Numerical simulations with pseudo-potential for Reissner-Nordström naked singularity

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Outline

- More than hundred years of black holes and naked singularities
- Why naked singularities are suddenly so interesting to astronomers
- Numerical simulations with pseudo-Newtonian potential
- Conclusion

More than hundred years of black holes and naked singularities

The name "black hole" is catchy, but misleading. These objects are not "holes", at least not empty ones: in fact, it is a large mass gathered in some volume (not necessarily large density). The escape velocity for these objects is greater than the speed of light! In this sense, the previous descriptive name "dark stars" was more exact, but they are not “stars”, either.

The solutions of the equations from which they "popped out" (Schwarzschild, 1916) were published almost simultaneously with the Theory of Relativity (1916). They were the solutions for a non-rotating black hole, but in the same year the solutions with the electric charge added to the description were published by Reissner (1916), for a non-rotating case. Others, like Wigner and Nordström (1918) followed, and those, for the sufficiently large charges exhibited the objects without event horizon, properly named “naked singularities”.

The next step came only in the 1960s, with solutions for rotating objects: uncharged (Kerr 1963) and charged (Kerr-Newman, 1965) black holes & naked singularities.

Very fast rotating black holes can become naked singularities and have no charge, which is more physically realistic - black holes are difficult to charge electrically, it is generally accepted that they should be electrically neutral: timescales of restoring the neutrality by the fast movement of the electrons in the plasma (because of large electric fields arising with any charge inequality) is much smaller than dynamical timescale of the ions motion.

Naked singularities, a competition to black holes

This was also the main reason why these electrically charged solutions, although simple, are not better known. They are actually not known to the level that I decided to present them in a local astro-amateur journal in north Croatia and also in public presentation there last week as old news. The only mention one person attending the lecture knew was in “Interstellar” movie. Ironically, in the journal and presentation I was presenting yet unpublished material.

VEGA HORIZONTI — br 7. / studeni - prosinac 2024.

ZNANSTVENA OTKRIĆA

Gole singularnosti, konkurencija crnim rupama

Piše:
dr. sc. Miljenko Čemeljić

Napomena: "Vega Horizonti" ovdje ima priliku prikazati novi, još neobjavljeni sadržaj. Autor članka radi s grupama u Poljskoj i Češkoj koje rade na tim istraživanjima i rezultat sa slike je prva simulacija tankog diska oko nerotirajuće gole singularnosti u pseudo-potencijalu sa geometrijom područja nulte gravitacije jednake onom kod Reissner-Nordström metrike.

Crne rupe su na velika vrata ušle u modernu znanost sa Schwarzschildovim rješenjima Einsteinovih jednadžbi gravitacije. Debate o njihovom postojanju ili nepostojanju traju i dalje, ali nedavni uspjeh u njihovom promatranju ju, čini se, privodi kraju. Kolaboracija Event Horizon Telescope (EHT) je 2019. objavila prvu sliku materije upadajuće u supermasivnu crnu rupu u centru galaksije M87 i nedugo nakon toga je ponovila sličan uspjeh sa centrom naše Galaksije, objektom Sgr A*. Manje je poznato da zapravo još nismo do kraja sigurni da se u tim slučajevima radi o crnim rupama, debata još traje!

Karl Schwarzschild

Za veliku masu u središtima galaksija, reda veličine miliona i, ponekad, milijardi masa Sunca, sakupljenu u vrlo malom području, veličine unutrašnjeg dijela Sunčevog sustava, nemamo boljeg objašnjenja nego da se radi o vrlo kompaktnom objektu. Ali osim supermasivne crne rupe postoji i druga mogućnost: gola singularnost. Zanimljivo je da je to rješenje manje poznato znanstvenicima i općoj javnosti, iako su rješenja jednadžbi Einsteinove Opće teorije relativnosti (1915) koja ukazuju na mogućnost njihova postojanja, bila objavljena gotovo istovremeno sa Schwarzschildovim rješenjima (1916) za nerotirajuće crne rupe: Hans Reissner je 1916. dao rješenje za električki nabijene kompaktne objekte, a isto je neovisno postigao i Gunnar Nordström 1918., kao i još nekoliko istraživača. Metriku koja opisuje to rješenje danas nazivamo Reissner-Nordström metrikom. Kasnije je nađeno mnogo sličnih rješenja i bez uvođenja naboja, npr. uvođenjem vrlo brze rotacije kompaktnog objekta.

Zbog svojih intrigantnih svojstava, gole singularnosti prisutne su u teorijskim istraživanjima od supermasivnih crnih rupa do elementarnih čestica-u jednoj od ranih verzija takvih teorija elektron je opisan kao električki nabijena gola singularnost!

Gunnar Nordström
(Atelier Apollo, CC BY 4.0)

Hans Reissner i supruga

Rješenja za gole singularnosti se razlikuju od Schwarzschildovih time da nemaju horizont događaja, nego sfernu ili spljoštenu ljusku područja nulte gravitacije na nekoj konačnoj udaljenosti od centra, tako da je unutar tog područja gravitacija usmjerena prema van. To znači da i svjetlo može izaći prema vanjskom promatraču, pa je moguć direktni pogled u singularnost, gdje je gustoća ekstremno velika. Otuda i pridjev "gola" u nazivu, jer nije zakrivena horizontom događaja, kao što je to slučaj kod crne rupe, koji onemogućava vanjskom promatraču da vidi što je unutar tog horizonta.

