
- We will be searching for the rotationally invariant stationary solutions, so that stationarity, $\partial/\partial t = 0$, and axial symmetry $\partial/\partial\varphi = 0$ both hold. [In some cases we write $\partial/\partial x=\partial_x$ for simplification.]
- Another assumption is that the structure of the disk is symmetric under reflection about the z=0 midplane. From this follows that physical quantities such as Ω, ρ, P , η, $u=v_r$, and c_s are even functions of z, while $v=v_z$ is odd under reflections through the equatorial plane. When we expand an even/odd function (e.g. Ω) in powers of $\varepsilon \ll 1$, we require each term in the expansion (e.g. Ω_i ; i = 0, 1, 2,...) to be independently even/odd. **This means that e.g. for $\Omega=\Omega_0+\varepsilon\Omega_1+\varepsilon^2\Omega_2+\varepsilon^3\Omega_3+\ldots$, when we have Ω=even, all the *terms* [including ε in ($\varepsilon\Omega_n$)] should be even=> $\varepsilon\Omega_1=0=\Omega_1=0$ and so on for all the odd terms. This is generalized in Rebusco et al. (2009).**

Magnetic Kluźniak-Kita disk solution (MKK)

- We work in the cylindrical coordinates (r, φ, z). The normalization is defined with the following equations: ε= $\tilde{c}_{\rm s}/(\tilde{R}\tilde{\Omega}) = \tilde{H}/\tilde{R} \ll 1$, so that $\tilde{c}_{\rm s} = \epsilon\tilde{R}\tilde{\Omega}$, and then $c_s' = c_s/\tilde{c}_{\rm s} = c_s/(\epsilon\tilde{R}\tilde{\Omega})$. Twiddles denote characteristic values of the variables, and primes the scaled variables. Further, $\Omega' = \Omega/\tilde{\Omega}$, $\tilde{\Omega} = \Omega_{\rm K} = \sqrt{GM_\star/\tilde{R}^3}$, $r' = r/\tilde{R}$, $z' = z/(\epsilon\tilde{R})$, $v_r' = v_r/\tilde{c}_{\rm s} = v_r/(\epsilon\tilde{R}\tilde{\Omega})$, $v_z' = v_z/\tilde{c}_{\rm s} = v_z/(\epsilon\tilde{R}\tilde{\Omega})$, $v_\varphi' = v_\varphi/(\tilde{R}\tilde{\Omega})$. The magnetic field we normalize with the Alfvén speed $\tilde{v}_{\rm A} = \tilde{B}/\sqrt{4\pi\tilde{\rho}}$ as a characteristic speed, and $\rho' = \rho/\tilde{\rho}$. Then we have $B' = B/\tilde{B} = B/(\tilde{v}_{\rm A}^2\sqrt{4\pi\tilde{\rho}})$, and $\tilde{B}$ is the normalization for all the magnetic field components: $B_r' = B_r/\tilde{B}$, $B_z' = B_z/\tilde{B}$, $B_\varphi' = B_\varphi/\tilde{B}$.

  The beta plasma parameter $\beta = P_{\rm gas}/P_{\rm mag} = 8\pi P_{\rm gas}/B^2$. With $P = P_{\rm gas}$ we can write $c_s^2 = \gamma P/\rho = \gamma\beta B^2/(8\pi\rho) = \gamma\beta v_{\rm A}^2/2$, so that $\tilde{v}_{\rm A}^2/\tilde{c}_{\rm s}^2 = 2/(\gamma\tilde{\beta})$.

  The viscosity scales with the sound speed as a characteristic velocity and the height of the disk H, so that the normalization for the kinetic viscosity is $\tilde{\nu}_{\rm v} = \tilde{c}_{\rm s}\tilde{H} = \epsilon^2\tilde{R}^2\tilde{\Omega}$, and then $\tilde{\eta} = \tilde{\rho}\tilde{\nu}_{\rm v} = \tilde{\rho}\epsilon^2\tilde{R}^2\tilde{\Omega}$. Then $\eta' = \eta/\tilde{\eta} = \eta/(\tilde{\rho}\epsilon^2\tilde{R}^2\tilde{\Omega})$. For the resistivity we choose the normalization with the Alfvén speed as a characteristic speed, so that $\tilde{\eta}_{\rm m} = \tilde{v}_{\rm A}\tilde{H} = \epsilon\tilde{R}\tilde{v}_{\rm A}$. Then $\eta_{\rm m}' = \eta_{\rm m}/\tilde{\eta}_{\rm m} = \eta_{\rm m}/(\epsilon\tilde{R}\tilde{v}_{\rm A}) = \eta_{\rm m}\sqrt{\gamma\tilde{\beta}/2}\,/(\tilde{c}_{\rm s}\epsilon\tilde{R}) = \eta_{\rm m}\sqrt{\gamma\tilde{\beta}/2}\,/(\epsilon^2\tilde{R}^2\tilde{\Omega})$.

MKK-continuity equation example

We illustrate the asymptotic approximation method in detail by deriving all the terms through the second order in the continuity equation. Other equations are derived by following the same method.

The continuity equation is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \qquad (7)$$

In the stationary case $\partial_t \rho = 0$. With the condition of axial symmetry $\partial_\varphi(\rho \mathbf{v}) = 0$:

$$\frac{1}{r}\partial_r(r\rho v_r) + \partial_z(\rho v_z) = 0. \qquad (8)$$

The normalized equation, with the terms in the order of a small parameter ϵ:

$$\frac{1}{\tilde{R}r'}\frac{1}{\tilde{R}}\partial_{r'}(r'\tilde{R}\tilde{\rho}\rho'\epsilon\tilde{\Omega}\tilde{R}v_r') + \frac{1}{\epsilon\tilde{R}}\partial_{z'}(\rho\tilde{\rho}\epsilon\tilde{\Omega}\tilde{R}v_z') = 0. \qquad (9)$$

Dividing through $\tilde{\rho}\tilde{\Omega}$ and removing the primes, we can write:

$$\frac{\epsilon}{r}\partial_r(r\rho v_r) + \partial_z(\rho v_z) = 0. \qquad (10)$$

With the expansion in ϵ in each quantity:

$$\frac{\epsilon}{r}\partial_r[r(\rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \ldots)(v_{r0} + \epsilon v_{r1} + \epsilon^2\rho_2 + \ldots)]$$
$$+\partial_z[(\rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \ldots)(v_{z0} + \epsilon v_{z1} + \epsilon^2 v_{z2} + \ldots)] = 0. \qquad (11)$$

Now we can write the terms in the different orders in ϵ.

Order ϵ^0:

$$\frac{\partial}{\partial z}(\rho_0 v_{z0}) = 0 \ \Rightarrow v_{z0} \equiv 0. \qquad (12)$$

Since ρ_0 is an even function, and v_z is odd with respect to z, at the disk equatorial plane this product is $\rho_0 v_{z0} = 0$. Since it does not depend on z, and $\rho_0 \neq 0$, we conclude that $v_{z0} = 0$ throughout the disk.

Order ϵ^1:

In the first order in ϵ it is:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho_0 v_{r0}) + \frac{\partial}{\partial z}(\rho_0 v_{z1}) = 0 \ \Rightarrow v_{z1} \equiv 0. \qquad (13)$$

See the next slide for this.

Since $v_{r0} \equiv 0$ (KK00), we have $\partial_z(\rho_0 v_{z1}) = 0 \Rightarrow \rho_0 v_{z1} =$ const along z. Since v_z is odd with respect to z, following the same argumentation as above in the zeroth order term in ϵ, we conclude that $v_{z1} = 0$ everywhere.

Order ϵ^2:

In the second order in ϵ:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho_0 v_{r1}) + \frac{\partial}{\partial z}(\rho_0 v_{z2}) = 0. \qquad (14)$$

Finding the solution for v_{r1} will give us the vertical dependence of v_{z2}.

The same procedure is carried in each of the following equations.

MKK-radial momentum

Radial momentum:

$$\epsilon^2 v_r \frac{\partial v_r}{\partial r} + \epsilon v_z \frac{\partial v_r}{\partial z} - \Omega^2 r = -\frac{1}{r^2}\left[1+\epsilon^2\left(\frac{z}{r}\right)^2\right]^{-3/2}$$
$$-\epsilon^2 n \frac{\partial c_s^2}{\partial r} + \frac{2}{\gamma\tilde{\beta}}\frac{1}{\rho}\left(\epsilon^2 B_r \frac{\partial B_r}{\partial r} + \epsilon B_z \frac{\partial B_r}{\partial z} - \epsilon^2 \frac{B_\varphi^2}{r}\right)$$
$$-\frac{\epsilon^2}{\gamma\tilde{\beta}}\frac{1}{\rho}\frac{\partial B^2}{\partial r} + \frac{\epsilon^3}{\rho r}\frac{\partial}{\partial r}\left(2\eta r \frac{\partial v_r}{\partial r}\right) + \frac{\epsilon}{\rho}\frac{\partial}{\partial z}\left(\eta \frac{\partial v_r}{\partial z}\right) \qquad (19)$$
$$+\frac{\epsilon^2}{\rho}\frac{\partial}{\partial z}\left(\eta\frac{\partial v_z}{\partial r}\right) - \epsilon^3 \frac{2\eta v_r}{\rho r^2} - \frac{2\epsilon^3}{3\rho}\frac{\partial}{\partial r}\left[\eta \frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right]$$
$$-\frac{2}{3}\frac{\epsilon^2}{\rho}\frac{\partial}{\partial r}\left(\eta\frac{\partial v_z}{\partial z}\right).$$

For an ideal gas with the polytropic index n, if adiabatic index $\gamma = 5/3$, we have $n = 3/2$.

