Numerical simulations of accretion discs

Tomasz Krajewski Cosmological simulations

The birth of modern cosmology

The Einstein equation:

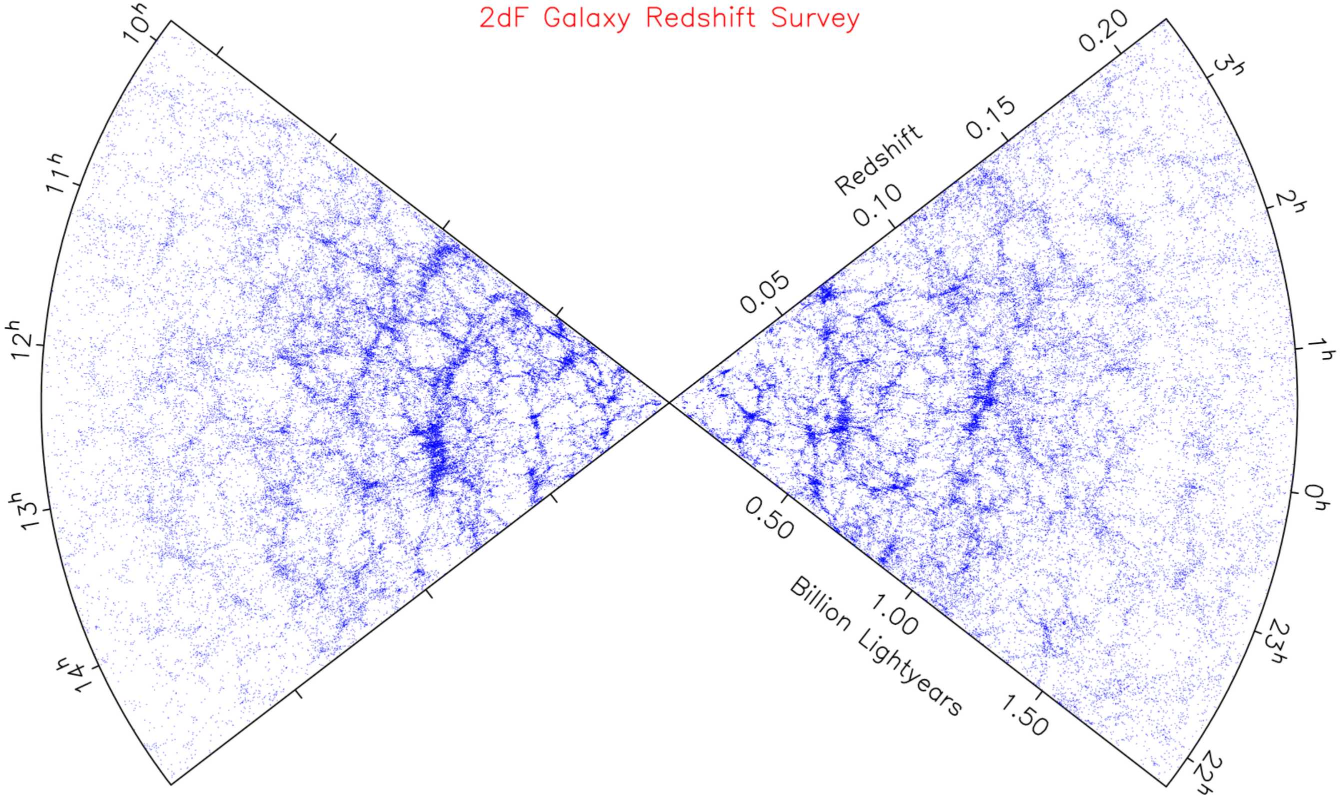
$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu},$$

has homogeneous and isotropic solution known as Friedmann–Lemaître–Robertson–Walker metric:

$$g = dt \otimes dt - a^2(t)\left(\frac{dr \otimes dr}{1 - kr^2} + r^2 d\Omega\right),$$

where k belongs to $\{-1, 0, 1\}$.