Treba razlikovati golu singularnost, koja je točka u prostor vremenu, od Einstein-Rosenovog mosta, gdje se prostor-vrijeme iskrivljuje u lijevak, koji s druge strane izlazi u isti takav lijevak. Tako se uglavnom u znanstvenoj fantastici rješava putovanje svemirom brzinama većim od brzine svjetlosti: materija se prelije iz jednog dijela svemira u drugi, kao da napravimo rupu u stranicama atlasa i kroz nju prelijemo tintu među stranicama. Promatrač s izlazne strane takvog mosta bi vidio bijelu rupu iz koje bi materija samo izlazila a ne i ulazila-takve objekte u svemiru astronomi još nisu pronašli.

Kolaboracija EHT je sliku crne rupe dobila tako da su napravili tisuće numeričkih simulacija crnih rupa i statističkim metodama usporedili promatračke podatke dobivene mrežom radioteleskopa postavljenih na velike udaljenosti, reda veličine promjera Zemlje, da se postigne što veća rezolucija. Među tisućama računalno dobivenih slika odabrana je kombinacija koja najviše odgovara promatranjima. Neki istraživači iz te kolaboracije su uočili da je moguće da bi simulacije golih singularnosti mogle dati vrlo slične rezultate, što bi značilo da za sada nemamo načina da promatranjima utvrdimo koja pretpostavka je točna, ona sa crnim rupama ili sa golim singularnostima. "Houston, imamo problem!". Potrebno je provjeriti promatračke rezultate i sa rješenjima za gole singularnosti.

Istraživačka grupa u kojoj radim u Varšavi se, u suradnji sa jednim od članova iz tima EHT, prihvatila zadatka i već smo izvršili analitičke izračune i niz simulacija golih singularnosti sa diskom materije oko nje i radimo na tome da nađemo detalje koji su različiti kod crnih rupa i golih singularnosti. Promatranjima i uočavanjem tih razlika, moglo bi se konačno nepobitno utvrditi da li su u središtima galaksija crne rupe ili gole singularnosti.

Jedan primjer rezultata simulacije električki nabijene gole singularnosti je prikazan na slici, sa naznačenim promjerom nulte gravitacije. Zanimljivo je da se oko područja nulte gravitacije formira zadebljanje u disku, tvoreći torus materijala. Zračenje iz tog područja bi moglo razriješiti dilemu bar što se tiče ove vrste golih singularnosti, jer takvog područja nema u simulacijama sa crnim rupama. Ovakva rješenja sada još treba provući kroz obradu u kojoj ćemo uključiti zakrivljenje prostor-vremena u blizini velike mase i napraviti predviđanje za promatrani intenzitet zračenja. Konačnu riječ će, naravno, imati promatranja, kad će naši rezultati biti uspoređeni s onima iz stvarnih promatranja.

Preliminarni rezultat simulacije za disk oko električki nabijene gole singularnosti, sa rastućom gustoćom materije prikazanom u obojenoj skali od plave, preko žute prema crvenoj, vektorima koji pokazuju smjer brzine i kružnicom koja naznačuje područje nulte gravitacije. Unutar tog područja gravitacija djeluje u obratnom smjeru, izbacujući materiju prema van.

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Reissner-Nordström metric and singularity

They remained as an exercise in MTW book and I could not find mention in Landau-Lifshitz II (maybe added in some edition). I performed them on a couple of pages for the exercise of Christoffel symbols, I recommend them to doctoral students to refresh the General Theory of Relativity – I suggest to do it on paper, without using Maple or similar computer packages.

The Reissner-Nordström solutions, which are relatively simple, are nevertheless useful because they are similar to much more complicated solutions for rotating objects. That's why we started to study them in a pseudo-Newtonian approach, so that we can run simulations faster and do the first checks of ideas, which we can then study in more detail in relativistic simulations, which are more demanding, and do not have "physical" viscosity and resistivity.

Exercise 31.8. REISSNER-NORDSTRØM GEOMETRY

(a) Solve the Einstein field equations for a spherically symmetric, static gravitational field

$$ds^2 = -e^{2\Phi(r)}\,dt^2 + e^{2\Lambda(r)}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2),$$

§31.6. DYNAMICS OF THE SCHWARZSCHILD GEOMETRY 841

with no matter present, but with a radial electric field $\boldsymbol{B} = 0$, $\boldsymbol{E} = f(r)\boldsymbol{e}_{\hat{r}}$ in the static orthonormal frame

$$\boldsymbol{\omega}^{\hat{t}} = e^{\Phi}\,\boldsymbol{d}t, \qquad \boldsymbol{\omega}^{\hat{r}} = e^{\Lambda}\,\boldsymbol{d}r, \qquad \boldsymbol{\omega}^{\hat{\theta}} = r\,\boldsymbol{d}\theta, \qquad \boldsymbol{\omega}^{\hat{\phi}} = r\sin\theta\,\boldsymbol{d}\phi.$$

Use as a source in the Einstein field equations the stress-energy of the electric field. [*Answer:*

$$\boldsymbol{E} = (Q/r^2)\boldsymbol{e}_{\hat{r}}, \qquad (31.24a)$$

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2). \qquad (31.24b)$$

This is called the "Reissner (1916)-Nordstrøm (1918) metric".]