Order ϵ^0:

$$\Omega_0 = r^{-3/2} \qquad (20)$$

Order ϵ^1:

In HD case, from vertical symmetry and also parity consideration from the bottom of slide 94, Ω_1=0, and here we obtain u_0=v_{r0}=0.

$$-2r\Omega_0\Omega_1 = \frac{2}{\gamma\tilde{\beta}}\frac{1}{\rho_0}B_{z0}\frac{\partial B_{r0}}{\partial z} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\eta_0\frac{\partial v_{r0}}{\partial z}\right) \qquad (21)$$

Since $v_{r0} \equiv 0$, from the vertical symmetry $\Omega_1 = 0$ follows, as shown in KK00 for the HD disk, see also Appendix A in Rebusco et al. (2009) for a more formal derivation. ***If this is maintained in the MHD case, we can write, with $v_{r0} = 0$:

$$B_{z0}\frac{\partial B_{r0}}{\partial z} = 0. \qquad (22)$$

Order ϵ^2:

$$2r\rho_0\Omega_0\Omega_2 = \frac{3\rho_0}{2}\frac{z^2}{r^4} + n\rho_0\frac{\partial c_{s0}^2}{\partial r} - \frac{\partial}{\partial z}\left(\eta_0\frac{\partial v_{r1}}{\partial z}\right)$$
$$-\frac{2}{\gamma\tilde{\beta}}\left(B_{r0}\frac{\partial B_{r0}}{\partial r} + B_{z0}\frac{\partial B_{r1}}{\partial z} + B_{z1}\frac{\partial B_{r0}}{\partial z} - \frac{B_{\varphi 0}^2}{r}\right) \qquad (23)$$
$$+\frac{1}{\gamma\tilde{\beta}}\frac{\partial B_0^2}{\partial r}.$$

***In Rebusco et al. (2009) is given a non-axisymmetric solution. It is obtained with help of Gegenbauer (or hyper-spherical) equation and its solutions are known in terms of combinations of the associated Legendre functions (also known as Gegenbauer polynomials), with use of Wolfram Mathematica 6 software. As mentioned above, there is also given a general discussion of eqs.for the first order in angular velocity, I copy it in the next slide:

Rebusco et al.(2009), Appendix A

In the previous order the explicit form of the viscosity η was not needed, but now the situation is not as comfortable. Following Shakura and Sunyaev [9], we posit the following form of the (ϕ, r) viscous stress tensor component

$$|\tau_{\phi r}| = \alpha P_0, \quad (25)$$

where α is an adimensional parameter. This gives

$$\eta r \left| \frac{d\Omega_0}{dr} \right| = \alpha P_0 \quad \rightarrow \quad \eta = \frac{2}{3} \alpha r^{3/2} \rho_0^{5/3} \quad (26)$$

tion is vertically dependent, i.e., the vertical distribution of stress is assumed to follow
We now substitute this η together with the polytropic relation, $\Omega_0 = r^{-3/2}$ and use
ion (12)) in equations (17-18), which become, after some straightforward algebra

$$\frac{2\alpha}{3} \rho_0^{2/3} r^3 \frac{\partial^2 u_0}{\partial z^2} - \frac{2\alpha}{3} z \frac{\partial u_0}{\partial z} + (2r\Omega_1) = 0 \quad (27)$$

$$\frac{2\alpha}{3} \rho_0^{2/3} r^3 \frac{\partial^2 (2r\Omega_1)}{\partial z^2} - \frac{2\alpha}{3} z \frac{\partial (2r\Omega_1)}{\partial z} - u_0 = 0 \quad (28)$$

$$-2r\Omega_0\Omega_1 = \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\eta \frac{\partial u_0}{\partial z}\right) \quad (17)$$

$$\frac{u_0}{r^2}\frac{\partial (r^2\Omega_0)}{\partial r} = \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\eta \frac{\partial \Omega_1}{\partial z}\right) \quad (18)$$

APPENDIX A: THE VANISHING OF Ω_1 AND u_0

Rewriting the first order equations (27) and (28) with the definitions $U \equiv u_0$ and $V \equiv 2r\Omega_1$ gives

$$\left[\left(\frac{2\alpha}{3}\right) r^3 \rho_0^{2/3}\right] \frac{\partial^2 U}{\partial z^2} - \left(\frac{2\alpha}{3}\right) z \frac{\partial U}{\partial z} + V = 0 \quad (A1)$$

$$\left[\left(\frac{2\alpha}{3}\right) r^3 \rho_0^{2/3}\right] \frac{\partial^2 V}{\partial z^2} - \left(\frac{2\alpha}{3}\right) z \frac{\partial V}{\partial z} - U = 0. \quad (A2)$$

Substituting now the zeroth order solution $\rho_0^{2/3} = (h^2 - z^2)/(5r^3)$ and rearranging leads to

$$U_{zz} - \frac{5z}{h^2 - z^2} U_z + \frac{a}{h^2 - z^2} V = 0 \quad (A3)$$

$$V_{zz} - \frac{5z}{h^2 - z^2} V_z - \frac{a}{h^2 - z^2} U = 0, \quad (A4)$$

where the subscripts z denote here the differentiation with respect to z and $a \equiv 15/(2\alpha)$ is a constant.
Using now $Q(z) \equiv (h^2 - z^2)^{5/2}$ as an integrating factor for the first two terms in both of the above equations, we see that

$$\frac{\partial}{\partial z}\left(Q \frac{\partial U}{\partial z}\right) = -a(h^2 - z^2)^{3/2}\, V \quad (A5)$$

$$\frac{\partial}{\partial z}\left(Q \frac{\partial V}{\partial z}\right) = a(h^2 - z^2)^{3/2}\, U. \quad (A6)$$

Multiplying the first equation by U, the second by V, adding and integrating over the domain $[-h, h]$, gives, after dropping the integrated parts,

$$\int_{-h}^{h} \left[\left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right] Q(z)dz = 0. \quad (A7)$$

Because $Q(z) \neq 0$, except at $z = \pm h$ and the functions U, V are bound, they must be equal to *constants*. Thus, it follows from equations (A3-A4), that $U = V = 0$, except perhaps at $z = \pm h$. However, since they are bound and constant in all the domain, they (and hence u_0 and Ω_1) must be zero identically.

MKK-azimuthal momentum

Azimuthal momentum:

$$\epsilon\frac{\rho v_r}{r^2}\frac{\partial}{\partial r}\left(r^2\Omega\right)+\rho v_z\frac{\partial\Omega}{\partial z}=\frac{\epsilon^2}{r^3}\frac{\partial}{\partial r}\left(r^3\eta\frac{\partial\Omega}{\partial r}\right)+\frac{\partial}{\partial z}\left(\eta\frac{\partial\Omega}{\partial z}\right)$$
$$+\frac{2}{\gamma\tilde{\beta}}\frac{1}{r}\left(\epsilon^2 B_r\frac{\partial B_\varphi}{\partial r}+\epsilon B_z\frac{\partial B_\varphi}{\partial z}+\epsilon^2\frac{B_\varphi B_r}{r}\right) \tag{24}$$

Order ϵ^0:

$$0=\frac{\partial}{\partial z}\left(\eta_0\frac{\partial\Omega_0}{\partial z}\right), \tag{25}$$

consistent with Eq. (20).

Order ϵ^1:

$$\frac{\rho_0 v_{r0}}{r^2}\frac{\partial}{\partial r}\left(r^2\Omega_0\right)=\frac{\partial}{\partial z}\left(\eta_0\frac{\partial\Omega_1}{\partial z}\right)+\frac{2}{\gamma\tilde{\beta}}\frac{1}{r}B_{z0}\frac{\partial B_{\varphi 0}}{\partial z} \tag{26}$$

Since $v_{r0}=\Omega_1=0$, we obtain that:

$$B_{z0}\frac{\partial B_{\varphi 0}}{\partial z}=0. \tag{27}$$

Order ϵ^2:

$$\frac{\rho_0 v_{r1}}{r}\frac{\partial}{\partial r}\left(r^2\Omega_0\right)=\frac{2}{\gamma\tilde{\beta}}\left(B_{r0}\frac{\partial B_{\varphi 0}}{\partial r}+B_{z0}\frac{\partial B_{\varphi 1}}{\partial z}+\frac{B_{r0}B_{\varphi 0}}{r}\right)$$
$$+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\eta_0\frac{\partial\Omega_0}{\partial r}\right)+\frac{\partial}{\partial z}\left(\eta_0\frac{\partial\Omega_2}{\partial z}\right).$$

Vertical momentum:

$$\epsilon v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}=-\frac{z}{r^3}\left[1+\epsilon^2\left(\frac{z}{r}\right)^2\right]^{-3/2}$$
$$-n\frac{\partial c_s^2}{\partial z}+\frac{2}{\gamma\tilde{\beta}}\frac{1}{\rho}\left(\epsilon B_r\frac{\partial B_z}{\partial r}+B_z\frac{\partial B_z}{\partial z}\right)-\frac{1}{\gamma\tilde{\beta}}\frac{1}{\rho}\frac{\partial B^2}{\partial z}$$
$$+\frac{2}{\rho}\frac{\partial}{\partial z}\left(\eta\frac{\partial v_z}{\partial z}\right)+\frac{\epsilon^2}{\rho r}\frac{\partial}{\partial r}\left(r\eta\frac{\partial v_z}{\partial r}\right) \tag{29}$$
$$-\frac{2}{3}\frac{\epsilon}{\rho}\frac{\partial}{\partial z}\left[\frac{\eta}{r}\frac{\partial}{\partial r}\left(rv_r\right)\right]-\frac{2}{3\rho}\frac{\partial}{\partial z}\left(\eta\frac{\partial v_z}{\partial z}\right)$$
$$+\frac{\epsilon}{\rho r}\frac{\partial}{\partial r}\left(\eta r\frac{\partial v_r}{\partial z}\right)$$

Order ϵ^0:

$$0=-\frac{z}{r^3}-n\frac{\partial c_{s0}^2}{\partial z}-\frac{1}{\gamma\tilde{\beta}}\frac{1}{\rho_0}\frac{\partial B_0^2}{\partial z}. \tag{30}$$

From Eqs. (16), (22) and (27), with $B_{z0}=f(r)$ it follows that $B_{r0}=f(r)$ and $B_{\varphi 0}=f(r)$. With $B_0^2=B_{r0}^2+B_{z0}^2+B_{\varphi 0}^2$ it gives $\partial_z B_0^2=0=\partial_z B_0$. For a disk in the vertical equilibrium, components of the magnetic field do not contribute in the zeroth order in ϵ to the vertical gradient of the magnetic field.

We remain with the vertical hydrostatic equilibrium equation identical to the purely HD case:

$$\frac{z}{r^3}=-n\frac{\partial c_{s0}^2}{\partial z}. \tag{31}$$

This is consistent with the demand that, for a quasi-stationary disk, the lowest order in ϵ of the magnetic field components does not contribute to the solution:

$$B_{r0}=B_{z0}=B_{\varphi 0}=0\quad\Rightarrow B_0=0. \tag{32}$$

MKK-HD solutions

Order ϵ^1:

$$\frac{2}{\gamma\tilde{\beta}}\left[B_{z0}\frac{\partial B_{z1}}{\partial z}-\frac{\partial}{\partial z}\left(B_0B_1\right)\right]-\frac{2}{3r}\frac{\partial}{\partial z}\left[\eta_0\frac{\partial}{\partial r}\left(rv_{r1}\right)\right]+\frac{1}{r}\frac{\partial}{\partial r}\left(\eta_0 r\frac{\partial v_{r1}}{\partial z}\right)=0. \quad (33)$$

With $B_{z0}=B_0=0$ we obtain:

$$\frac{\partial}{\partial z}\left[\eta_0\frac{\partial}{\partial r}\left(rv_{r1}\right)\right]=\frac{3}{2}\frac{\partial}{\partial r}\left(\eta_0 r\frac{\partial v_{r1}}{\partial z}\right). \quad (34)$$

Order ϵ^2:

$$0=B_{r0}\frac{\partial B_{z1}}{\partial r}+B_{r1}\frac{\partial B_{z0}}{\partial r}+B_{z0}\frac{\partial B_{z2}}{\partial z}+B_{z1}\frac{\partial B_{z1}}{\partial z}+\frac{1}{\gamma\tilde{\beta}}\frac{\partial}{\partial z}\left(B_1^2+2B_0B_2\right)+\frac{2}{3}\frac{\partial}{\partial z}\left[\frac{\eta_1}{r}\frac{\partial}{\partial r}\left(rv_{r1}\right)\right] \quad (35)$$

$$-2\frac{\partial}{\partial z}\left(\eta_0 v_{z2}\right)+\frac{2}{3}\frac{\partial}{\partial z}\left(\eta_0\frac{\partial v_{z2}}{\partial z}\right)-\frac{1}{r}\frac{\partial}{\partial r}\left(\eta_0 r\frac{\partial v_{r1}}{\partial z}\right).$$