Large Scale Structures



Visualization of data from 2dF galaxy catalog.[1]

1. Colless, M. *et al. Mon. Not. Roy. Astron. Soc.* **328,** 1039. arXiv: `astro-ph/0106498 [astro-ph]` (2001).

Test particle in FRW metric

Geodesic equations are

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0.$$

The only non-zero Christoffel symbols for FRW metric are $\Gamma^0{}_{ii}$, $\Gamma^i{}_{0i}$, $\Gamma^i{}_{i0}$ and $\Gamma^i{}_{jk}$.
For spacelike components of 4-velocity geodesic equations take form:

$$\frac{d^2x^i}{d\tau^2} + 2\Gamma^i{}_{0i}\frac{dx^0}{d\tau}\frac{dx^i}{d\tau} + \Gamma^i{}_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0.$$

If initially all spacelike components are zero $\frac{dx^i}{d\tau} = 0$: $i = 1, 2, 3$, the trajectory with constant coordinates $x^i = const$ is the solution.
Coordinates are comoving with bodies in rest.

Conformal horizon

Solving for trajectories of light rays we obtain:

$$\eta(t_2, t_1) := \int_{t_1}^{t_2} \frac{dt}{a(t)} = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} \text{arcsin}\, r_2 - \text{arcsin}\, r_1 & \text{if } k = 1, \\ r_2 - r_1 & \text{if } k = 0, \\ \text{arsinh}\, r_2 - \text{arsinh}\, r_1 & \text{if } k = -1, \end{cases}$$

where we defined the conformal horizon η.
FRW metric in terms of conformal horizon η takes the (conformal) form:

$$g = a^2(t(\eta)) \left(d\eta \otimes d\eta - \frac{dr \otimes dr}{1-kr^2} + r^2 d\Omega \right).$$

Redshift

Let us consider a source with frequency $\nu_1 = \delta t_1^{-1}$ at comoving constant distance r_1. Then we have

$$\eta(t_0, t_1) = \eta(t_0 + \delta t_0, t_1, +\delta t_1),$$

where by δt_0 we denoted the period of the received signal.
Expanding in δt_0 and δt_1 which we assume are much shorter than the timescale of the evolution of the Universe we get

$$\nu_1 := \delta t_1^{-1} = \frac{a(t_0)}{a(t_1)} \delta t_0^{-1} = \frac{a(t_0)}{a(t_1)} \nu_0.$$

For convenience we denote $\frac{a(t_1)}{a(t_0)} = 1 + z$, so z is the redshift of the signal.

Hubble's law

Let us assume that this source emitted the during δt_0 the energy of $L\delta t_0$. When received at comoving distance r_1 the energy of redshifted signal will be distributed at sphere of surface $4\pi r_1^2 a(t_0)^2$, but will be received during δt_1 period, thus the power will be suppressed by $(1+z)^{-1}$. The energy flux of the received signal will be

