Pseudo-Newtonian simulations with Reissner–Nordström naked singularity

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Outline

- More than hundred years of black holes and naked singularities
- Why naked singularities are suddenly so interesting to astronomers
- Numerical simulations with pseudo-Newtonian potential
- Conclusion

More than hundred years of black holes and naked singularities

The solutions of the Einstein equations for non-rotating “dark stars”(Schwarzschild, 1916), later named black holes, were published almost simultaneously with the Theory of Relativity (1916). In the same year the solutions with the electric charge added to the description were published by Reissner (1916), followed by Wigner and also Nordström (1918). For the sufficiently large charges, these solutions introduced the objects without event horizon, properly named “naked singularities”.

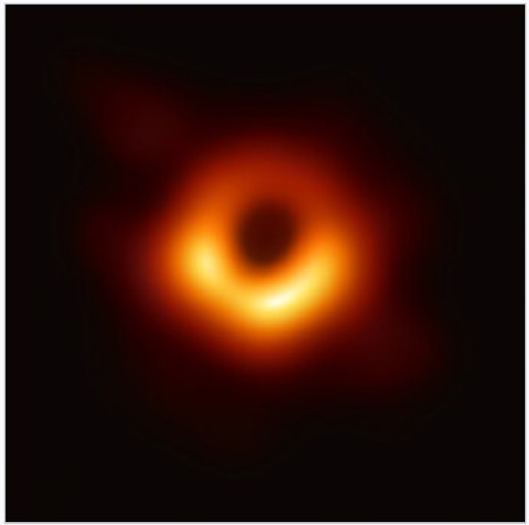
The next step came only in the 1960s, with solutions for rotating objects: uncharged (Kerr 1963) and charged (Kerr-Newman, 1965) black holes & naked singularities.

Black holes are difficult to charge electrically, it is generally accepted that they should be electrically neutral: timescales of restoring the neutrality by the fast movement of the electrons in the plasma (because of large electric fields arising with any charge inequality) is much smaller than dynamical timescale of the ions motion.

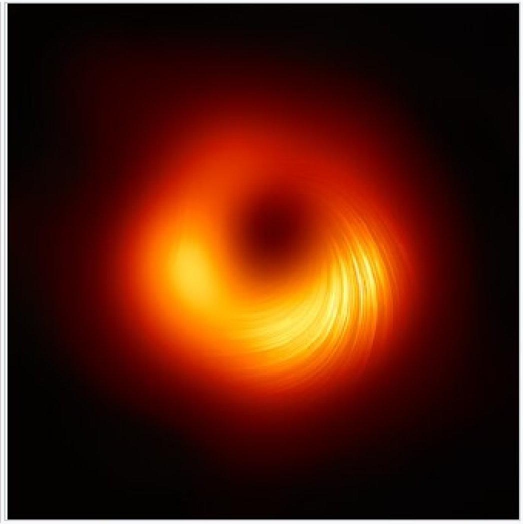
More physically realistic is to have very fast rotating black holes, which can become naked singularities even without electric charge. The solutions for such fast rotating objects are similar to RN non-rotating electrically charged objects, so we can use them as a proxy.

How we see the BHs?

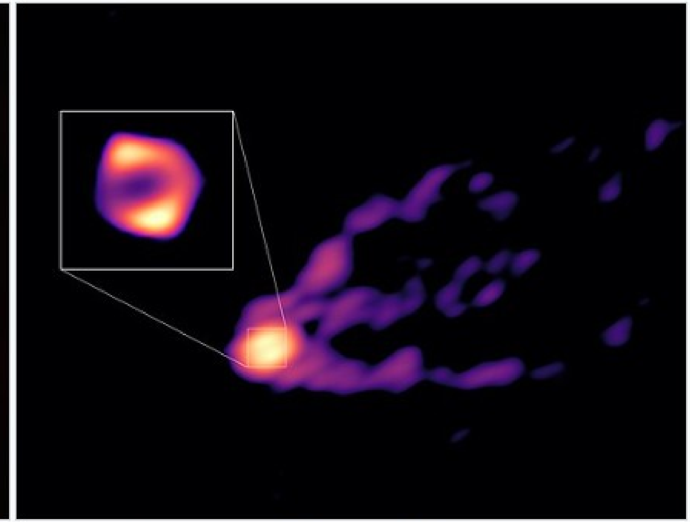
Supermassive black hole M87* [edit]



The Event Horizon Telescope image of the core of M87 using 1.3 mm microwaves. The central dark spot is the shadow of M87* and is larger than the black hole's event horizon.