If now we use Eq. (17) with $B_{r0}=0$, giving

$$\frac{\partial B_{z1}}{\partial z}=0, \quad (36)$$

we obtain:

$$\frac{3\rho_0}{2}\frac{z^3}{r^7}=2n\rho_0\frac{\partial}{\partial z}\left(c_{s0}c_{s2}\right)+n\rho_2\frac{\partial c_{s0}^2}{\partial z}+\frac{1}{\gamma\tilde{\beta}}\frac{\partial B_1^2}{\partial z}-2\frac{\partial}{\partial z}\left(\eta_0 v_{z2}\right)+\frac{2}{3}\frac{\partial}{\partial z}\left[\frac{\eta_1}{r}\frac{\partial}{\partial r}\left(rv_{r1}\right)\right]+\frac{2}{3}\frac{\partial}{\partial z}\left(\eta_0\frac{\partial v_{z2}}{\partial z}\right)-\frac{1}{r}\frac{\partial}{\partial r}\left(\eta_0 r\frac{\partial v_{r1}}{\partial z}\right). \quad (37)$$

We review already obtained results with the vanishing components of B_0 , when magnetic field influences the solution only in the higher orders in . Then Eqs. (14), (23) and (28) become the same as in a HD case:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_0 v_{r1}\right)+\frac{\partial}{\partial z}\left(\rho_0 v_{z2}\right)=0.$$

$$2r\rho_0\Omega_0\Omega_2=\frac{3\rho_0}{2}\frac{z^2}{r^4}+n\rho_0\frac{\partial c_{s0}^2}{\partial r}-\frac{\partial}{\partial z}\left(\eta_0\frac{\partial v_{r1}}{\partial z}\right)$$

$$\frac{\rho_0 v_{r1}}{r}\frac{\partial}{\partial r}\left(r^2\Omega_0\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\eta_0\frac{\partial\Omega_0}{\partial r}\right)+\frac{\partial}{\partial z}\left(\eta_0\frac{\partial\Omega_2}{\partial z}\right). \quad (38)$$

In the disk solution in Hōshi (1977) and KK00, those equations were solved by assuming that the disk density decreases towards the surface, $\rho_0\to 0$. If, instead, we supply at the disk surface a value at the boundary with the coronal density $\rho_{\rm cd}$, we obtain:

$$\rho_0=\left(\rho_{\rm cd}^{2/3}+\frac{h^2-z^2}{5r^3}\right)^{3/2}, \quad (39)$$

where h is the disk semi-thickness. The pressure and sound speed now become:

$$P_0=\left(\rho_{\rm cd}^{2/3}+\frac{h^2-z^2}{5r^3}\right)^{5/2},\ c_{s0}=\sqrt{\frac{5}{3}\left(\rho_{\rm cd}^{2/3}+\frac{h^2-z^2}{5r^3}\right)}. \quad (40)$$

MKK-HD and MHD solutions

The Hōshi (1977) solution is recovered by setting $\rho_{\rm cd} = 0$, for the boundary at the disk maximal height.

In our case, since $h \propto r$, we can write, with the proportionality constant h', $h = h'r$. Assuming the corona at the surface of the disk to be in the hydrostatic equilibrium, with $\rho_{\rm cd} \propto (\rho_{\rm c0}/r)^{3/2}$ we can write:

$$c_{\rm s0}^2 = \frac{5}{3}\left(\rho_{\rm cd}^{2/3} + \frac{h'^2 r^2 - z^2}{5r^3}\right) = \frac{5}{3}\left(\frac{k_\rho \rho_{\rm c0}}{r} + \frac{h'^2}{5r} - \frac{z^2}{5r^3}\right)$$
$$= \frac{5k_\rho \rho_{\rm c0} + h'^2}{3r}\left[1 - \left(\zeta\frac{z}{r}\right)^2\right] \propto \frac{1}{r}\left[1 - \left(\zeta\frac{z}{r}\right)^2\right], \quad (41)$$

with $\zeta^2 = 1/(5k_\rho \rho_{\rm c0} + h'^2)$, where k_ρ is the proportionality constant, and $\rho_{\rm c0} \sim 0.01$ is the ratio between the initial corona and disk density.

Now we can continue with the rest of equations.

Magnetic field solenoidality ($\nabla \cdot \mathbf{B} = 0$):

$$\frac{\epsilon}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$

Order ϵ^0:

$$\frac{\partial B_{z0}}{\partial z} = 0 \;\Rightarrow\; B_{z0} = f(r) \text{ or } B_{z0} = 0.$$

Order ϵ^1:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_{r0}) + \frac{\partial B_{z1}}{\partial z} = 0.$$

Order ϵ^2:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_{r1}) + \frac{\partial B_{z2}}{\partial z} = 0$$

MKK-MHD solutions

Radial induction equation:

$$0 = B_z \frac{\partial v_r}{\partial z} + v_r \frac{\partial B_z}{\partial z} - B_r \frac{\partial v_z}{\partial z} - v_z \frac{\partial B_r}{\partial z} + \sqrt{\frac{2}{\gamma\tilde{\beta}}} \left(\frac{\partial \eta_{\rm m}}{\partial z} \frac{\partial B_r}{\partial z} - \epsilon \frac{\partial \eta_{\rm m}}{\partial z} \frac{\partial B_z}{\partial r} \right) + \eta_{\rm m} \sqrt{\frac{2}{\gamma\tilde{\beta}}} \left(\frac{\partial^2 B_r}{\partial z^2} - \epsilon \frac{\partial^2 B_z}{\partial r \partial z} \right). \tag{42}$$

Order ϵ^0:

$$\frac{\partial \eta_{\rm m0}}{\partial z} \frac{\partial B_{r0}}{\partial z} + \eta_{\rm m0} \frac{\partial^2 B_{r0}}{\partial z^2} = 0. \tag{43}$$

If we multiply this with B_{z0}, the first term equals zero because of Eq. (22), and we remain with the second term:

$$\eta_{\rm m0} B_{z0} \frac{\partial^2 B_{r0}}{\partial z^2} = 0. \tag{44}$$

If all the zeroth-order magnetic field components are zero, $B_{r0} = 0$ and we remain with

$$0 = 0. \tag{45}$$

Order ϵ^1:

$$B_{z0} \frac{\partial v_{r1}}{\partial z} + \sqrt{\frac{2}{\gamma\tilde{\beta}}} \left(\frac{\partial \eta_{\rm m0}}{\partial z} \frac{\partial B_{r1}}{\partial z} + \frac{\partial \eta_{\rm m1}}{\partial z} \frac{\partial B_{r0}}{\partial z} - \frac{\partial \eta_{\rm m0}}{\partial z} \frac{\partial B_{z0}}{\partial r} \right) + \sqrt{\frac{2}{\gamma\tilde{\beta}}} \left(\eta_{\rm m0} \frac{\partial^2 B_{r1}}{\partial z^2} + \eta_{\rm m1} \frac{\partial^2 B_{r0}}{\partial z^2} - \eta_{\rm m0} \frac{\partial^2 B_{z0}}{\partial r \partial z} \right) = 0. \tag{46}$$

With the components of B_0 vanishing, we remain with:

$$\frac{\partial}{\partial z} \left(\eta_{\rm m0} \frac{\partial B_{r1}}{\partial z} \right) = 0. \tag{47}$$

Order ϵ^2:

$$0 = B_{z0} \frac{\partial v_{r2}}{\partial z} + B_{z1} \frac{\partial v_{r1}}{\partial z} + v_{r1} \frac{\partial B_{z1}}{\partial z} - B_{r0} \frac{\partial v_{z2}}{\partial z} - v_{z2} \frac{\partial B_{r0}}{\partial z} + \sqrt{\frac{2}{\gamma\tilde{\beta}}} \left(\frac{\partial \eta_{\rm m0}}{\partial z} \frac{\partial B_{r2}}{\partial z} + \frac{\partial \eta_{\rm m1}}{\partial z} \frac{\partial B_{r1}}{\partial z} + \frac{\partial \eta_{\rm m2}}{\partial z} \frac{\partial B_{r0}}{\partial z} - \frac{\partial \eta_{\rm m1}}{\partial r} \frac{\partial B_{z0}}{\partial r} - \frac{\partial \eta_{\rm m0}}{\partial z} \frac{\partial B_{z1}}{\partial r} \right) + \sqrt{\frac{2}{\gamma\tilde{\beta}}} \left(\eta_{\rm m0} \frac{\partial^2 B_{r2}}{\partial z^2} + \eta_{\rm m1} \frac{\partial^2 B_{r1}}{\partial z^2} - \eta_{\rm m0} \frac{\partial^2 B_{z1}}{\partial r \partial z} - \eta_{\rm m1} \frac{\partial^2 B_{z0}}{\partial r \partial z} \right). \tag{48}$$

Without the components of B_0, we remain with:

$$0 = B_{z1} \frac{\partial v_{r1}}{\partial z} + \sqrt{\frac{2}{\gamma\tilde{\beta}}} \left(\frac{\partial \eta_{\rm m0}}{\partial z} \frac{\partial B_{r2}}{\partial z} + \frac{\partial \eta_{\rm m1}}{\partial z} \frac{\partial B_{r1}}{\partial z} - \frac{\partial \eta_{\rm m0}}{\partial z} \frac{\partial B_{z1}}{\partial r} + \eta_{\rm m0} \frac{\partial^2 B_{r2}}{\partial z^2} + \eta_{\rm m1} \frac{\partial^2 B_{r1}}{\partial r^2} - \eta_{\rm m0} \frac{\partial B_{z2}}{\partial r \partial z} \right).$$

MKK-MHD solutions

Azimuthal induction equation:

$$0 = \epsilon r B_r \frac{\partial \Omega}{\partial r} + \epsilon r \Omega \frac{\partial B_r}{\partial r} + \epsilon \Omega B_r + r B_z \frac{\partial \Omega}{\partial z} + r \Omega \frac{\partial B_z}{\partial z} - \epsilon^2 v_r \frac{\partial B_\varphi}{\partial r} - \epsilon v_z \frac{\partial B_\varphi}{\partial z} - \epsilon^2 B_\varphi \frac{\partial v_r}{\partial r} - \epsilon B_\varphi \frac{\partial v_z}{\partial z} + \sqrt{\frac{2}{\gamma\tilde{\beta}}} \left(\frac{\epsilon^3 B_\varphi}{r} \frac{\partial \eta_{\rm m}}{\partial r} + \epsilon^3 \frac{\partial \eta_{\rm m}}{\partial r} \frac{\partial B_\varphi}{\partial r} + \epsilon \frac{\partial \eta_{\rm m}}{\partial z} \frac{\partial B_\varphi}{\partial z} \right) + \eta_{\rm m} \sqrt{\frac{2}{\gamma\tilde{\beta}}} \left(\frac{\epsilon^3}{r} \frac{\partial B_\varphi}{\partial r} - \frac{\epsilon^3 B_\varphi}{r^2} + \epsilon^3 \frac{\partial^2 B_\varphi}{\partial r^2} + \epsilon \frac{\partial^2 B_\varphi}{\partial z^2} \right). \quad (50)$$

Order ϵ^0:

$$r \Omega_0 \frac{\partial B_{z0}}{\partial z} = 0 \Rightarrow \frac{\partial B_{z0}}{\partial z} = 0, \quad (51)$$

in agreement with Eq. (16).