(b) Show that the constant Q is the total charge as measured by a distant observer ($r \gg 2M$ and $r \gg Q$), who uses a Gaussian flux integral, or who studies the coulomb-force-dominated orbits of test charges with charge-to-mass ratio $e/\mu \gg M/Q$. What is the charge-to-mass ratio, in dimensionless units, for an electron? Show that the constant M is the total mass as measured by a distant observer using the Keplerian orbits of electrically neutral particles.

(c) Show that for $Q > M$, the Reissner-Nordstrøm coordinate system is well-behaved from $r = \infty$ down to $r = 0$, where there is a physical singularity and infinite tidal forces.

(d) Explore the nature of the spacetime geometry for $Q < M$, using all the techniques of this chapter (coordinate transformations, Kruskal-like coordinates, studies of particle orbits, embedding diagrams, etc.).
[*Solution:* see Graves and Brill (1960); also Fig. 34.4 of this book.]

(e) Similarly explore the spacetime geometry for $Q = M$. [*Solution:* see Carter (1966b).]

(f) For the case of a large ratio of charge to mass [$Q > M$ as in part (c)], show that the region near $r = 0$ is unphysical. More precisely, show that any spherically symmetric distribution of charged stressed matter that gives rise to the fields (31.24) outside its boundary must modify these fields for $r < r_0 = Q^2/2M$. [Hint: Study the quantity $m(r)$ defined in equations (23.18) and (32.22h), noting its values deduced from equation (31.24), on the one hand, and from the appropriate Einstein equation within the matter distribution, on the other hand. See Figure 26 of Misner (1969a) for a similar argument.]

The first exact solutions in GR

Schwarzschild metric (1916) for nonrotating BHs

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta^2 d\phi^2\right).$$

Reissner-Nordström metric for nonrotating **charged** BHs,or, if Q>M, naked singularities (1916, 1918)

$$ds^2 = f(r)\,dt^2 - \frac{1}{f(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta^2 d\phi^2\right), \tag{13}$$

$$\text{with} \quad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \equiv 1 - 2\frac{M}{r} + q^2\frac{M^2}{r^2}. \tag{14}$$

Q is the electric charge of the gravitating body, and we define the dimensionless charge parameter $q = Q/M$.

Pseudo-Newtonian potentials

In a classical Newtonian description, we can describe the motion of a test particle of unit mass in the orbital plane $\theta = \pi/2$ around a much larger point mass M as a motion in the central gravitational potential $\Phi(r) = -GM/r$. The energy equation is:

$$E = \frac{1}{2}(v_r^2 + v_\phi^2) + \Phi(r). \quad (1)$$

Introducing the effective potential $U_{\rm eff}(r, L) = \Phi(r) + L^2/(2r^2)$, we can write:

$$E - \frac{1}{2}v_r^2 = U_{\rm eff}(r, L), \quad (2)$$

where E, L and v_r are the total energy, angular momentum and radial velocity, respectively.

For the circular orbits, the condition $(\partial U_{\rm eff}/\partial r)_L = 0$ is satisfied, so that we obtain the condition:

$$\frac{\partial\Phi(r)}{\partial r} - \frac{L^2}{r^3} = 0. \quad (3)$$

Paczyński & Wiita (1980), for a non-rotating BH:

$$\frac{d}{dr}\left(\frac{1}{1-2M/r}\right) + \frac{2\ell}{r^3} = 0. \quad (10)$$

Employing $r/(r-2M) = 2M/(r-2M) + 1$, we can write:

$$\frac{d}{dr}\left(1 + \frac{2M}{r-2M}\right) + \frac{2\ell}{r^3} = 0 \Rightarrow \frac{d}{dr}\left(\frac{2M}{r-2M}\right) + \frac{2\ell^2}{r^3} = 0. \quad (11)$$

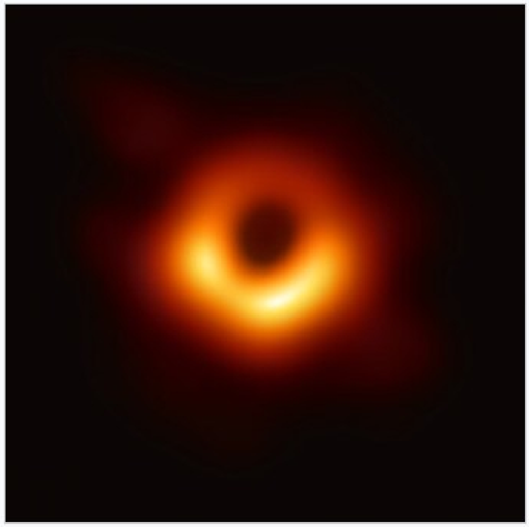
Comparison with the Eq. 3 gives the pseudo-Newtonian potential

$$V_{\rm PW} = -\frac{M}{r-2M}. \quad (12)$$

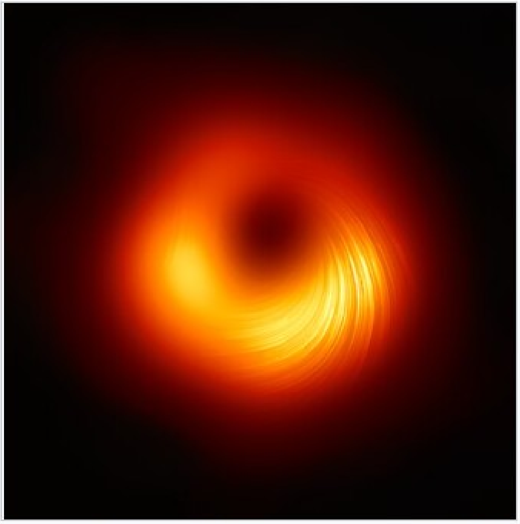
The Keplerian angular momentum $\ell = \sqrt{Mr^3}/(r-2M)$ in the PW potential is the same as in its exact Schwarzschild GR solution, but the value of the Keplerian angular velocity is different: $\Omega_{K,PW} =???fix\ it \sqrt{M/r^3}/(r-2GM/r)$, while in the Schwarzschild solution it is the usual Newtonian $\Omega_K = \sqrt{GM/r^3}$

How we see the BHs?