A view of the M87* supermassive black hole released by the Event Horizon Telescope Collaboration with lines overlaid to mark the orientation of polarization of the magnetic field



A view of the jet and shadow of M87's black hole. Observations from the Global Millimetre VLBI Array (GMVA), the Atacama Large Millimeter/submillimeter Array (ALMA), and the Greenland Telescope.[70]

Results of such observations are intrinsically dependent on the model used for the BH. Kerr solutions for a rotating BH were used here, but the possibility of naked singularity remained unexplored.

This is the reason for the recent interest in naked singularities.

In GR, if the simulations are to capture the accretion disk, they have to capture also the MRI, so full 3D and good resolution are needed=computationally expensive.

Some breakthroughs in the BH accretion disk theory were done by using the PW pseudo-Newtonian potential. Could a pseudo-Newtonian potential for RN naked singularity help us? We could perform more simulations, maybe get some additional insights.

The first exact solutions in GR

Schwarzschild metric (1916) for nonrotating BHs

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta^2 d\phi^2\right).$$

Reissner-Nordström metric for nonrotating **charged** BHs,or, if Q>M, naked singularities (1916, 1918)

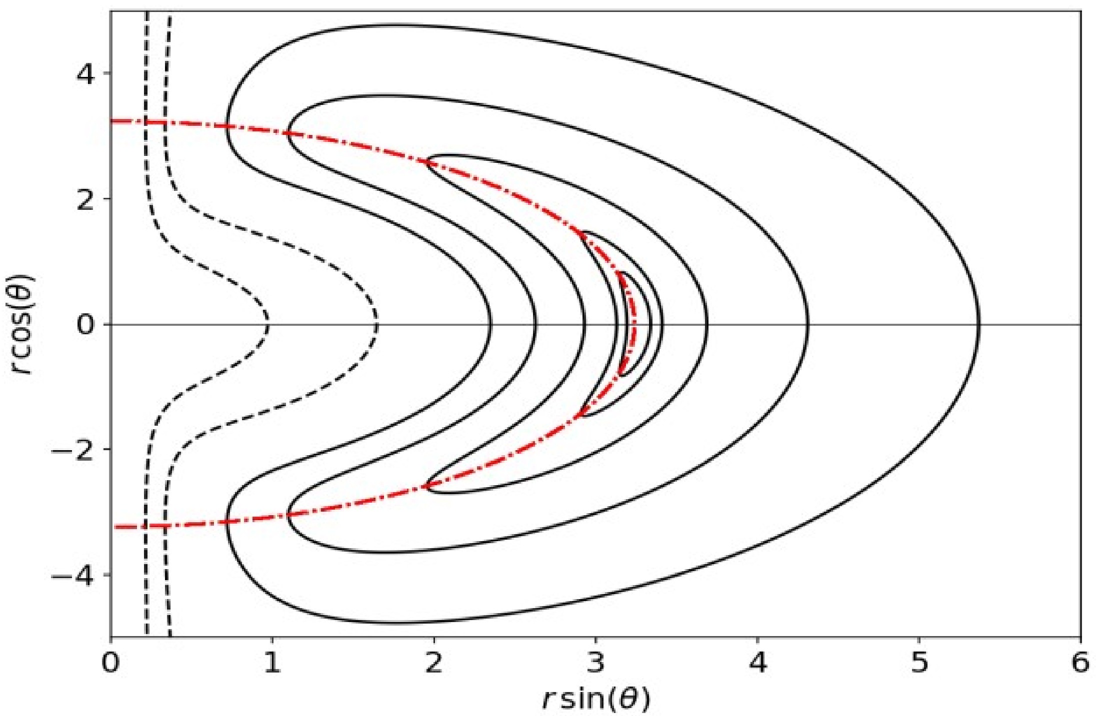
$$ds^2 = f(r)\,dt^2 - \frac{1}{f(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta^2 d\phi^2\right), \tag{13}$$

$$\text{with} \quad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \equiv 1 - 2\frac{M}{r} + q^2\frac{M^2}{r^2}. \tag{14}$$