Order ϵ^1:

$$r B_{r0} \frac{\partial \Omega_0}{\partial r} + r \Omega_0 \frac{\partial B_{r0}}{\partial r} + r \Omega_0 \frac{\partial B_{z1}}{\partial z} + \sqrt{\frac{2}{\gamma\tilde{\beta}}} \left(\frac{\partial \eta_{\rm m0}}{\partial z} \frac{\partial B_{\varphi 0}}{\partial z} + \eta_{\rm m0} \frac{\partial^2 B_{\varphi 0}}{\partial z^2} \right) = 0, \quad (52)$$

which, with vanishing components of B_0, becomes:

$$r \Omega_0 \frac{\partial B_{z1}}{\partial z} = 0. \quad (53)$$

This confirms Eq. (36).

Order ϵ^2:

$$0 = r \frac{\partial B_{r1}}{\partial r} \frac{\partial \Omega_0}{\partial r} + r \Omega_0 \frac{\partial B_{r1}}{\partial r} + r B_{z0} \frac{\partial \Omega_2}{\partial z} + r \Omega_0 \frac{\partial B_{z2}}{\partial z} + \sqrt{\frac{2}{\gamma\tilde{\beta}}} \left(\frac{\partial \eta_{\rm m1}}{\partial z} \frac{\partial B_{\varphi 0}}{\partial z} + \frac{\partial \eta_{\rm m0}}{\partial z} \frac{\partial B_{\varphi 1}}{\partial z} + \eta_{\rm m0} \frac{\partial^2 B_{\varphi 1}}{\partial z^2} + \eta_{\rm m1} \frac{\partial^2 B_{\varphi 0}}{\partial z^2} \right). \quad (54)$$

Without the vanishing components of B_0 it becomes:

$$r \frac{\partial B_{r1}}{\partial r} \frac{\partial \Omega_0}{\partial r} + r \Omega_0 \frac{\partial B_{r1}}{\partial r} + r \Omega_0 \frac{\partial B_{z2}}{\partial z} + \sqrt{\frac{2}{\gamma\tilde{\beta}}} \left(\frac{\partial \eta_{\rm m0}}{\partial z} \frac{\partial B_{\varphi 1}}{\partial z} + \eta_{\rm m0} \frac{\partial^2 B_{\varphi 1}}{\partial z^2} \right) = 0. \quad (55)$$

Vertical induction equation:

$$0 = \frac{v_z B_r}{r} - \frac{v_r B_z}{r} + B_r \frac{\partial v_z}{\partial r} - v_r \frac{\partial B_z}{\partial r} + v_z \frac{\partial B_r}{\partial r} - B_z \frac{\partial v_r}{\partial r} + \sqrt{\frac{2}{\gamma\tilde{\beta}}} \left(\epsilon \frac{\partial \eta_{\rm m}}{\partial r} \frac{\partial B_z}{\partial r} - \frac{\partial \eta_{\rm m}}{\partial r} \frac{\partial B_r}{\partial z} \right) + \eta_{\rm m} \sqrt{\frac{2}{\gamma\tilde{\beta}}} \left(\frac{\epsilon}{r} \frac{\partial B_z}{\partial r} - \frac{1}{r} \frac{\partial B_r}{\partial r} - \frac{\partial^2 B_r}{\partial r \partial z} + \epsilon \frac{\partial^2 B_z}{\partial r^2} \right). \quad (56)$$

MKK-MHD solutions

Order ϵ^0:

$$\eta_{\mathrm{m}0}\left(\frac{1}{r}\frac{\partial B_{r0}}{\partial r}+\frac{\partial^2 B_{r0}}{\partial r\partial z}\right)+\frac{\partial \eta_{\mathrm{m}0}}{\partial r}\frac{\partial B_{r0}}{\partial z}=0, \tag{57}$$

giving, with $B_{r0}=0$:

$$0=0. \tag{58}$$

Order ϵ^1:

$$\begin{aligned} v_{r1}\left(\frac{B_{z0}}{r}+\frac{\partial B_{z0}}{\partial r}\right)+B_{z0}\frac{v_{r1}}{\partial r} \\ =\sqrt{\frac{2}{\gamma\tilde{\beta}}}\left[\frac{\partial \eta_{\mathrm{m}0}}{\partial r}\left(\frac{\partial B_{z0}}{\partial r}-\frac{\partial B_{r1}}{\partial z}\right)-\frac{\partial \eta_{\mathrm{m}1}}{\partial r}\frac{\partial B_{r0}}{\partial z}\right. \\ +\frac{\eta_{\mathrm{m}0}}{r}\left(\frac{\partial B_{z0}}{\partial r}-\frac{\partial B_{r1}}{\partial r}\right)-\eta_{\mathrm{m}0}\frac{\partial^2 B_{r1}}{\partial r\partial z} \\ \left.-\eta_{\mathrm{m}1}\left(\frac{1}{r}\frac{\partial B_{r0}}{\partial r}+\frac{\partial^2 B_{r0}}{\partial r\partial z}\right)\right]. \end{aligned} \tag{59}$$

With vanishing components of B_0:

$$\frac{\partial}{\partial r}\left(\eta_{\mathrm{m}0}\frac{\partial B_{r1}}{\partial z}\right)+\frac{\eta_{\mathrm{m}0}}{r}\frac{\partial B_{r1}}{\partial r}=0. \tag{60}$$

Order ϵ^2:

$$\begin{aligned} &\frac{1}{r}\Big(v_{z2}B_{r0}-v_{r1}B_{z1}-v_{r2}B_{z0}\Big)+B_{r0}\frac{\partial v_{z2}}{\partial r}+v_{z2}\frac{\partial B_{r0}}{\partial r} \\ &-\left(B_{z0}\frac{\partial v_{r2}}{\partial r}+v_{r2}\frac{\partial B_{z0}}{\partial r}\right)-\left(B_{z1}\frac{\partial v_{r1}}{\partial r}+v_{r1}\frac{\partial B_{z1}}{\partial r}\right) \\ &=\sqrt{\frac{2}{\gamma\tilde{\beta}}}\left[\frac{\partial \eta_{\mathrm{m}0}}{\partial r}\left(\frac{\partial B_{r2}}{\partial z}-\frac{\partial B_{z1}}{\partial r}\right)+\frac{\partial \eta_{\mathrm{m}1}}{\partial r}\left(\frac{\partial B_{r1}}{\partial z}\right.\right. \\ &\left.-\frac{\partial B_{z0}}{\partial r}\right)+\frac{\partial \eta_{\mathrm{m}2}}{\partial r}\frac{\partial B_{r0}}{\partial z}+\frac{\eta_{\mathrm{m}0}}{r}\left(\frac{\partial B_{r2}}{\partial r}-\frac{\partial B_{z1}}{\partial r}\right) \\ &+\frac{\eta_{\mathrm{m}1}}{r}\left(\frac{\partial B_{r1}}{\partial r}-\frac{\partial B_{z0}}{\partial r}\right)+\frac{\eta_{\mathrm{m}2}}{r}\frac{\partial B_{r0}}{\partial r}+\eta_{\mathrm{m}0}\frac{\partial^2 B_{r2}}{\partial r\partial z} \\ &\left.+\eta_{\mathrm{m}1}\frac{\partial^2 B_{r1}}{\partial r\partial z}+\eta_{\mathrm{m}2}\frac{\partial^2 B_{r0}}{\partial r\partial z}-\eta_{\mathrm{m}0}\frac{\partial^2 B_{z1}}{\partial r^2}-\eta_{\mathrm{m}1}\frac{\partial^2 B_{z0}}{\partial r^2}\right], \end{aligned} \tag{61}$$

which with vanishing components of B_0 becomes:

$$\begin{aligned} 0=-\frac{1}{r}v_{r1}B_{z1}-B_{zr1}\frac{\partial v_{r1}}{\partial r}-v_{r2}\frac{\partial B_{z2}}{\partial r} \\ =\sqrt{\frac{2}{\gamma\tilde{\beta}}}\left[\frac{\partial \eta_{\mathrm{m}0}}{\partial r}\left(\frac{\partial B_{r2}}{\partial z}-\frac{\partial B_{z1}}{\partial r}\right)\right. \\ +\frac{\eta_{\mathrm{m}0}}{r}\left(\frac{\partial B_{r2}}{\partial r}-\frac{\partial B_{z1}}{\partial r}\right)+\frac{\partial \eta_{\mathrm{m}1}}{\partial r}\frac{\partial B_{r1}}{\partial z} \\ \left.+\frac{\eta_{\mathrm{m}1}}{r}\frac{\partial B_{r1}}{\partial r}+\eta_{\mathrm{m}0}\frac{\partial^2 B_{r2}}{\partial r\partial z}+\eta_{\mathrm{m}1}\frac{\partial^2 B_{r1}}{\partial r\partial z}-\eta_{\mathrm{m}0}\frac{\partial^2 B_{z1}}{\partial r^2}\right]. \end{aligned} \tag{62}$$

MKK-MHD solutions

Energy equation:

$$\epsilon^4 \tilde{R}^2\tilde{\Omega}^2 n\rho v_r \frac{\partial c_s^2}{\partial r} + \epsilon^3 \tilde{R}^2\tilde{\Omega}^2 n\rho v_z \frac{\partial c_s^2}{\partial z} + \epsilon\rho\tilde{v}^2 v_z \frac{\partial v^2/2}{\partial z}$$
$$+\epsilon^2\rho\tilde{v}_{\rm A}^2 v_r \frac{\partial v_{\rm A}}{\partial r} + \epsilon\rho\tilde{v}_{\rm A}^2 v_z \frac{\partial v_{\rm A}^2}{\partial z} + \epsilon^2\rho\tilde{v}^2 v_r \frac{\partial v/2}{\partial r}$$
$$+\left[\epsilon^2\rho\tilde{\Omega}^2\tilde{R}^2\frac{v_r}{r^2} + \epsilon^3\rho\tilde{\Omega}^2\tilde{R}^2\frac{v_z z}{r^3}\right]\left[1+\epsilon^2\left(\frac{z}{r}\right)^2\right]^{-3/2}$$
$$= \frac{\tilde{v}_{\rm A}^2 B_r}{r}\left(\epsilon^2 v_r B_r + \epsilon\Omega r B_\varphi\right)$$
$$+\tilde{v}_{\rm A}^2\frac{\partial}{\partial r}\left(\epsilon^2 v_r B_r^2 + \epsilon^2 v_z B_z B_r + \epsilon\Omega r B_\varphi B_r\right)$$
$$+\tilde{v}_{\rm A}^2\frac{\partial}{\partial z}\left(\epsilon v_r B_r B_z + \epsilon v_z B_z^2 + \Omega r B_\varphi B_z\right). \quad (63)$$

Order ϵ^0:

$$0 = \tilde{v}_{\rm A}^2\frac{\partial}{\partial z}(\Omega_0 r B_{\varphi 0} B_{z0}) \Rightarrow B_{\varphi 0}\frac{\partial B_{z0}}{\partial z} + B_{z0}\frac{\partial B_{\varphi 0}}{\partial z} = 0,$$

which, with the first term vanishing by Eq. (16), confirms the Eq. (27).