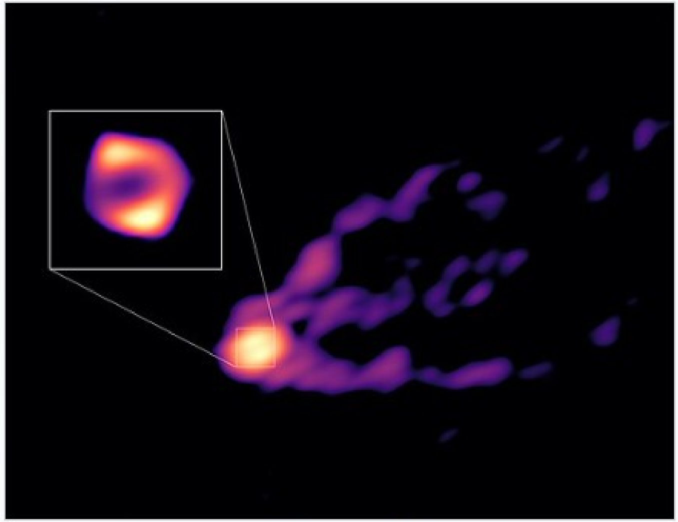
Supermassive black hole M87* [edit]



The Event Horizon Telescope image of the core of M87 using 1.3 mm microwaves. The central dark spot is the shadow of M87* and is larger than the black hole's event horizon.



A view of the M87* supermassive black hole released by the Event Horizon Telescope Collaboration with lines overlaid to mark the orientation of polarization of the magnetic field



A view of the jet and shadow of M87's black hole. Observations from the Global Millimetre VLBI Array (GMVA), the Atacama Large Millimeter/submillimeter Array (ALMA), and the Greenland Telescope.[70]

Results of such observations are intrinsically dependent on the model used for the BH. Kerr solutions for a rotating BH were used here, but the possibility of naked singularity remained unexplored.

This is the reason for the recent interest in naked singularities, both from the side of astronomers and simulations people.

There are many GR codes, but if the simulations are to capture the accretion disk, they have to capture also the MRI, so full 3D and good resolution are needed=computationally expensive.

Some breakthroughs in the BH accretion disk theory were done by using the PW pseudo-Newtonian potential. Could a pseudo-Newtonian potential for RN naked singularity help us? We could perform more simulations, maybe get some additional insights...

[Vlak Uherske Hradište-Katowice.]

PLUTO code and naked singularities

In CAMK summer program, interns worked with me on numerical simulations of thin accretion disc with Paczyński-Wiita and Kluźniak-Lee potentials. It took me no time to introduce the Kluźniak-RN potential.

```
    I A  init.c (Modified)(c)  return -1./x1+0.5*(q*q/x1/x1);
 *
 * \return The body force potential \f$ \Phi(x_1,x_2,x_3) \f$.
 *
 ***************************************************************
{
 //ccm--060824--uncomment the wanted pseudo-potential
 //set in definitions.h #define BODY_FORCE POTENTIAL
 double q;
 q=1.25;
  //return -1./(x1-2.);//0.0;//ccm--Paczynski-Wiita
//  return -(1.0/6.0)*(exp(6.0/x1)-1.0);//KluzniakLee
return -1./x1+0.5*(q*q/x1/x1);//KluzniakRN
}
#endif
```