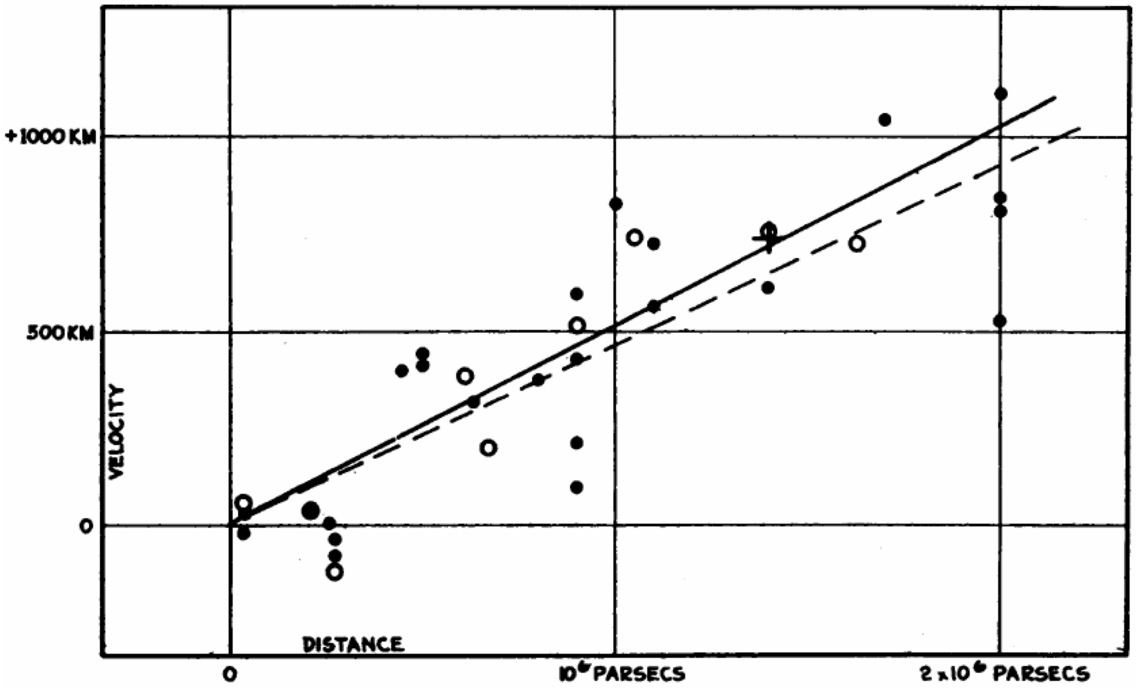
$$F = \frac{L(1+z)^2}{4\pi r_1^2 a(t_0)^2} = \frac{L}{4\pi d_L^2},$$

where d_L is the luminosity distance which we found equal to $d_L = a(t_0) r_1 (1+z)$. Expanding in small $t_0 - t_1$ (so also small r_1) one gets

$$r_1 = a(t_0)^{-1} H(t_0)^{-1} z + \mathcal{O}(z^2),$$

where $H := a^{-1}\frac{\partial a}{\partial t}$ is Hubble parameter, thus $d_L = H(t_0)^{-1} z + \mathcal{O}(z^2)$.

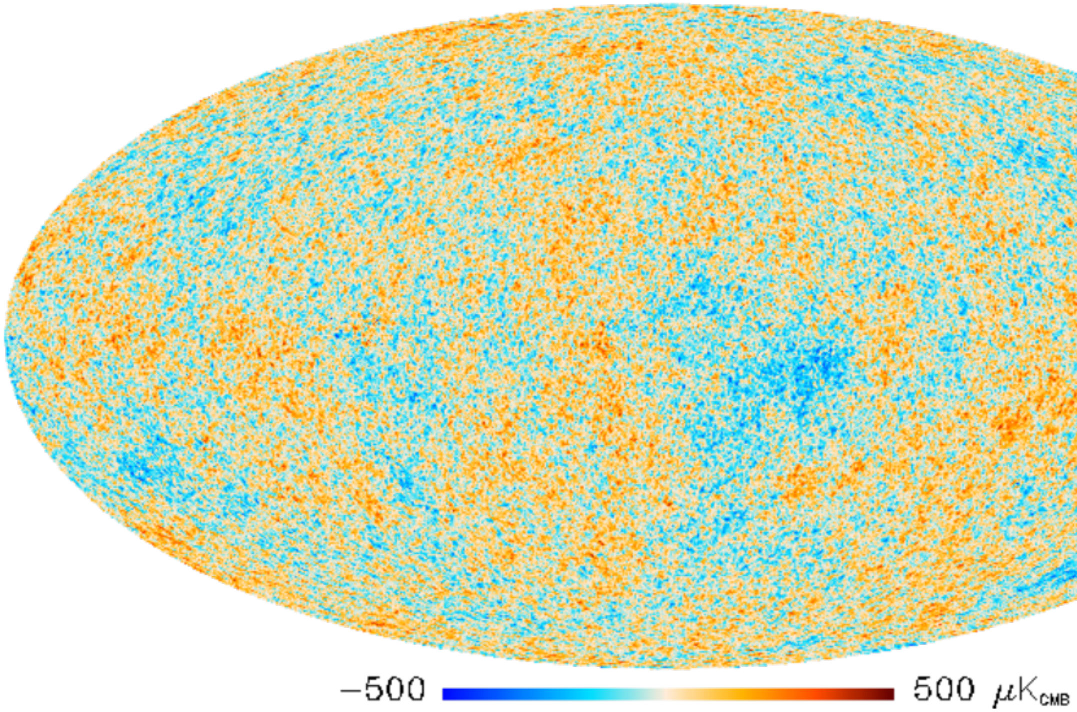
The Universe is expanding



Radial velocities are plotted against distances estimated from involved stars and mean luminosities of nebulae in a cluster.[2]