Order ϵ^1:

$$B_{\varphi 0}\left(\frac{B_{r0}}{2r} + \frac{\partial B_{r0}}{\partial r} + \frac{\partial B_{z1}}{\partial z}\right) + B_{z0}\frac{\partial B_{\varphi 1}}{\partial z}$$
$$+B_{r0}\frac{\partial B_{\varphi 0}}{\partial r} + B_{z1}\frac{\partial B_{\varphi 0}}{\partial z} = 0. \quad (64)$$

With Eq. 17, we can write:

$$B_{r0}\left(\frac{\partial B_{\varphi 0}}{\partial r} - \frac{B_{\varphi 0}}{2r}\right) + B_{z0}\frac{\partial B_{\varphi 1}}{\partial z} + B_{z1}\frac{\partial B_{\varphi 0}}{\partial z} = 0. \quad (65)$$

Order ϵ^2:

$$\Omega_0 B_{r0}B_{\varphi 1} + \frac{\partial}{\partial r}\left[r\Omega_0\left(B_{r0}B_{\varphi 1} + B_{r1}B_{\varphi 0}\right)\right]$$
$$+\frac{\partial}{\partial z}\left[v_{r1}B_{r0}B_{z0} + r\Omega_0\Big(B_{z2}B_{\varphi 0} + B_{z1}B_{\varphi 1}\right.$$
$$\left.+B_{z0}B_{\varphi 2}\Big) + r\Omega_2 B_{z0}B_{\varphi 0}\right] = 0, \quad (66)$$

which with $\partial_z B_{z0} = 0$ can be recast into:

$$\Omega_0 B_{r0}B_{\varphi 1} + \frac{\partial}{\partial r}\left[r\Omega_0\left(B_{r0}B_{\varphi 1} + B_{r1}B_{\varphi 0}\right)\right]$$
$$+B_{z0}\frac{\partial}{\partial z}\left[v_{r1}B_{r0} + r\Big(\Omega_0 B_{\varphi 2} + \Omega_2 B_{\varphi 0}\Big)\right] = 0, \quad (67)$$

In all three orders in ϵ, with $B_{i0} = 0$ with $i = (r, \varphi, z)$ we obtain identities 0=0, confirming that our assumptions and results are in agreement with the energy equation.

MKK-MHD solutions

- We list now the solutions. Far away from the star, where we expect a small effect of the magnetic field, solutions in the simulations should not differ much from the HD solutions. Closer to the star, the magnetic field influence increases and the change in results will be larger. Higher order terms in the MHD solution may differ from those in KK00.

• $v_{r0} = v_{z0} = v_{z1} = \Omega_1 = c_{s1} = \rho_1 = 0$, as found in HD case.

we readily obtain $\Omega_0 = r^{-3/2}$. This solution is valid equally in the HD and MHD cases.
• $B_0 = 0$ and also $B_{i0} = 0$, with $i = (r, \varphi, z)$. Magnetic field influences the disk only in the higher orders in a small parameter ϵ.
• $\partial_z B_{z1} = 0$, vertical dependence of the leading component of the magnetic field in the vertical direction is f(r) only.
• Vertical hydrostatic equilibrium condition gives the same solutions for the lowest order in ϵ for the density (see Eq. 39), pressure and the sound speed as in the HD solution. The difference from KK00 is that now the disk surface boundary condition is not vacuum, but a corona with the density $\rho_{\rm cd}(r)$ at the disk interface. The zeroth order profile of density, pressure, and the sound speed are:

$$\rho_0(r,z) = \left[\rho_{\rm cd}^{2/3}(r) + \frac{h^2 - z^2}{5r^3}\right]^{3/2},$$

$$P_0(r,z) = \left[\rho_{\rm cd}^{2/3}(r) + \frac{h^2 - z^2}{5r^3}\right]^{5/2},$$

$$c_{s0}(r,z) = \sqrt{\frac{5}{3}\left[\rho_{\rm cd}^{2/3}(r) + \frac{h^2 - z^2}{5r^3}\right]}. \qquad (68)$$

Clearly, $\rho_0(r,h) = \rho_{\rm cd}(r)$.

MKK-HD and MHD numerical solutions

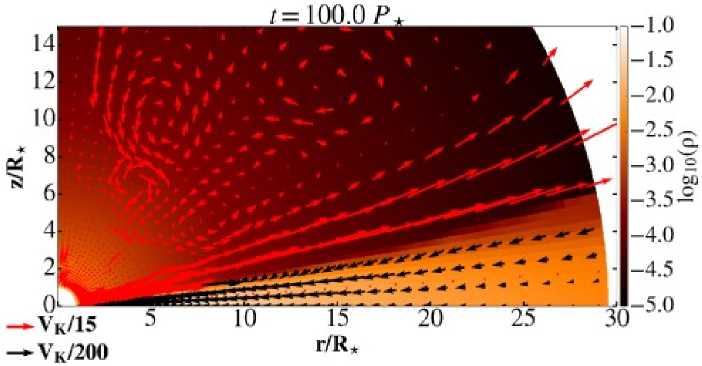

Fig. 2. Capture of our hydrodynamic simulation after t=100

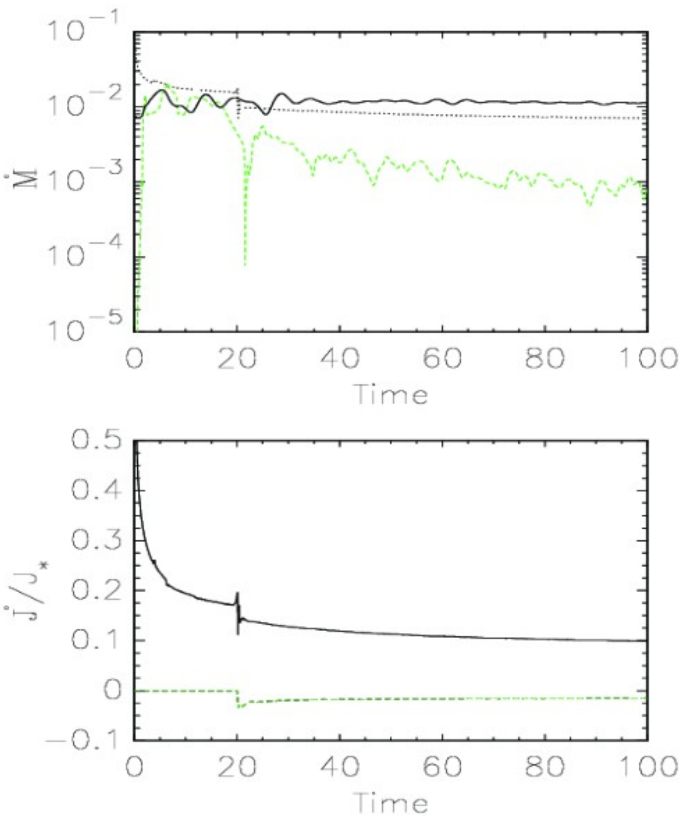

Fig. 3. Illustration of the quasi-stationarity of our solution ir

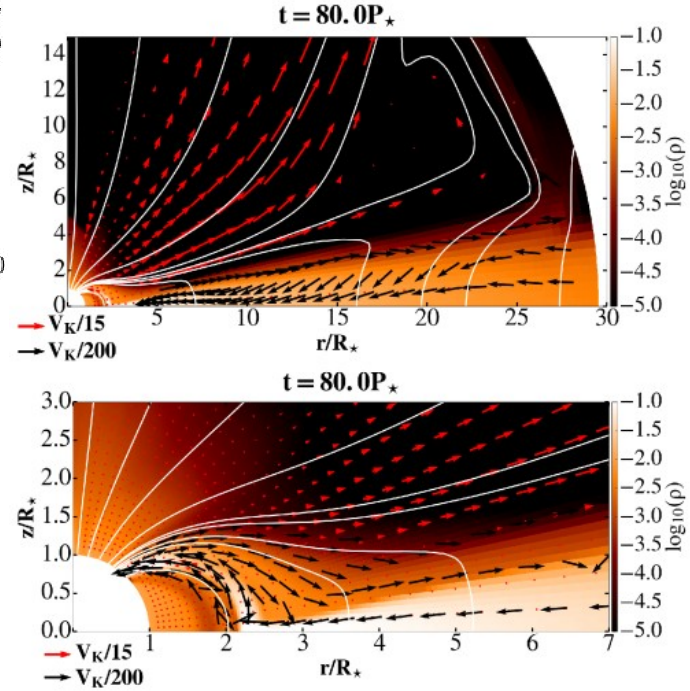

Fig. 4. Captures of our magnetic simulation after t=80 stellar rotations (top panel), and a zoom closer to the star (bottom panel) to better show the accretion column. Colors and vectors have the same meaning as in Fig. 2. Note the different scale of the poloidal velocity (arrows below the panels). A sample of the poloidal magnetic field lines is shown with the solid lines.

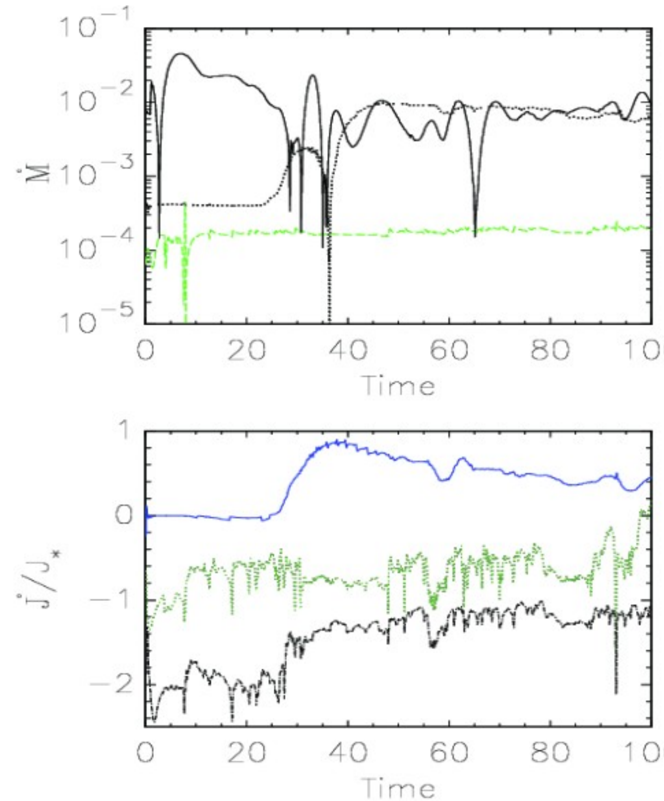

Fig. 5. Illustration of the quasi-stationarity in our magnetic case solutions, in the same units as in Fig. 3. *Top panel:* the

MKK- analytical expressions from the numerical solutions

We can write the results in our simulations as simple functions obtained in KK00, with coefficients of proportionality we find from our simulations:

$$\rho(r,z) = \frac{k_1}{r^{3/2}}\left[1-\left(\zeta_1\frac{z}{r}\right)^2\right]^{3/2}, \tag{69}$$

$$v_r(r,z) = \frac{k_2}{r^{1/2}}\left[1+(\zeta_2 z)^2\right], \tag{70}$$

$$v_z(r,z) \approx \frac{z}{r}v_r(r,z) = k_3\frac{z}{r^{3/2}}\left[1+(\zeta_3 z)^2\right],$$

$$v_\varphi(r,z) = \frac{k_4}{\sqrt{r}},\ \Omega = \frac{v_\varphi}{r} = \frac{k_4}{r^{3/2}}.$$