NkS in pseudo-Newtonian potential

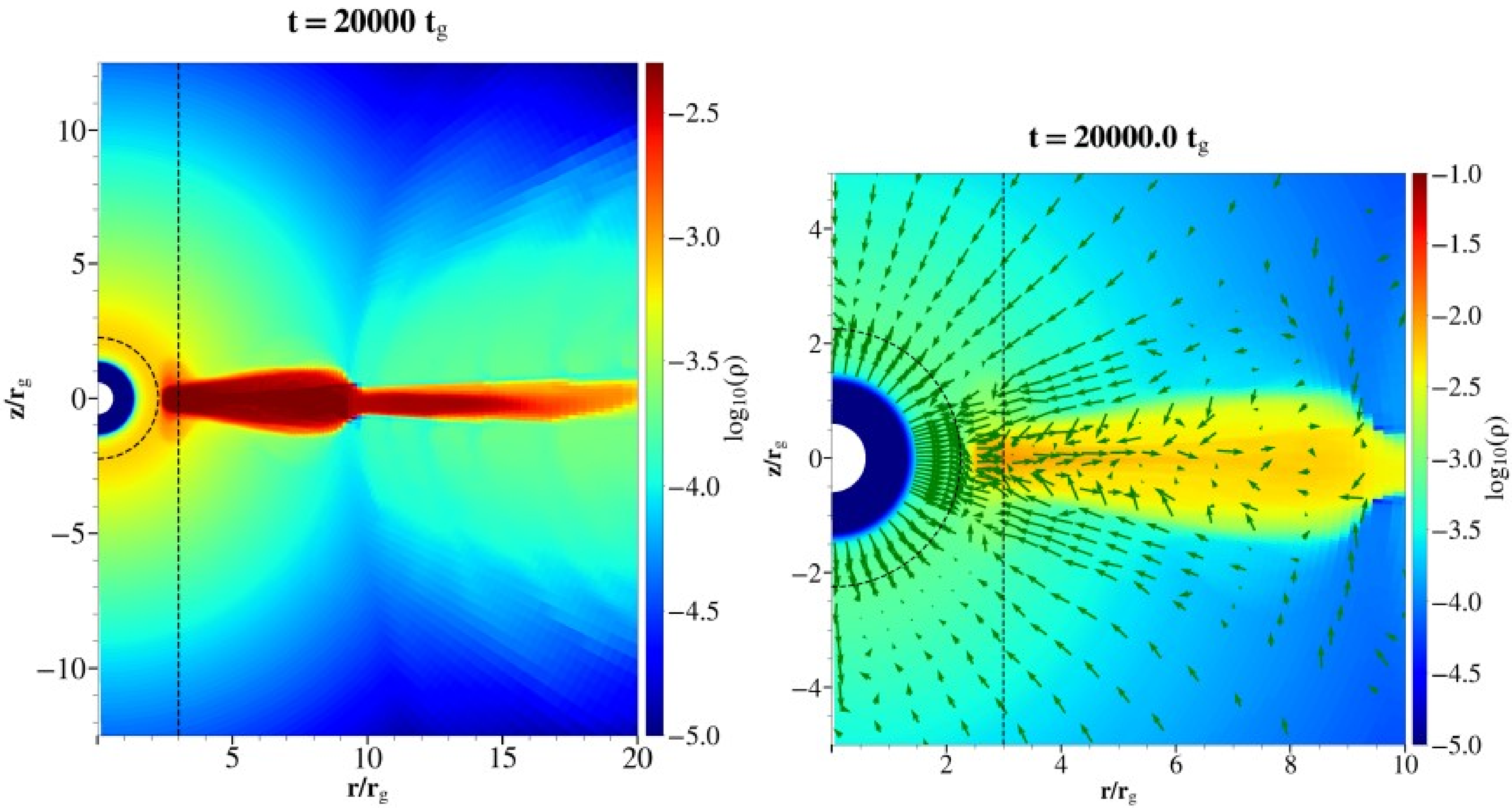


Fig. 2. Gas density in the simulation with $q = 1.5$. *Left panel*: a snapshot result at t=20000 t_g, where $t_g = r_g/c$. The zero-gravity radius r_0 is marked with the dashed half-circle and the radius, $4r_0/3$, at which test-particle Ω attains a maximum is marked with the straight black dashed line. *Right panel*: a zoom into the inner region of the accretion flow, obtained as an average over the time interval of $t \in [19000, 21000]$ t_g. Poloidal gas velocity vectors are indicated with green arrows.