Q is the electric charge of the gravitating body, and we define the dimensionless charge parameter $q = Q/M$.

Analytical solutions in GR

In Mishra et al. (2024) it is shown that an observer could see the structure inside a NkS.



ure 4. Equilibrium tori around RN naked singularity with Q/M=1.8 with $l_0 < l_{\Omega\max}$. L

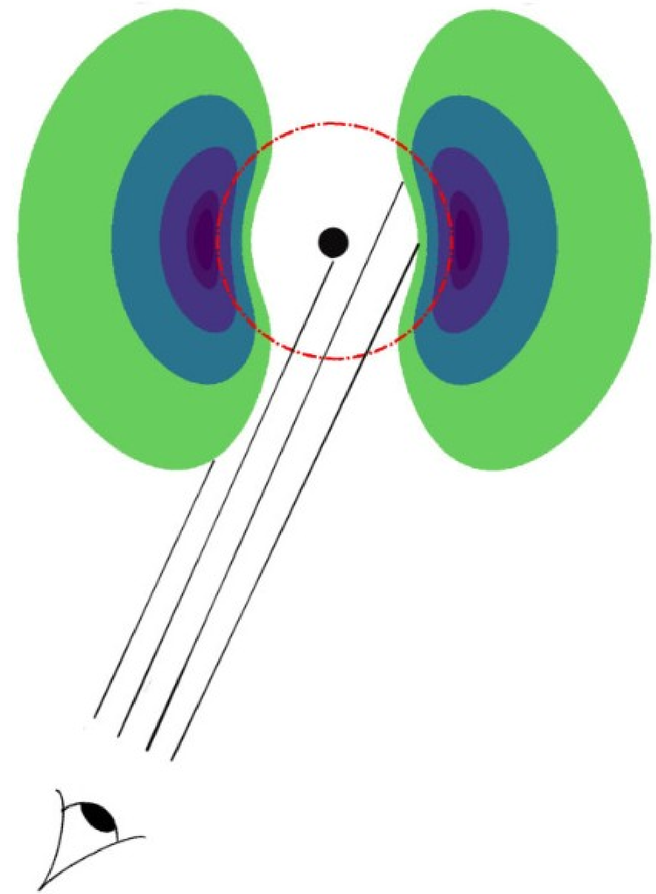


Figure A1. Contours of equipotential fluid surfaces around RN naked singularity with charge $Q/M = 1.8$ and specific angular momentum $l = 0.7$. The zero gravity sphere has been indicated as dotted-dashed line.

Pseudo-Newtonian potentials

In a classical Newtonian description, we can describe the motion of a test particle of unit mass in the orbital plane $\theta = \pi/2$ around a much larger point mass M as a motion in the central gravitational potential $\Phi(r) = -GM/r$. The energy equation is:

$$E = \frac{1}{2}(v_r^2 + v_\phi^2) + \Phi(r). \tag{1}$$

Introducing the effective potential $U_{\rm eff}(r, L) = \Phi(r) + L^2/(2r^2)$, we can write:

$$E - \frac{1}{2}v_r^2 = U_{\rm eff}(r, L), \tag{2}$$

where E, L and v_r are the total energy, angular momentum and radial velocity, respectively.