Magnetic field components are proportional to r^{-3}, as expected for the dipole stellar field, and depend linearly on height above the disk midplane:

$$B_r(r,z) = \frac{k_5}{r^3}z,\ B_z(r,z) = \frac{k_6}{r^3}z,\ B_\varphi(r,z) = \frac{k_7}{r^3}z, \tag{71}$$

In the case of B_r, the linear dependence is a consequence of the boundary condition at the disk equatorial plane, where the magnetic field components are reflected, with the change in sign of the component tangential to the boundary. This means that the radial magnetic field component $B_r \rightarrow 0$ at the equatorial plane, and is slowly increasing above (and below) that plane, in the densest parts of the disk.

$$\eta(r,z) = \frac{k_8}{r}\left[1-\left(\zeta_8\frac{z}{r}\right)^2\right]^2, \tag{72}$$

$$\eta_{\rm m}(r,z) = k_9\sqrt{r}\left[1-\left(\zeta_9\frac{z}{r}\right)^2\right]^{1/2}.$$

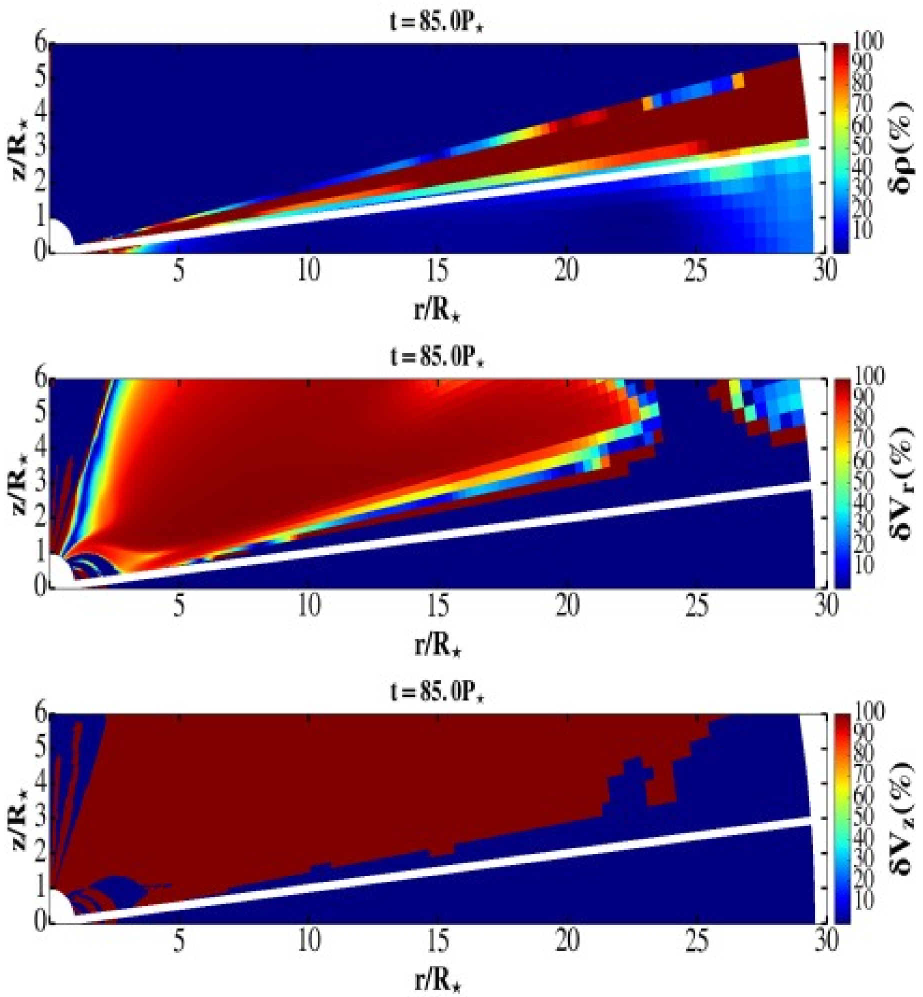

Fig. A.1. Difference between our numerical solutions and analytical expressions, in percentage of the value in the simulations. Our analytical solution is mostly inside the 10% margin everywhere inside the thin disk region, below the thick white solid line demarcating $h = 0.1r$ dependence, where h is the disk height. Close to the star and accretion column footpoint, our simulations are in the ideal MHD regime, so the analytical expressions fail there. It is also failing close to the outer boundary, where the material is fed into the disk by the amount based on the analytical solution in purely HD approach.

Umurhan generalized γ solutions

Global axisymmetric dynamics of thin viscous accretion disks*

O. M. Umurhan[1,3], A. Nemirovsky[1], O. Regev[1,2,4,5], and G. Shaviv[1,2]

[1] Department of Physics, Technion-Israel Institute of Technology, 32000 Haifa, Israel
e-mail: mumurhan@physics.technion.ac.il
[2] Asher Space Research Institute, Technion-Israel Institute of Technology, 32000 Haifa, Israel
[3] Department of Astronomy, City College of San Francisco, San Francisco, CA 94112, USA
[4] Department of Astrophysics, American Museum of Natural History, New York, NY 10024, USA
[5] Department of Astronomy, Columbia University, New York, NY 10028, USA

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ABSTRACT

The purpose of this paper is to explore the steady state and dynamical behavior of thin, axisymmetric, viscous accretion disks. To facilitate an analytical treatment we replace the energy equation with a general polytropic assumption. The asymptotic expansion of Kluźniak & Kita (2000, Three-dimensional structure of an alpha accretion disk [arXiv:astro-ph/0006266]), which extended the method of Regev (1983, A&A, 126, 146) to a full steady polytropic disk (with $n = 3/2$), is further developed and implemented for both the steady (for any polytropic index) and time-dependent problems. The spatial form and temporal behavior of selected dynamical disturbances are studied in detail. It is shown that the transient dynamics resulting from initial perturbations on the linearly stable steady state gives rise to substantial growth of perturbations. We identify the initial perturbation space which leads to such transient growth and provide analytical solutions which manifest this behavior three terms (physical causes) responsible for the appearance of transient dynamics are identified. Two depend explicitly on the viscosity while the third one is relevant also for inviscid disks. The main conclusion we draw is that transient dynamics and, in particular, significant perturbation energy amplification occurs in disks on a global scale. We speculate on the possible implications of these findings to accretion disk theory.

Umurhan generalized γ solutions

Our starting point are the general Navier-Stokes equations

$$\rho\frac{\partial \boldsymbol{V}}{\partial t} + \rho(\boldsymbol{V}\cdot\nabla)\boldsymbol{V} = -\nabla P + \rho\boldsymbol{b} + \nabla\cdot\boldsymbol{\sigma}, \qquad (1)$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{V}) = 0, \qquad (2)$$

where $\boldsymbol{V}$ is the three dimensional velocity vector, ρ is the density, P is the pressure and $\boldsymbol{b}$ is the body force per unit mass.

Also, the disk matter's self gravity is neglected and therefore the body force derives from the gravitational potential of a central accreting object, whose mass is M, say. Thus we have $\boldsymbol{b} = -\nabla\Phi$ with

$$\Phi = -\frac{GM}{\sqrt{r^2+z^2}}. \qquad (4)$$

Regarding the viscosity of the accretion disk flow, we shall assume that the situation is such that the viscosity coefficient is greatly enhanced relatively to the microscopic one (see above in the Introduction). We will thus use one of the standard α-prescriptions (see below). If this effective viscosity enhance-

- Here we shall not assume a steady flow: we retain the time derivative terms. This will enable us to analyze the dynamical evolution of deviations from the steady KK solution. -- we are not to go into this, we were interested in the steady disk only-so **we will take only the generalization in γ**

We assume throughout this work that the disk equation of state is polytropic – both in steady and dynamical states. This means that we can assume that the pressure and density always obey the relationship $P = P(\rho) \equiv K\rho^{(1+1/n)}$, where n is the (constant) polytropic index and K is a constant. It follows that the typical scale of the pressure is $\tilde{P} = P(\tilde{\rho})$. Also, we choose the typical sound speed to be $\tilde{c}_s = \sqrt{\tilde{P}/\tilde{\rho}}$. The assumption that

Thus the Keplerian angular velocity and the polytropic relations for the pressure and sound speeds are

$$\Omega_k = \frac{1}{r^{3/2}}, \qquad P = \rho^{1+1/n},$$

$$c_s^2 = \frac{\mathrm{d}P}{\mathrm{d}\rho} = \left(1+\frac{1}{n}\right)\frac{P}{\rho} = \left(1+\frac{1}{n}\right)\rho^{1/n}. \qquad (6)$$

To be consistent with previous works (e.g. KK) we make use of the non-dimensional sound speed c_s as the dependent variable instead of the pressure P. Consequently, we replace the pressure gradient terms in the equations in the following way:

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = n\frac{\partial c_s^2}{\partial r}, \qquad \frac{1}{\rho}\frac{\partial P}{\partial z} = n\frac{\partial c_s^2}{\partial z}. \qquad (7)$$

The standard α model of Shakura & Sunyaev (1973) is based on the assumption that the only non vanishing viscous stress component is $\sigma_{r\phi}$ and it is proportional to the pressure. Following KK, we adopt this assumption and derive from it the form of viscosity coefficient, but include in the dynamical equations all the components of the stress tensor. In lowest order in ϵ the angular velocity of a disk is Keplerian and we get (with the dynamic viscosity coefficient scaled by $\tilde{c}_s\tilde{H}$) the nondimensional relation (see also RG)

$$\eta = \frac{2}{3}\frac{\alpha P}{\Omega_k} = \frac{2}{3}\frac{\alpha c_s^2\rho}{\Omega_k\left(1+\frac{1}{n}\right)}, \qquad (8)$$

where α is the viscosity parameter.