NkS in pseudo-Newtonian potential

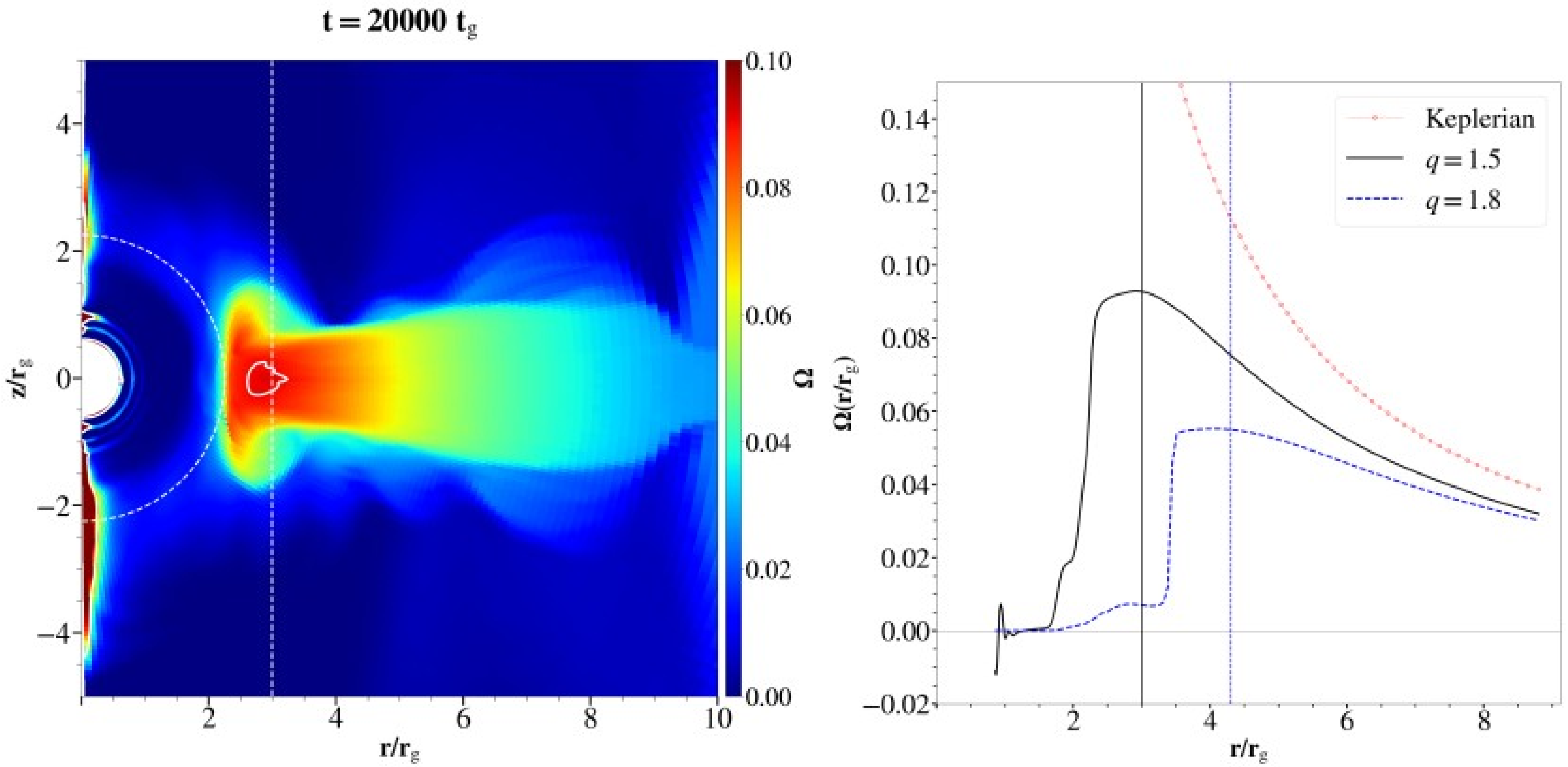


Fig. 3. *Left panel:* the angular velocity $\Omega = v_\phi/(r\cos\theta)$, in the linear color grading at $t = 20000\,t_g$ in our simulation with $q = 1.5$. The contour of $\Omega = 0.09/M$ within which the test-particle orbital frequency value of $\Omega_{\rm max}$ at $r/M = 4q^2/3$ is located is shown with the white solid curve. *Right panel:* $\Omega(r)$ in the equatorial plane for the RN metric with $q = 1.5$ and 1.8, in solid (black) and dashed (blue) lines, respectively. The dotted (red) line follows the Schwarzschild profile of ΩM, and is given for comparison. Vertical lines in the corresponding styles indicate the radial positions of $\Omega_{\rm max}$ test-particle orbital values for $q = 1.5$ and $q = 1.8$. MW: does it make sense to indicate the location of r_0 as well? Maybe with black/blue

There are many such approximations, each good for its purpose-some characteristic distances or surfaces are usually represented correctly in such pseudo-potentials.