2. Hubble, E. *Proc. Nat. Acad. Sci.* **15,** 168–173 (1929).

Cosmic Microwave Background (CMB) Radiation



CMB map reconstructed from Planck data.[3]

3. Ade, P. A. R. *et al.* *Astron. Astrophys.* **571,** A1. arXiv: `1303.5062 [astro-ph.CO]` (2014).

Ideal fluid and Friedman equation

If one assume the energy-momentum tensor in the form of one of the perfect fluid, the Einstein equation simplifies to the set of Friedman equations:

$$H^2 := \left(a^{-1}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho + \frac{c^2}{3}\Lambda - \frac{kc^2}{a^2},$$

$$a^{-2}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\left(\rho + \frac{3}{c^2}p\right) + \frac{c^2}{2}\Lambda,$$

where H is Hubble parameter, ρ is the energy density and p is the pressure. The critical energy density ρ_c is defined as:

$$\rho_c := \frac{3H^2}{8\pi G} =: 3M_{Pl}^2 H^2.$$

Dust, radiation, dark energy

Effective equations of state in the form

$$p = w\rho$$

are often used to characterize of components of the Universe:

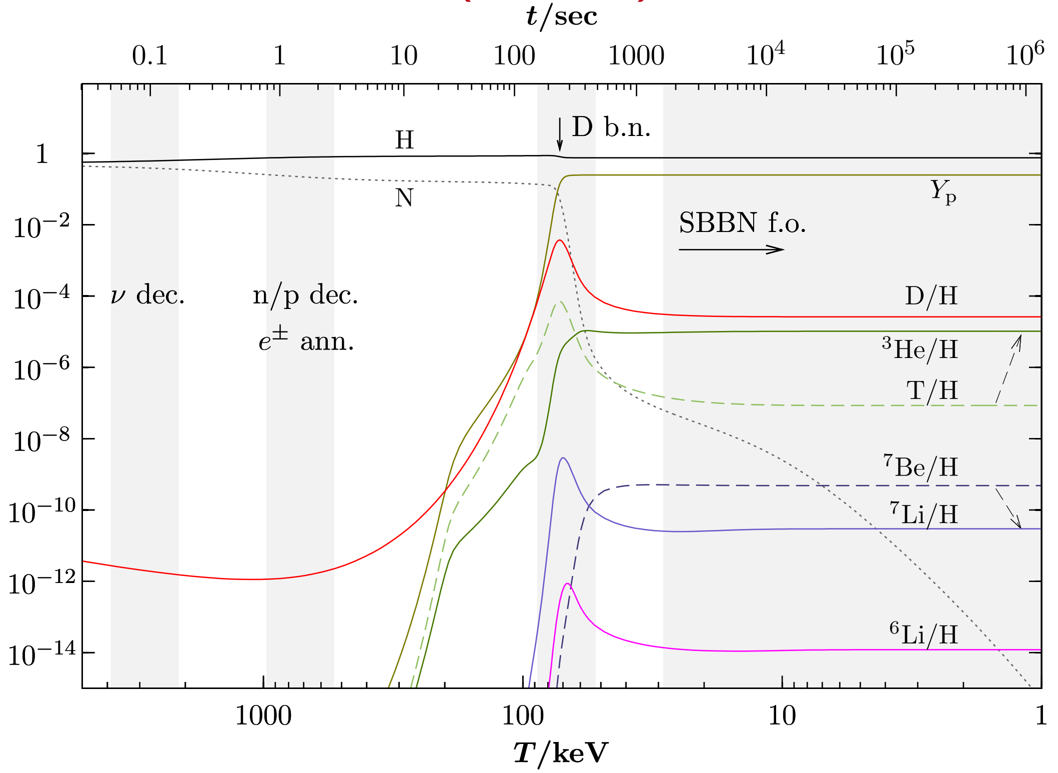
- $w = 1/3$ — radiation,
- $w = 0$ — dust,
- $w < -1/3$ — dark energy ($w = -1$ — cosmological constant).