Umurhan generalized γ solutions

Here $u=v_r$ and $v=v_z$

- the equations appear as they do in KK and RG except that here we allow for

time-dependence (the time unit is $1/\tilde{\Omega}$). Consequently time-derivatives are included and all the dependant variables are functions of r, z and t –

$$\epsilon\frac{\partial u}{\partial t} + \epsilon^2 u\frac{\partial u}{\partial r} + \epsilon v\frac{\partial u}{\partial z} - \Omega^2 r = -\frac{1}{r^2}\left[1+\epsilon^2\frac{z^2}{r^2}\right]^{-3/2} + \frac{\epsilon}{\rho}\frac{\partial}{\partial z}\left(\eta\frac{\partial u}{\partial z}\right) + \epsilon^2\left[-n\frac{\partial c_s^2}{\partial r} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\eta\frac{\partial v}{\partial r}\right) - \frac{1}{\rho}\frac{\partial}{\partial r}\left(\eta\frac{2}{3}\frac{\partial v}{\partial z}\right)\right] + \epsilon^3\left[-\frac{2\eta u}{\rho r^2} + \frac{1}{\rho r}\frac{\partial}{\partial r}\left(2\eta r\frac{\partial u}{\partial r}\right) - \frac{1}{\rho}\frac{\partial}{\partial r}\left(\frac{2}{3}\frac{\eta}{r}\frac{\partial(ru)}{\partial r}\right)\right], \quad (9)$$

$$\rho\frac{\partial\Omega}{\partial t} + \epsilon\frac{\rho u}{r^2}\frac{\partial(r^2\Omega)}{\partial r} + \rho v\frac{\partial\Omega}{\partial z} = \frac{\partial}{\partial z}\left(\eta\frac{\partial\Omega}{\partial z}\right) + \epsilon^2\frac{1}{r^3}\frac{\partial}{\partial r}\left(\eta r^3\frac{\partial\Omega}{\partial r}\right) \quad (10)$$

$$\frac{\partial v}{\partial t} + \epsilon u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial z} = -\frac{z}{r^3}\left(1+\epsilon^2\frac{z^2}{r^2}\right)^{-3/2} - n\frac{\partial c_s^2}{\partial z} + \frac{4}{3}\frac{1}{\rho}\frac{\partial}{\partial z}\left(\eta\frac{\partial v}{\partial z}\right) + \epsilon\frac{1}{\rho}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\eta r\frac{\partial u}{\partial z}\right) - \frac{2}{3}\frac{\partial}{\partial z}\left(\frac{\eta}{r}\frac{\partial(ru)}{\partial r}\right)\right] + \epsilon^2\frac{1}{\rho r}\frac{\partial}{\partial r}\left(\eta r\frac{\partial v}{\partial r}\right), \quad (11)$$

$$\frac{\partial\rho}{\partial t} + \frac{\epsilon}{r}\frac{\partial(r\rho u)}{\partial r} + \frac{\partial(\rho v)}{\partial z} = 0, \quad (12)$$

where the gravitational potential of the central star has been expanded only up to second order in ϵ.

It has been shown before (see R, KK, RG) that in steady state the lowest order nonzero components of the meridional velocities are u_1 and v_2 and, in addition, the choice $\Omega_1 = \rho_1 = 0$ (and thus also $c_{s1} = 0$) can be consistently made. Guided by these results (see also below) we retain in the expansions for Ω, c_s^2, ρ and v only even powers of ϵ (in the case of v starting from v_2), while for the u expansions only odd powers are kept, starting with u_1,

$$\Omega(r,z,t) = \Omega_0(r,z) + \epsilon^2\left[\Omega_2(r,z) + \Omega_2'(r,z,t)\right] + \epsilon^4\left[\Omega_4 + \Omega_4'(r,z,t)\right] + \cdots \quad (14)$$

$$u(r,z,t) = \epsilon\left[u_1(r,z) + u_1'(r,z,t)\right] + \epsilon^3\left[u_3(r,z) + u_3'(r,z,t)\right] + \cdots \quad (15)$$

$$v(r,z,t) = \epsilon^2\left[v_2(r,z) + v_2'(r,z,t)\right] + \epsilon^4\left[v_4(r,z) + v_4'(r,z,t)\right] + \cdots \quad (16)$$

$$c_s^2(r,z,t) = c_{s0}^2(r,z) + \epsilon^2\left[c_{s2}^2(r,z) + c_{s2}^{2\,\prime}(r,z,t)\right] + \epsilon^4\left[c_{s4}^2(r,z) + c_{s4}'(r,z,t)\right] + \cdots$$

$$\rho(r,z,t) = \rho_0(r,z) + \epsilon^2\left[\rho_2(r,z) + \rho_2'(r,z,t)\right] + \epsilon^4\left[\rho_4(r,z) + \rho_4'(r,z,t)\right] + \cdots$$

- Time dependence has been introduced into the expansions at O(ε^2) for the functions Ω, ρ, c_s^2 and v while it appears at O (ε) for u. At all orders in which time dependence is introduced we split the solutions up into a sum of a steady solution and a dynamical one, denoted by a prime, and this is true for all high orders as well. from now and on all the terms of the dependent variables expansions that are dependent on time are denoted by a prime, while those without prime are just space dependent. The time-dependent part is a perturbation (not necessarily infinitesimal!) on the steady state.

Umurhan generalized γ solutions

3. Steady state

3.1. $O(1)$

Neither the techniques nor the qualitative implications of this section differ from the results of KK. All that is substantively different is that the steady solutions are here derived in terms of an arbitrary polytropic index n.

The lowest order equations are straightforwardly solved to yield solutions quite well-known in the literature (see, e.g., Hoshi 1977). These are,

$$c_{s0}^2(r,z) = \frac{h^2(r) - z^2}{2nr^3}, \qquad \Omega_0 = \Omega_k = \frac{1}{r^{3/2}},$$
$$\rho_0(r,z) = \left[\frac{c_{s0}^2(r,z)}{1+\frac{1}{n}}\right]^n. \tag{19}$$

At $z = h(r)$, $c_{s0}^2 = \rho_0 = 0$, thus $h(r)$ is explicitly determined when solving the next order steady state equations. For values of $|z| \geq h$, c_{s0}^2, ρ_0 and all other associated quantities remain zero. Additionally the relationships between the sound speed, pressure and density, as well as their vertical gradients, are given by,

$$\frac{P_0}{\rho_0} = \frac{c_{s0}^2}{1+\frac{1}{n}}, \qquad \frac{1}{\rho_0}\frac{\partial P_0}{\partial z} = n\frac{\partial c_{s0}^2}{\partial z} = -\frac{z}{r^3}. \tag{20}$$

It useful to note that we shall use in the equations to all orders only the zeroth order value of the viscosity coefficient, that is, will not actually consider it as an expandable function, but rather as a prescribed function which is based on the zeroth order of the pressure. Thus

$$\eta \equiv \eta_0 = \frac{2}{3}\alpha P_0 r^{3/2}. \tag{21}$$

3.2. $O\left(\epsilon^2\right)$

The steady-state equations at the next non-trivial order of ϵ are,

$$n\frac{\partial c_{s0}^2}{\partial r} - \frac{3}{2}\frac{z^2}{r^4} = 2\Omega_0\Omega_2 r + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\eta\frac{\partial u_1}{\partial z}\right), \tag{22}$$

$$-\frac{1}{r^3\rho_0}\frac{\partial}{\partial r}\left(\eta r^3\frac{\partial \Omega_0}{\partial r}\right) = -\frac{u_1}{r^2}\frac{\partial(r^2\Omega_0)}{\partial r} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\eta\frac{\partial \Omega_2}{\partial z}\right), \tag{23}$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(r\rho_0 u_1) + \frac{\partial}{\partial z}(\rho_0 v_2), \tag{24}$$

$$0 = \frac{3}{2}\frac{z^3}{r^5} - n\frac{\partial c_{s2}^2}{\partial z} + \frac{4}{3}\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\eta\frac{\partial v_2}{\partial z}\right) + \frac{1}{\rho_0}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\eta r\frac{\partial u_1}{\partial z}\right) - \frac{2}{3}\frac{\partial}{\partial z}\left(\frac{\eta}{r}\frac{\partial(ru_1)}{\partial r}\right)\right]. \tag{25}$$

These equations have solutions exhibiting a steady meridional flow, something which we will formalize below.

Umurhan generalized γ solutions-time dependence

Finally, it is an interesting result that in the expression for the disk height for arbitrary polytropic index (see Eq. (A.5)) the disk flares for $n > 3/2$, namely $h(r) \sim r^m$ with $m > 1$.

The form of $h(r)$ given in KK is recovered by substituting $n = 3/2$ into the general expressions (see Eq. (A.5)) given in Appendix A and reads as follows,

$$h(r) = (2\Lambda)^{\frac{1}{6}} r^{\frac{11}{12}} \left(\sqrt{r} - \sqrt{r_*} \right)^{\frac{1}{6}},$$

$$\Lambda \equiv \frac{\dot{M}_1}{\alpha} \frac{\Gamma(4)}{\Gamma(5/2)} \cdot \frac{(5)^{\frac{3}{2}}}{3\pi^{3/2}(11/2)}. \tag{26}$$

The height of the disk at the fiducial radius $r = 1$ is denoted by h_1 and is

$$h_1 \equiv (2\Lambda)^{\frac{1}{6}} \left(1 - \sqrt{r_*}\right)^{\frac{1}{6}}. \tag{27}$$

- **Time dependence**: for the dynamical evolution they allow for time dependence and include in all equations at successive orders of ε and subject them to appropriate initial conditions.
- For the radial boundary conditions, the problems posed by them are avoided by considering only the portion of the disk with an inner radius r_{in} and an outer radius r_{max}, as in the steady-state case in KK disk.
- The "surface" of the disk, in the time-dependent case, is taken to consist of the last fluid parcel which, in a steady disk, is at z = h(r). This is where the boundary conditions are imposed.
- We are not interested here in this discussion, as we were interested in steady disks, so we stop here.

Python code for thin disk: Repeating slide 108: MKK- analytical expressions from numerical solutions

Idea here is that I give you some pieces of code which can be useful as a template for work with other models, too.

We will install and run two which I already briefly presented to you during the PLUTO lectures: **comparison of analytical and numerical model** and **DUSTER code for post-processing**.

We can write the results in our simulations as simple functions obtained in KK00, with coefficients of proportionality we find from our simulations:

$$\rho(r,z) = \frac{k_1}{r^{3/2}}\left[1-\left(\zeta_1\frac{z}{r}\right)^2\right]^{3/2}, \tag{69}$$

$$\begin{aligned} v_r(r,z) &= \frac{k_2}{r^{1/2}}\left[1+(\zeta_2 z)^2\right], \qquad (70)\\ v_z(r,z) \approx \frac{z}{r}v_r(r,z) &= k_3\frac{z}{r^{3/2}}\left[1+(\zeta_3 z)^2\right],\\ v_\varphi(r,z) &= \frac{k_4}{\sqrt{r}},\ \Omega=\frac{v_\varphi}{r}=\frac{k_4}{r^{3/2}}. \end{aligned}$$

Magnetic field components are proportional to r^{-3}, as expected for the dipole stellar field, and depend linearly on height above the disk midplane:

$$B_r(r,z)=\frac{k_5}{r^3}z,\ B_z(r,z)=\frac{k_6}{r^3}z,\ B_\varphi(r,z)=\frac{k_7}{r^3}z, \tag{71}$$

In the case of B_r, the linear dependence is a consequence of the boundary condition at the disk equatorial plane, where the magnetic field components are reflected, with the change in sign of the component tangential to the boundary. This means that the radial magnetic field component $B_r \to 0$ at the equatorial plane, and is slowly increasing above (and below) that plane, in the densest parts of the disk. It is catching-up with more dramatic changes only

$$\begin{aligned} \eta(r,z) &= \frac{k_8}{r}\left[1-\left(\zeta_8\frac{z}{r}\right)^2\right]^2, \qquad (72)\\ \eta_{\rm m}(r,z) &= k_9\sqrt{r}\left[1-\left(\zeta_9\frac{z}{r}\right)^2\right]^{1/2}. \end{aligned}$$