NkS in pseudo-Newtonian potential

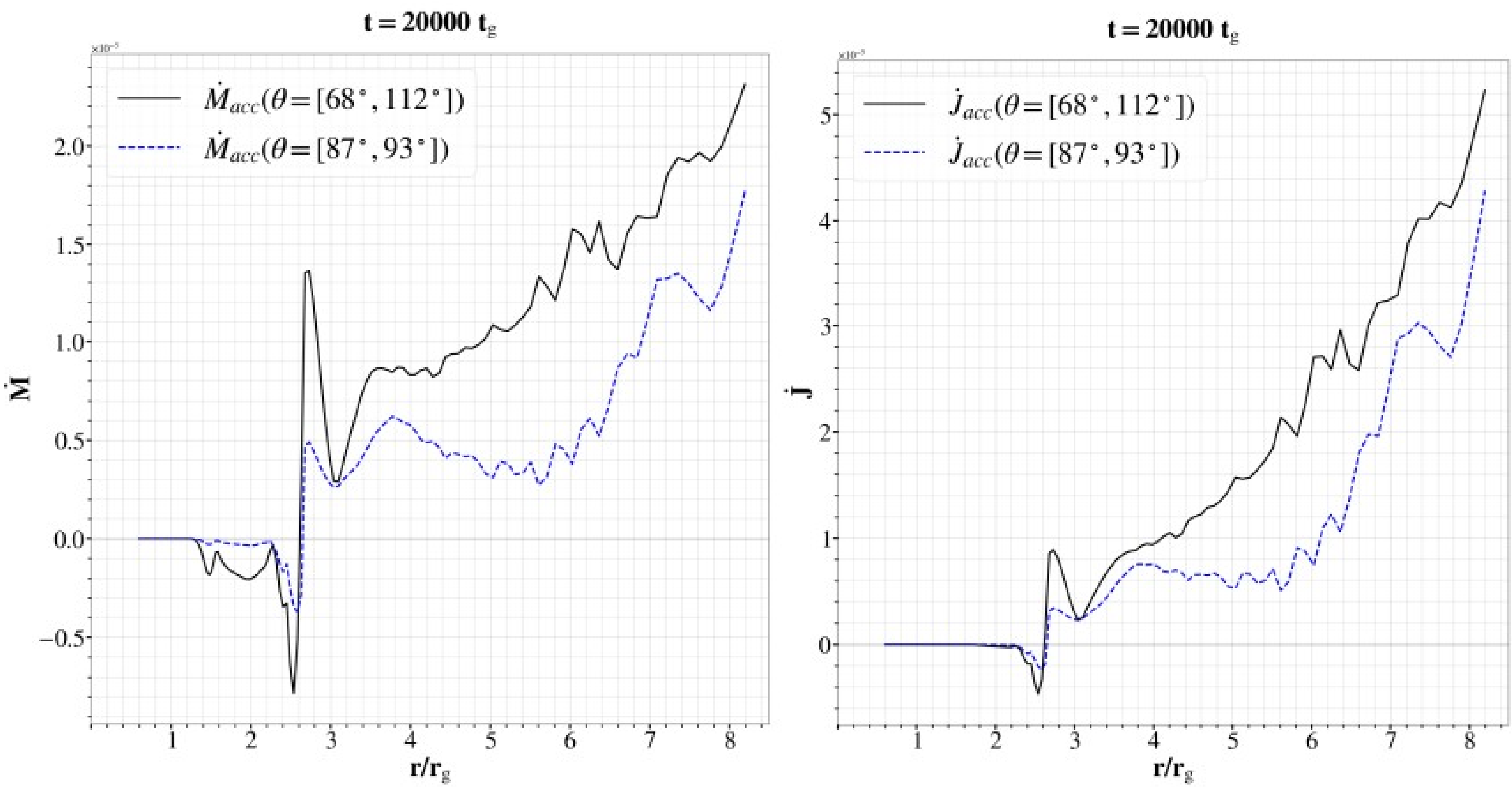
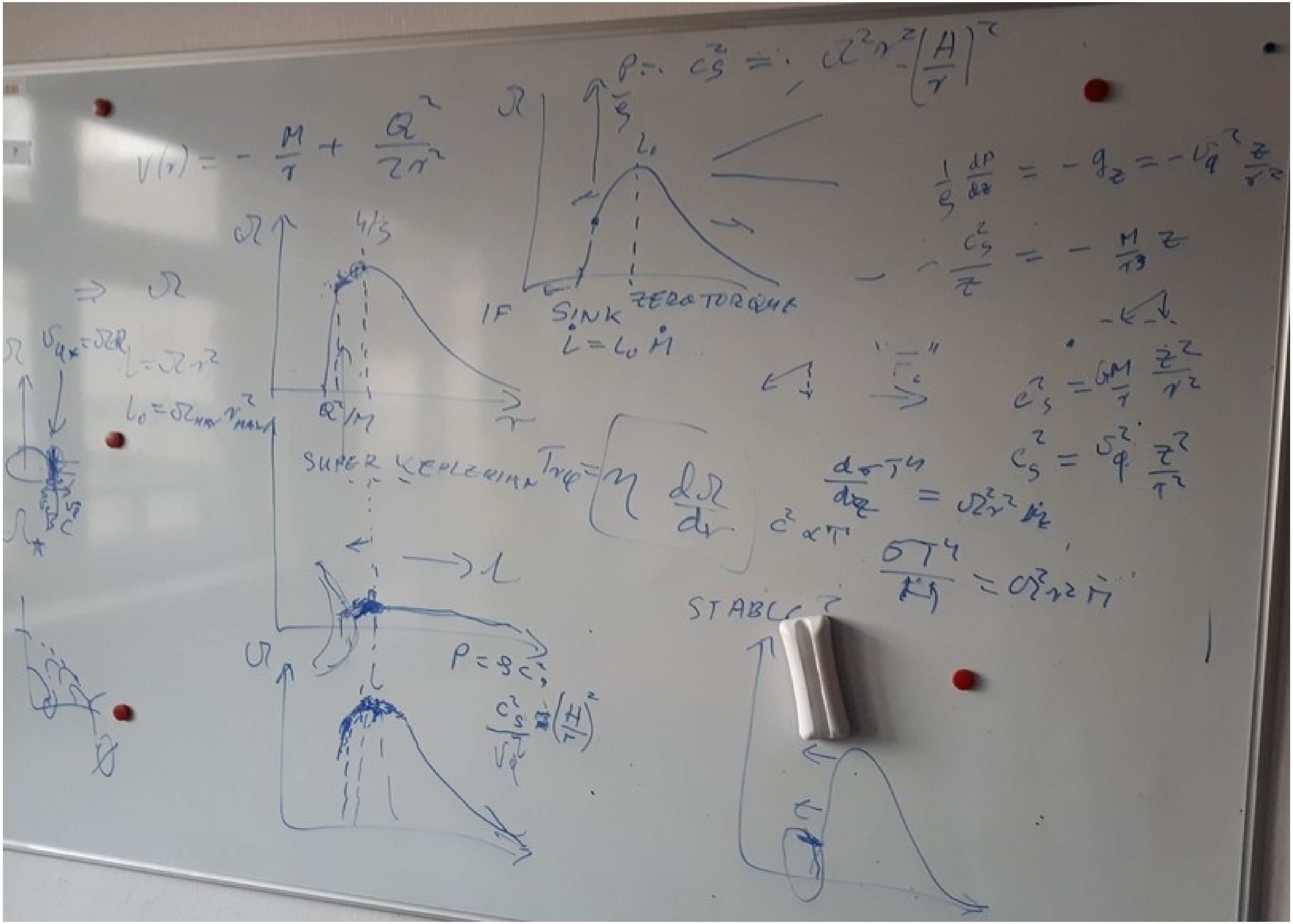