Energy conservation implicates that energy densities have to scale as:

$$\rho \propto a^{-3(1+w)},$$

- $\rho_R \propto a^{-4}$ — for radiation,
- $\rho_D \propto a^{-3}$ — for dust,
- $\rho_\Lambda \propto a^0 = const$ — for cosmological constant.

Big Bang Nucleosynthesis (BBN)



Production of light elements in BBN.[4]

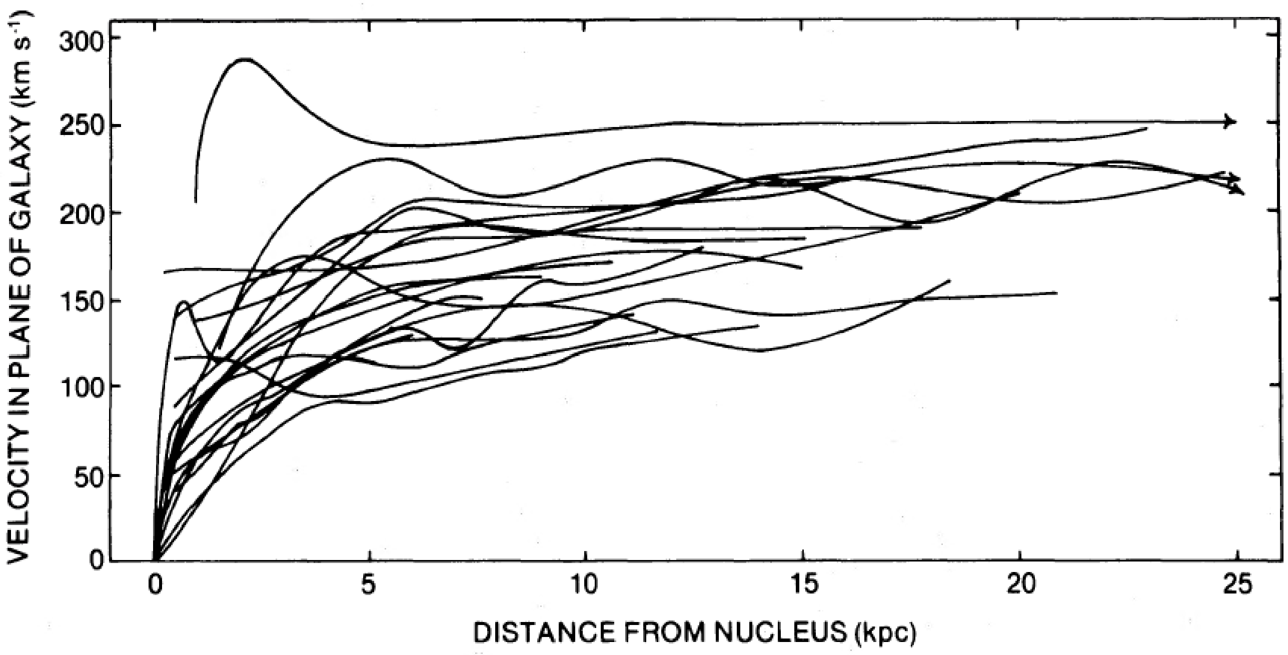
4. Pospelov, M. & Pradler, J. *Ann. Rev. Nucl. Part. Sci.* **60,** 539–568. arXiv: `1011.1054 [hep-ph]` (2010).

Effective number of neutrino species

- BBN is highly sensitive to the expansion history during it.
- Current observations are consistent with assumption that BBN took place during radiation domination era.
- Excess of relativistic degrees of freedom is parametrized by the so-called effective number of neutrino species N_{eff}, with current experimental estimation $N_{\text{eff}} = 3.27 \pm 0.15$.[5]
- Uncertainty of N_{eff} puts upper bound on the energy density of other forms of radiation like gravitational waves at the level of $h^2\Omega \leq 1.85 \times 10^{-6}$.