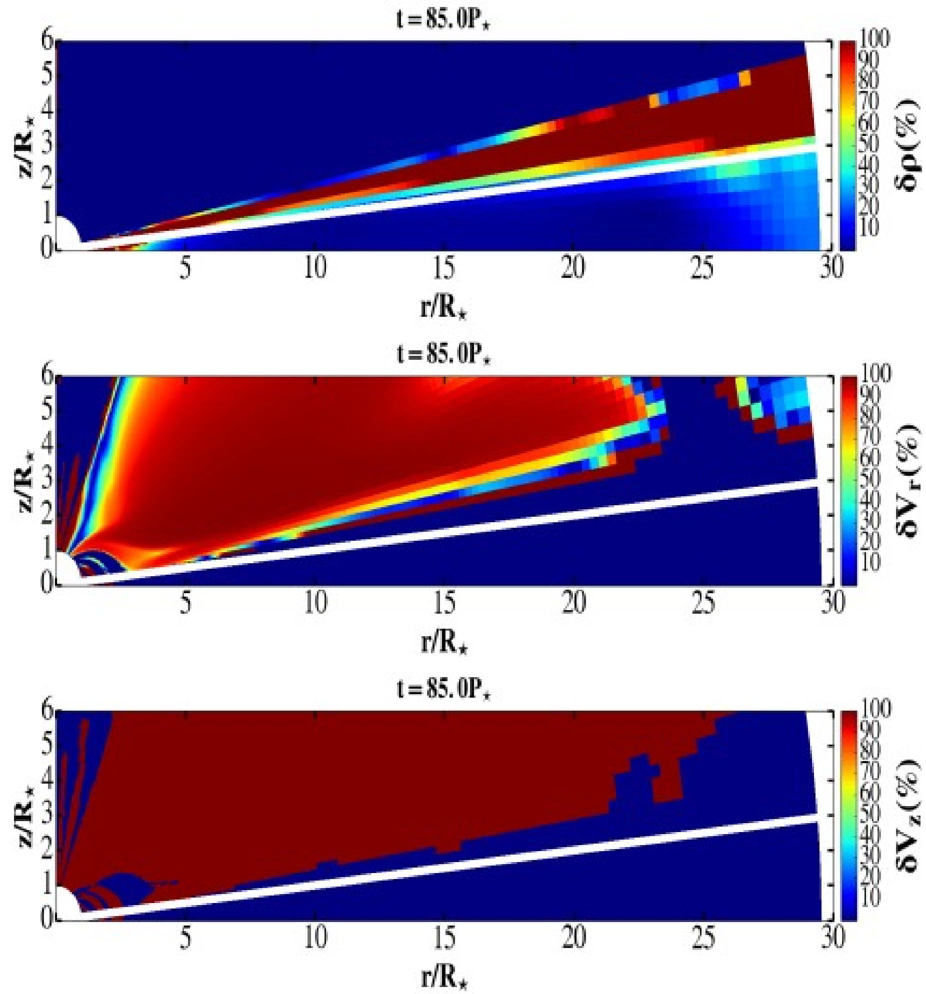

Fig. A.1. Difference between our numerical solutions and analytical expressions, in percentage of the value in the simulations. Our analytical solution is mostly inside the 10% margin everywhere inside the thin disk region, below the thick white solid line demarcating $h = 0.1r$ dependence, where h is the disk height. Close to the star and accretion column footpoint, our simulations are in the ideal MHD regime, so the analytical expressions fail there. It is also failing close to the outer boundary, where the material is fed into the disk by the amount based on the analytical solution in purely HD approach.

MKK- analytical expressions from the numerical solutions

Table 1. The proportionality coefficients in our simulations with $B_\star$=0.5 kG and 1 kG.

B(kG)	0.5	1
coef.	R=6 | R=15	R=6 | R=15
$k_{1i}\|k_{1o}$	0.9	1.2 | 0.29
$k_{2i}\|k_{2o}$	-0.01 | -0.006	$+1.2\times10^{-4}\|-2.9\times10^{-3}$
k_3	-2.65×10^{-4}	$-4.4\times10^{-3}\|-3.6\times10^{-5}$
k_4	0.255	0.255
$k_{5i}\|k_{5o}$	-0.69 | -0.41	-1.25
$k_{6i}\|k_{6o}$	-0.35 | -0.15	-0.29 | -0.19
$k_{7i}\|k_{7o}$	-2.8 | -1.1	-8.2 | -1.18
k_8	5.8×10^{-3}	$8.\times10^{-3}$
k_9	0.01	0.01
ζ_1	5.	5.
ζ_2	0.5	0.5
ζ_8	5.	6.8
ζ_9	6.	4.5

Using this table, we can “prescribe” the disk-it will fairly well describe the disk, as we saw in the previous slide.

- We could plug in any other disk model and compare with simulations or other computation result
- We write such a script, for comparison of solutions.
- I supply a template in mc_razlikaANnum.py, it can easily be modified. We test it with one of the results from simulations.

Python tool DUSTER for post-processing of our disk results

Transport of dust grain particles in the accretion disk

Robert Jaros[1], Miljenko Čemeljić[2,3], Włodek Kluźniak[2], Dejan Vinković[4] and Cezary Turski[5]

1. Nicolaus Copernicus University, Grudziadzka 5, 87-100 Toruń, Poland
2. Nicolaus Copernicus Astronomical Center, Polish Academy of Sciences, Bartycka 18, 00–716 Warsaw, Poland
3. Institute of Astronomy and Astrophysics, Academia Sinica, P.O. Box 23–141, Taipei 106, Taiwan
4. Science and Society Synergy Institute, Bana Josipa Jelačića 22,HR-40000, Čakovec, Croatia
5. Faculty of Physics, University of Warsaw, Pasteura 5, 02–093, Warsaw, Poland

Entrainment of dust particles in the flow inside and outside of the proto-planetary disk has implications for the disk evolution and composition of planets. Using quasi-stationary solutions in our star-disk simulations as a background, we add dust particles of different radii in post-processing of the results, using our Python tool DUSTER. The distribution and motion of particles in the disk is followed in the cases with and without the backflow in the disk. We also compare the results with and without the radiation pressure included in the computation.

Python tool DUSTER for post-processing of our disk results

used as input density and velocity for DUSTER code computation. Equation of motion solved by DUSTER is (Vinković, 2009, 2012):

$$\ddot{\vec{r}} = -G\frac{M_\star}{r^3}\vec{r} - \frac{\rho_{\rm gas}}{\rho_{\rm gr}}\frac{c_{\rm s}}{a}(\dot{\vec{r}} - \vec{v}_{\rm gas}) + \vec{\beta}G\frac{M_\star}{r^2}. \qquad (1)$$

We use the DUSTER tool to follow the dust under predefined forces. It is completely Newtonian, deterministic task, since we have all the needed data from simulation or numerical model.

- DUSTER can be used as a template for adding other forces, or modifying the existing equations.

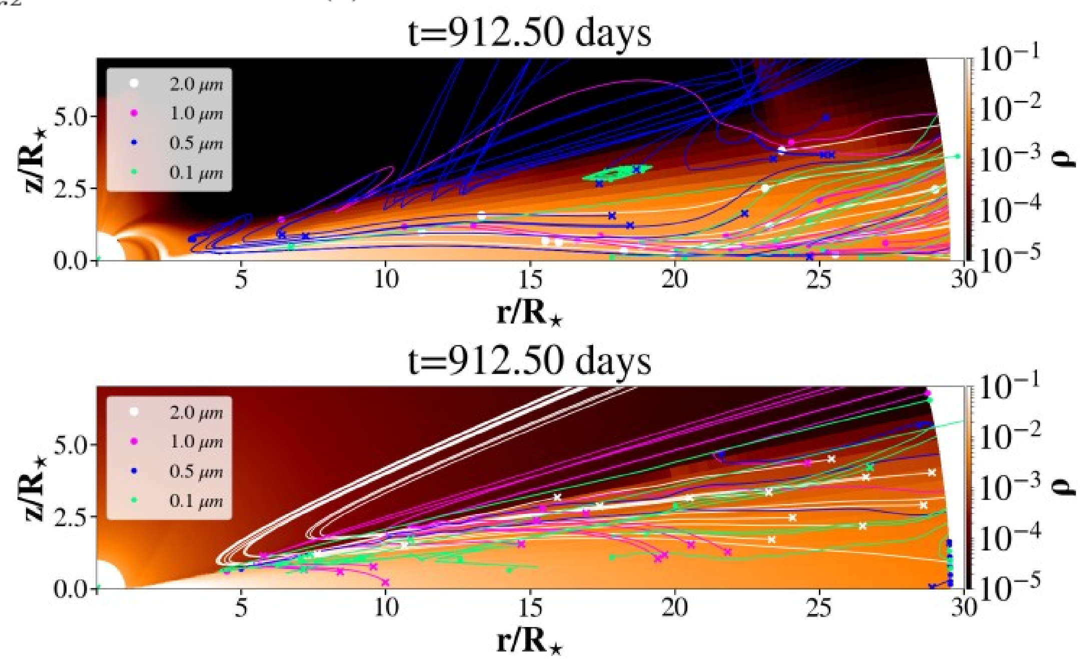

Fig. 1: Particles and their trajectories in a case without backflow in the disk (top panel) and with backflow in the disk (bottom panel). Radiation pressure is included in both cases, with a fully transparent disk. Paths of particles which melted after approaching the star to the critical distance of 4.5 stellar radii were erased, shown are only particles pushed away from the star, or those which remained inside the disk.

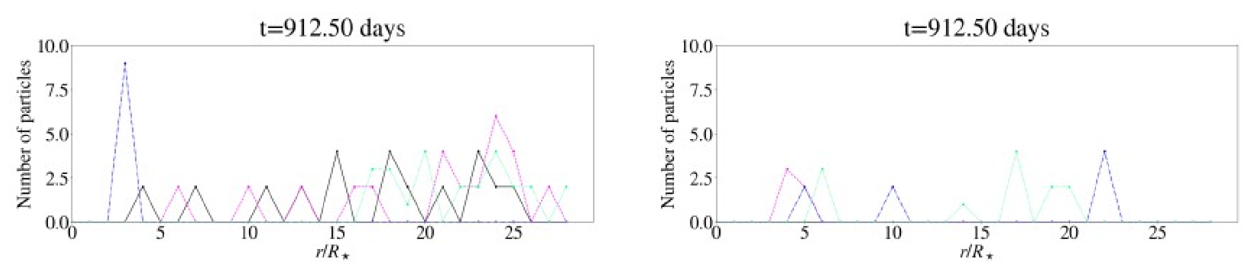

Fig. 2: The number of particles at different distances from the star. In the left panel is shown a case without backflow and in the right panel a case with backflow. Colors correspond to the particles in Fig. 1, with white particles shown in black color line.

Summary of the Part III

-Height-averaged solutions: Urpin (1984)

-Regev (1983) solution

- Kluźniak-Kita global solution for thin HD disk and its magnetic generalization
- Umurhan (2006) generalized γ solutions

- Python code for thin disk model
- Python tool DUSTER

Lectures summary & Concluding remarks

In 9 lectures, we went through a story of thin accretion disk.

- From spherical (Bondi) accretion, we went to
- accretion disk with viscosity.

We run a python code for Bondi accretion

Next we set the stage for steady disk solutions, with
- perturbative solutions,
- Shakura-Sunyaev solutions

We initialized the Python code for Shakura-Sunyaev disk

In the last block, we detailed general solutions for a thin disk:
- Urpin's and Regev's solutions as precursors, and then to
- Kluźniak-Kita 3D global solution for thin disk, with its magnetic generalization

We used those solutions in the Python code for thin disk and a tool DUSTER.

Thank you and let the Disk be with you!

Miljenko Čemeljić, Thin accretion disk, Feb & March 2022, Silesian University, Opava