For the circular orbits, the condition $(\partial U_{\rm eff}/\partial r)_L = 0$ is satisfied, so that we obtain the condition:

$$\frac{\partial\Phi(r)}{\partial r} - \frac{L^2}{r^3} = 0. \tag{3}$$

Paczyński & Wiita (1980), for a non-rotating BH:

$$\frac{d}{dr}\left(\frac{1}{1-2M/r}\right) + \frac{2\ell}{r^3} = 0. \tag{10}$$

Employing $r/(r-2M) = 2M/(r-2M) + 1$, we can write:

$$\frac{d}{dr}\left(1 + \frac{2M}{r-2M}\right) + \frac{2\ell}{r^3} = 0 \Rightarrow \frac{d}{dr}\left(\frac{2M}{r-2M}\right) + \frac{2\ell^2}{r^3} = 0. \tag{11}$$

Comparison with the Eq. 3 gives the pseudo-Newtonian potential

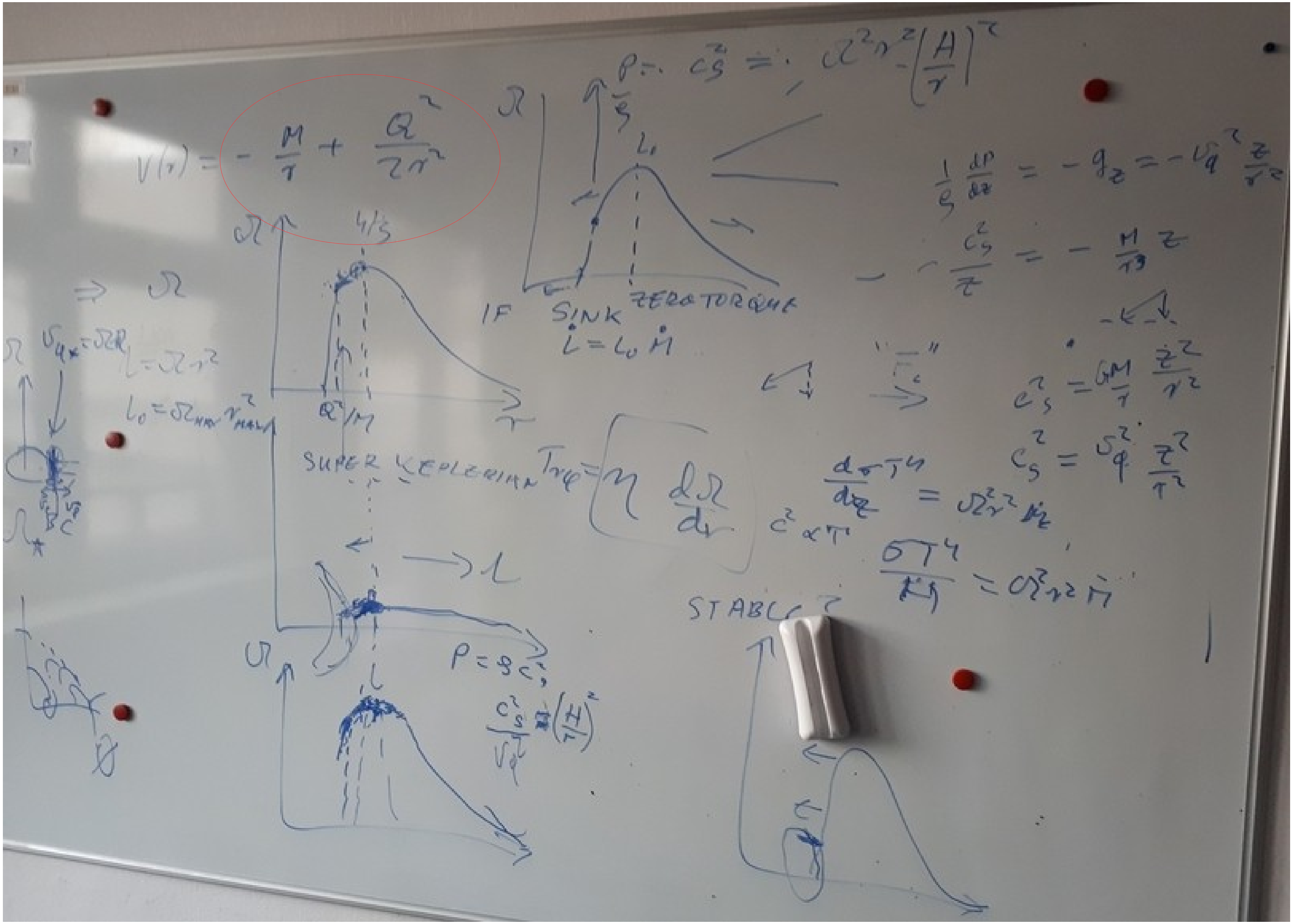
$$V_{\rm PW} = -\frac{M}{r-2M}. \tag{12}$$