5. Aghanim, N. *et al. Astron. Astrophys.* **641.** [Erratum: Astron.Astrophys. 652, C4 (2021)], A6. arXiv: `1807.06209 [astro-ph.CO]` (2020).

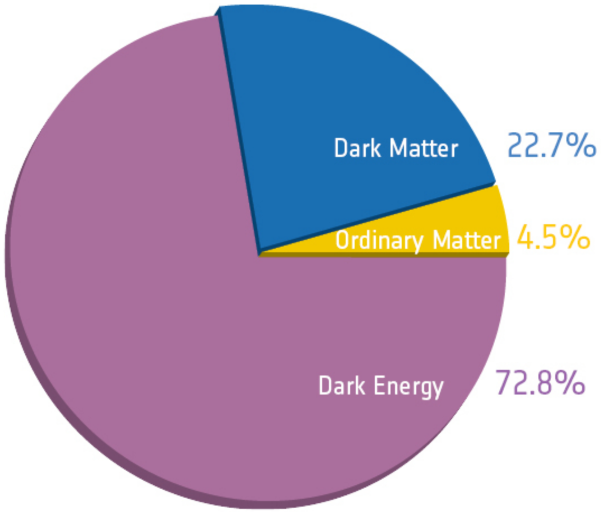
Rotational curves of galaxies



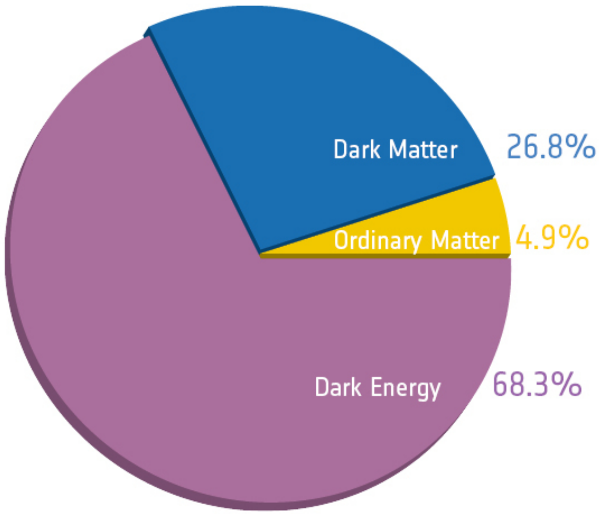
Rotational curves of 21 galaxies.[6]