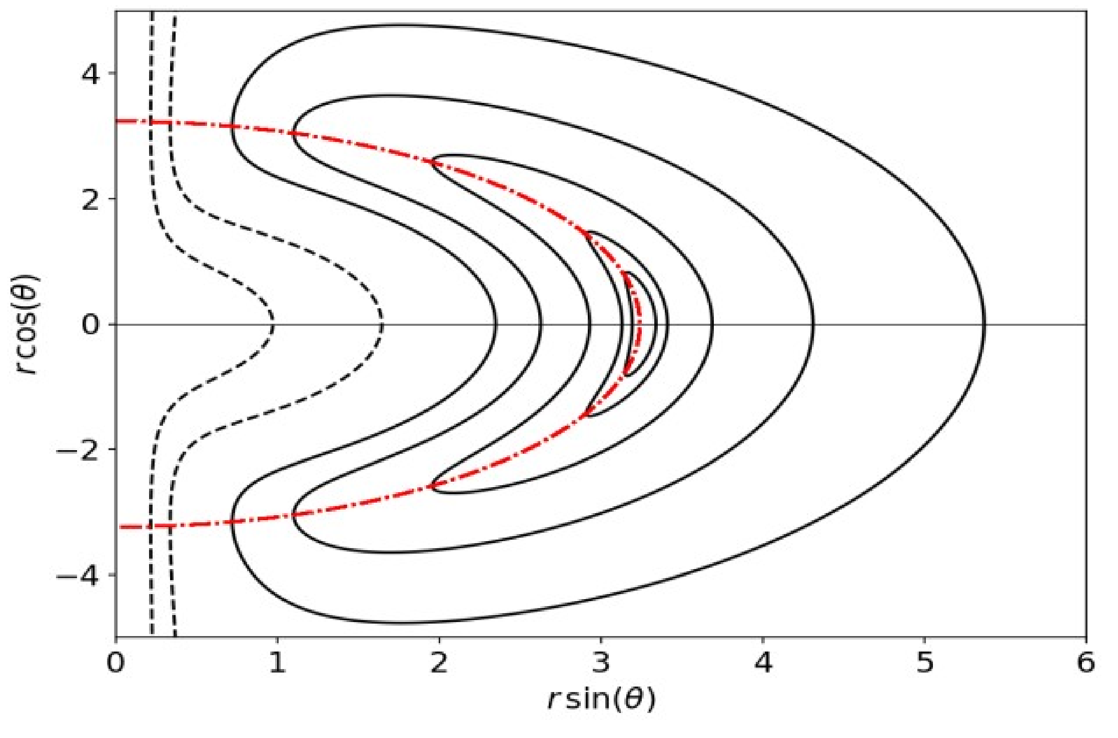
Fig. 4. Values of $\dot{M}$ and $\dot{J}$ computed in two co-latitudinal intervals as assigned in the legends, in dependence of radial distance from the origin in the simulation from Fig. 2, obtained as an average over the time interval of $t \in [19000, 21000]\ t_g$. MW: Can't the panels be of the same size?

Would naked singularities mimic BHs?



Analytical solutions in GR

In Mishra et al. (2024) it is shown that an observer could see the structure inside a NkS.



ure 4. Equilibrium tori around RN naked singularity with Q/M=1.8 with $l_0 < l_{\Omega\max}$. L

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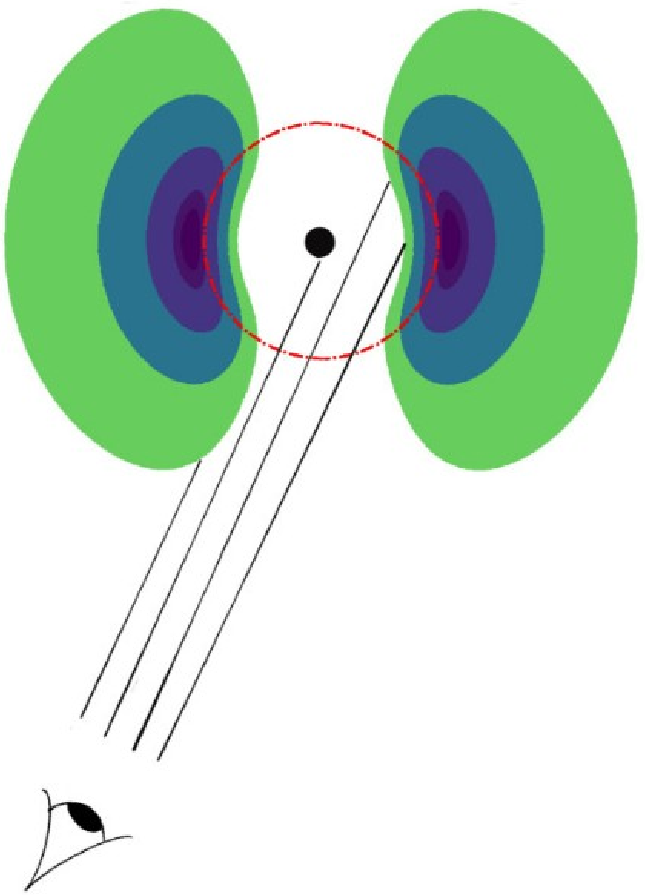

Figure A1. Contours of equipotential fluid surfaces around RN naked singularity with charge $Q/M = 1.8$ and specific angular momentum $l = 0.7$. The zero gravity sphere has been indicated as dotted-dashed line.

Conclusion

We should make series of simulations of naked singularities and use them as a model for creating the observational picture of M87 and Sgr A*.

Thank you!