The Keplerian angular momentum $\ell = \sqrt{Mr^3}/(r-2M)$ in the PW potential is the same as in its exact Schwarzschild GR solution, but the value of the Keplerian angular velocity is different:

$$\Omega_{{\rm K},PW} = \left(\frac{r}{r-2GM/c^2}\right)\sqrt{\frac{GM}{r^3}}, \tag{13}$$

while in the Schwarzschild solution it is the usual Newtonian $\Omega_{\rm K} = \sqrt{GM/r^3}$.

A novel pseudo-Newtonian potential for RN naked singularity



PLUTO code and naked singularities

In recent CAMK summer programs, interns worked with me on numerical simulations of thin accretion disc with Paczyński-Wiita and Kluźniak-Lee potentials. It took me no time to introduce the RN potential.

```
    I A  init.c (Modified)(c)  return -1./x1+0.5*(q*q/x1/x1);
 *
 * \return The body force potential \f$ \Phi(x_1,x_2,x_3) \f$.
 *
 ***************************************************************
{
 //ccm--060824--uncomment the wanted pseudo-potential
 //set in definitions.h #define BODY_FORCE POTENTIAL
 double q;
 q=1.25;
  //return -1./(x1-2.);//0.0;//ccm--Paczynski-Wiita
//  return -(1.0/6.0)*(exp(6.0/x1)-1.0);//KluzniakLee
return -1./x1+0.5*(q*q/x1/x1);//KluzniakRN
}
#endif
```

NkS in pseudo-Newtonian potential

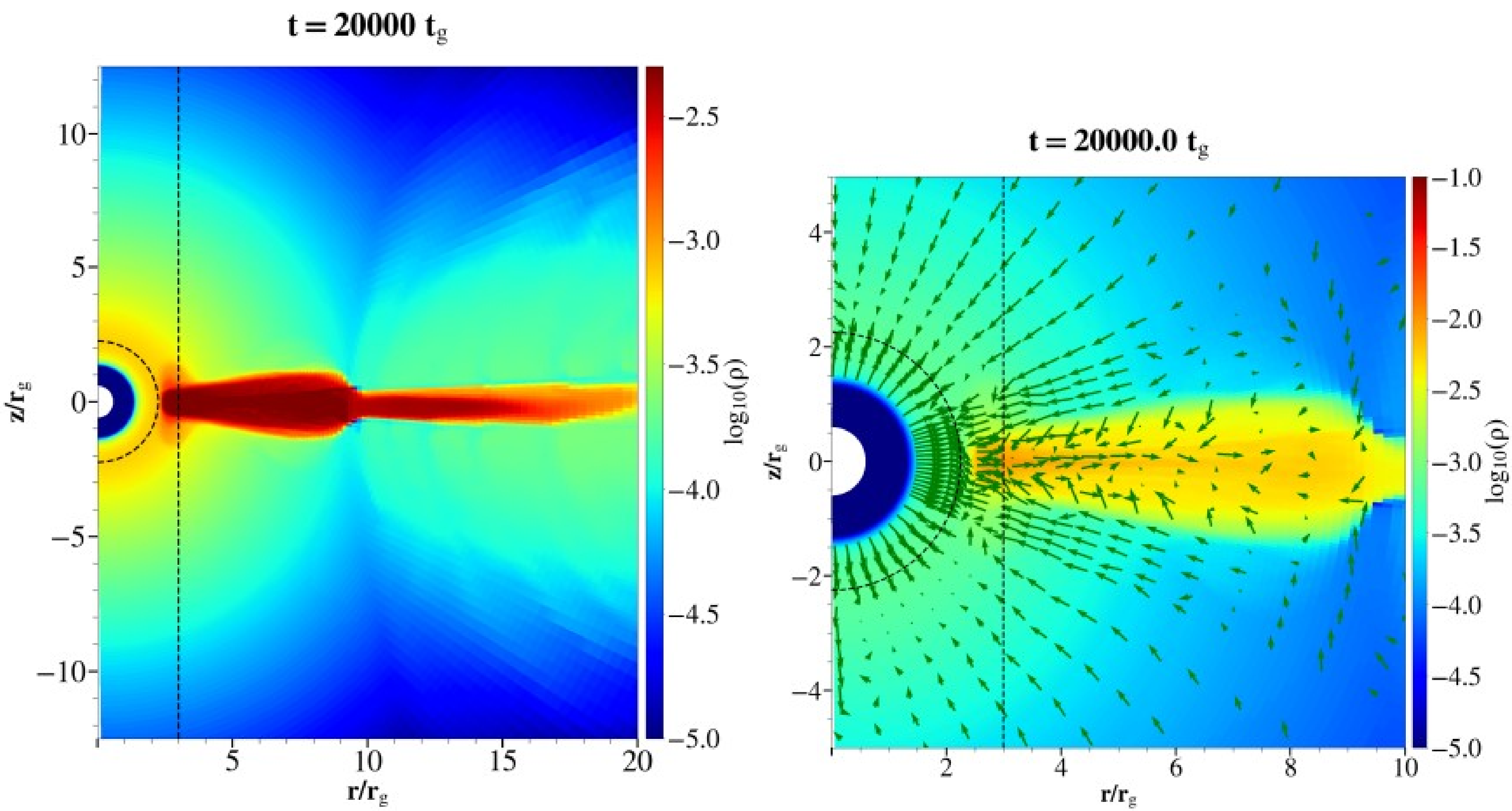


Fig. 2. Gas density in the simulation with $q = 1.5$. *Left panel*: a snapshot result at t=20000 t_g, where $t_g = r_g/c$. The zero-gravity radius r_0 is marked with the dashed half-circle and the radius, $4r_0/3$, at which test-particle Ω attains a maximum is marked with the straight black dashed line. *Right panel*: a zoom into the inner region of the accretion flow, obtained as an average over the time interval of $t \in [19000, 21000]$ t_g. Poloidal gas velocity vectors are indicated with green arrows.

NkS in pseudo-Newtonian potential

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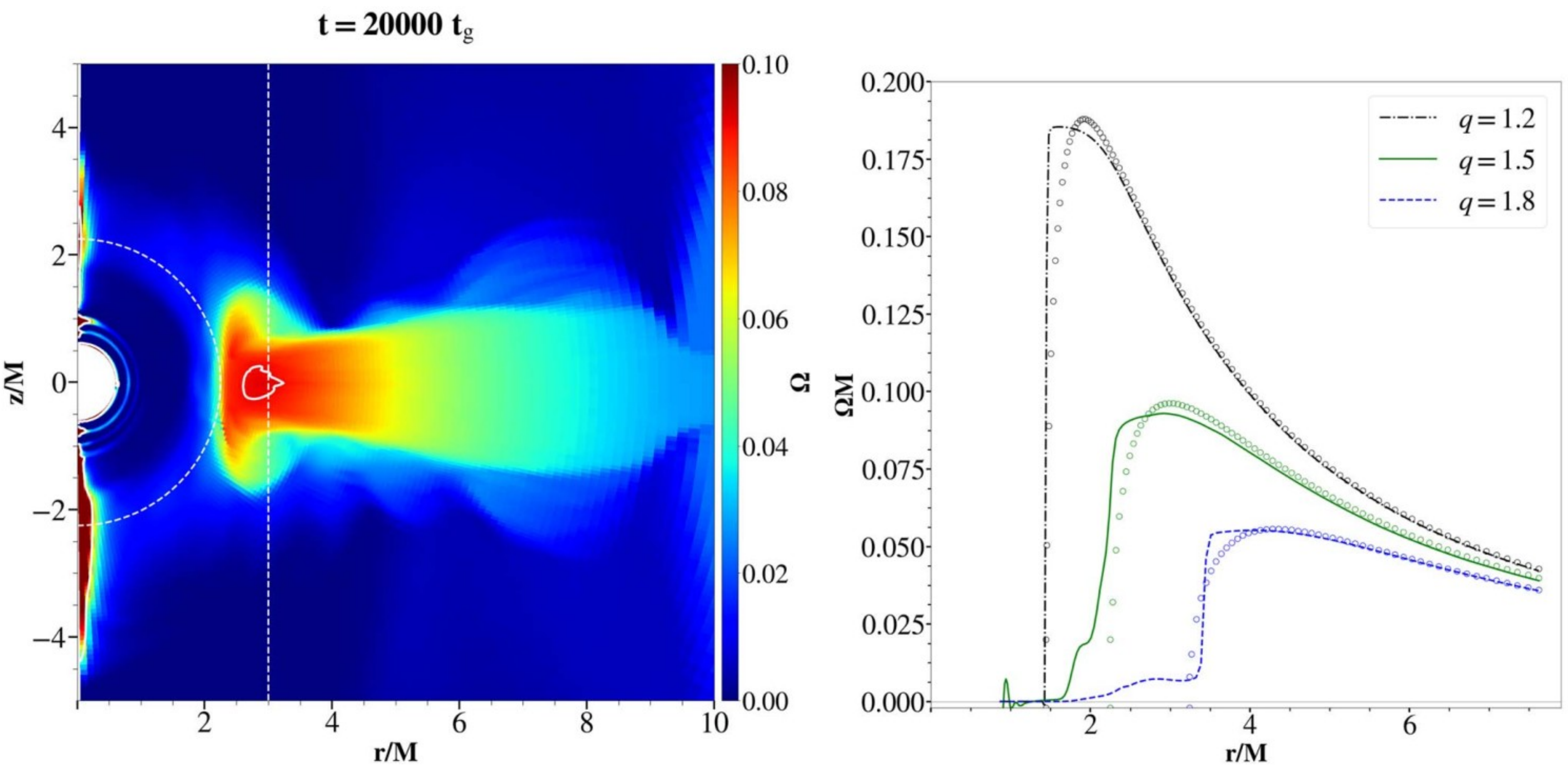

Figure 3. Left panel: angular velocity $\Omega = v_\phi/(r \sin\theta)$ in a linear color grading in a snapshot at $t = 20{,}000\, t_g$ in our simulation with $q = 1.5$. The contour of $\Omega = 0.09/M$, within which is located the test-particle orbital frequency value of $\Omega_{\max}$ at $r/M = 4q^2/3 = 3$, is shown with the white solid curve. The white dashed circular line indicates the zero-gravity sphere. Right panel: angular velocities in the equatorial plane in our numerical simulations, for NkS with $q = 1.2$, 1.5, and 1.8, are shown in dotted–dashed (black), solid (green), and dashed (blue) lines, respectively. Corresponding angular velocity profiles for test particles in RN metric Keplerian orbits are represented with small circles in the same color coding.

There are many such approximations, each good for its purpose-some characteristic distances or surfaces are usually represented correctly in such pseudo-potentials.

NkS in pseudo-Newtonian potential

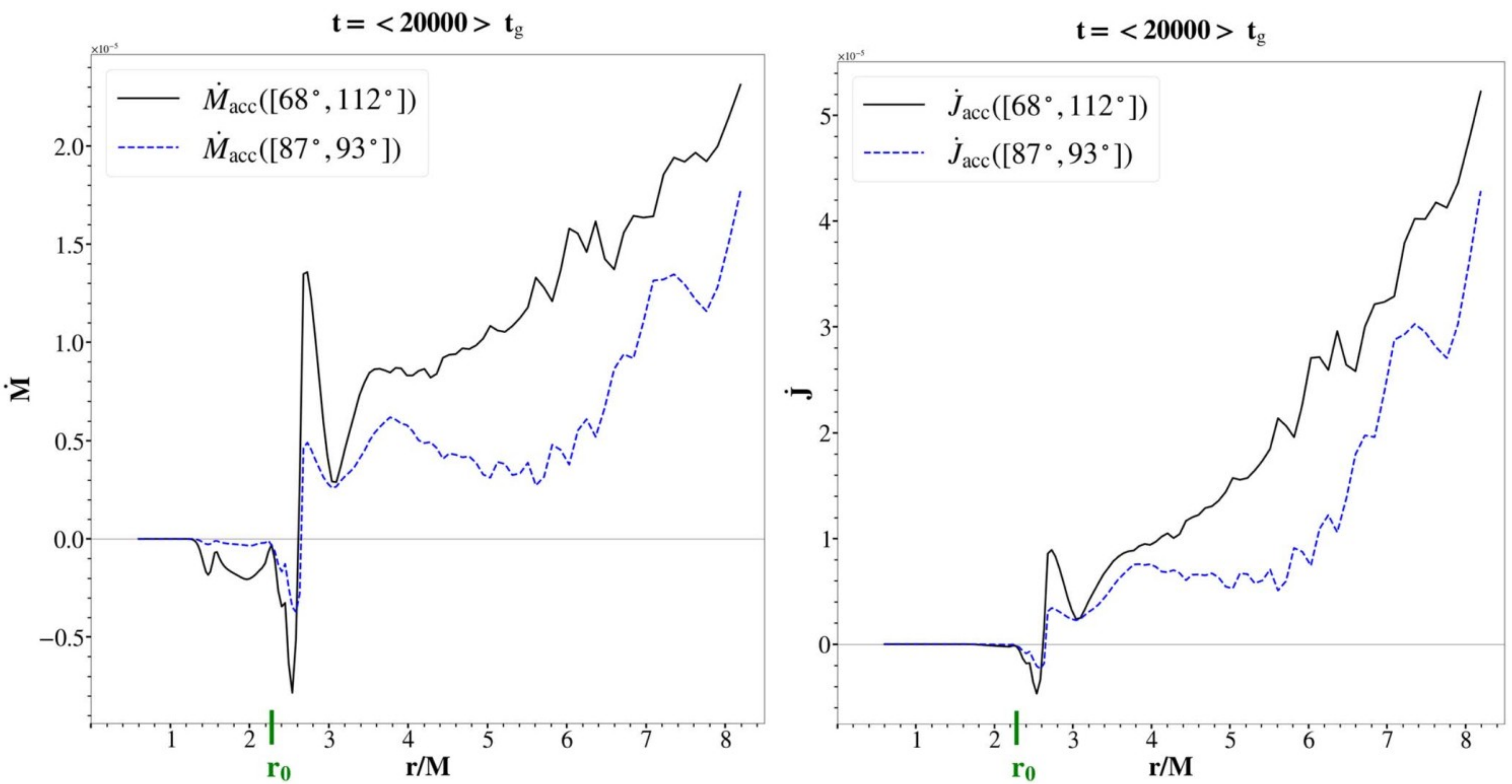

Figure 4. Values of $\dot{M}$ and $\dot{J}$ in code units, computed in two colatitudinal intervals as assigned in the legends, in dependence of radial distance from the origin in the simulation from Figure 2 with $q = 1.5$, obtained as an average over the time interval of $t \in [19{,}000, 21{,}000]\ t_g$. The zero-gravity radius r_0 is marked with the short green vertical line.

Conclusion

Series of simulations of naked singularities can be used as a model for creating the observational pictures of M87 and Sgr A*.

Magnetized solutions with NkS are much easier to investigate with this tool.

NkS in binary objects could also be simulated with such pseudo-potential.