6. Rubin, V. C. *et al.* *The Astrophysical Journal* **238,** 471–487 (1980).

Present content of the Universe



Before Planck



After Planck

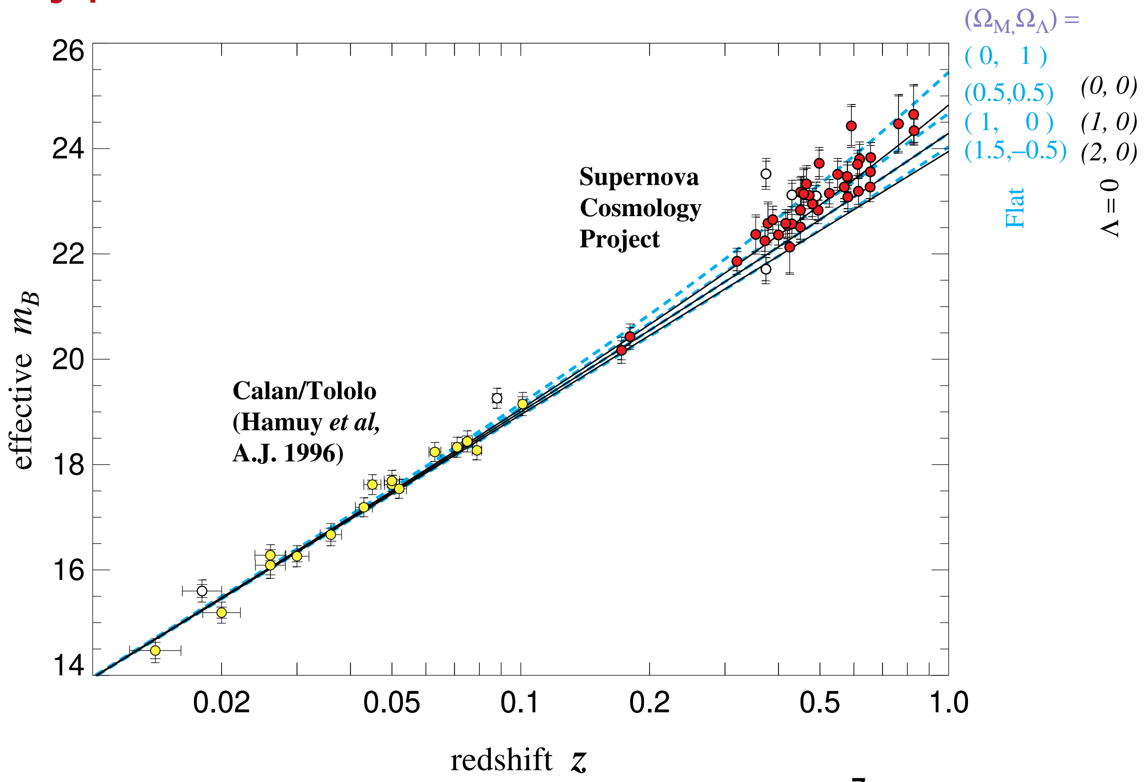
Equality of matter and radiation

Present energy density of radiation ρ_{R0} poses only a fraction of the present energy density of non-relativistic matter ρ_{M0}. However, in the past situation was different and radiation was a main component of the Universe. Time t_{EQ} (and scale factor a_{EQ}) when energy densities of matter and radiation were equal can be estimated from present values of these densities and Hubble constant:

$$a_{EQ} \approx a_0 \frac{\rho_{R0}}{\rho_{M0}} \qquad t_0 - t_{EQ} \approx (6\pi G \rho_{M0})^{-\frac{1}{2}} \left[1 - \left(\frac{\rho_{R0}}{\rho_{M0}} \right)^{\frac{3}{2}} \right],$$

where subscript 0 indicates present values.

Supernovae type Ia



Hubble diagram for Type Ia supernovae.[7]

7. Perlmutter, S. *et al. Astrophys. J.* **517,** 565–586. arXiv: `astro-ph/9812133` (1999).

Conclusions from observational data

- The current Universe is homogeneous at scales larger than $100\mathrm{Mpc}$.
- The Universe was (nearly) isotropic during recombination. Relative fluctuations of the CMB radiation are of the order of:

$$\frac{\Delta T}{T} \sim 10^{-5}.$$

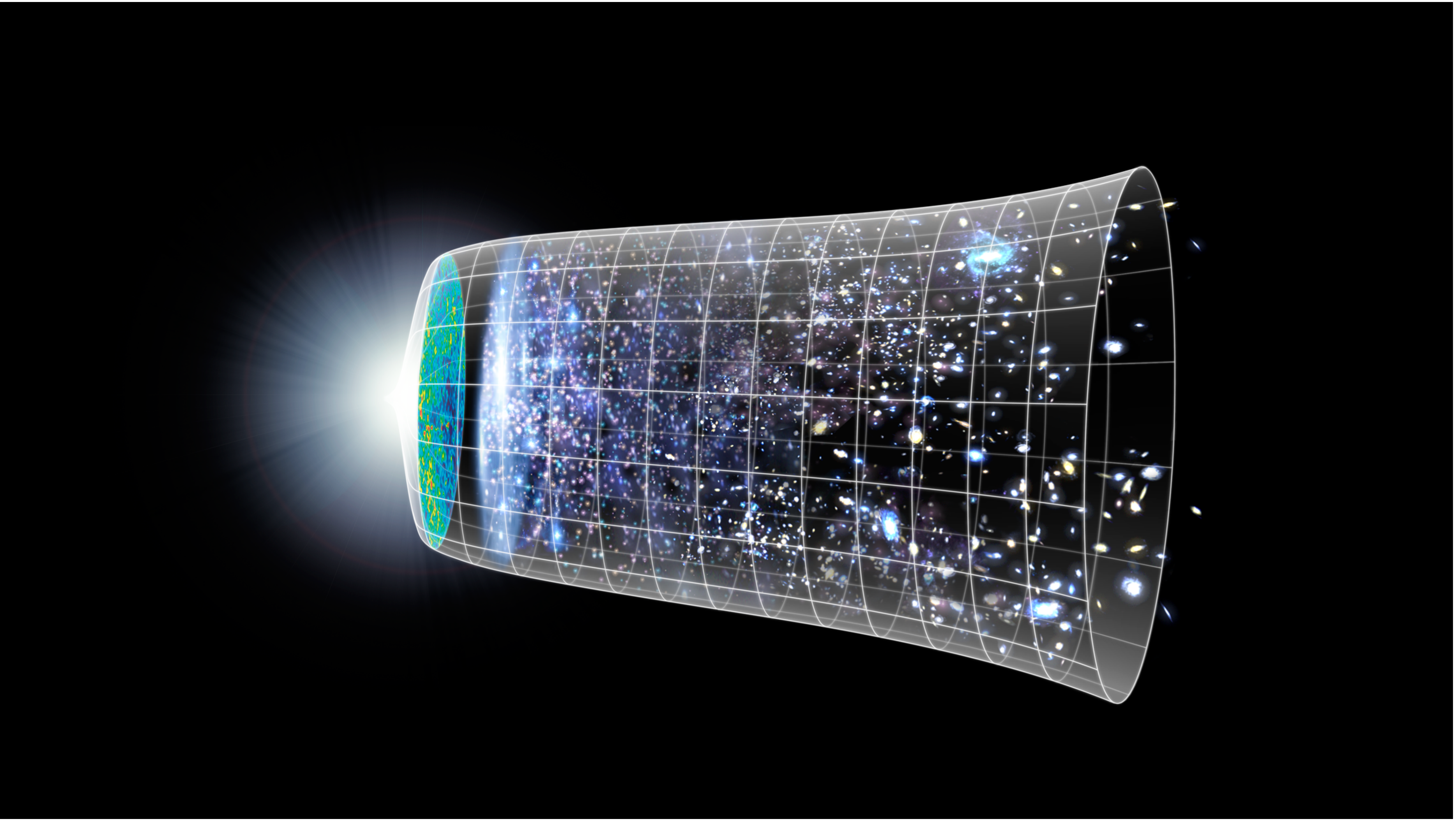
- The energy density of the Universe during BBN was dominated by radiation (particles moving with relativistic speeds).
- Currently, the expansion of the Universe accelerate.

ΛCDM model

In the standard cosmological scenario, it is assumed that the radiation domination epoch continued from the Big Bang up to the radiation-matter equality and was followed by the matter domination epoch. The Universe recently entered the cosmological constant domination epoch.

	Planck TT,TE,EE+lowE+lensing	+BAO
$\Omega_b h^2$	0.02237 ± 0.00015	0.02242 ± 0.00014
$\Omega_c h^2$	0.1200 ± 0.0012	0.1193 ± 0.0009
$100\,\theta_{MC}$	1.0409 ± 0.0003	1.0410 ± 0.0003
n_s	0.965 ± 0.004	0.966 ± 0.004
τ	0.054 ± 0.007	0.056 ± 0.007
$\ln(10^{10}\Delta^2_{\mathcal{R}})$	3.044 ± 0.014	3.047 ± 0.014

8. Lahav, O. & Liddle, A. R. arXiv: 2201.08666 [astro-ph.CO] (Jan. 2022).



Cosmological inflation

Cosmological inflation allows for simultaneous solution for many problems in cosmology:

- horizon problem
- flatness problem
- magnetic monopoles problem

Moreover, it provides a very natural explanation of CMB inhomogeneities.

Horizon problem

Comparing conformal horizon from time from the Big Bang $t = 0$ to emission of the CMB radiation t_{CMB} to one from emission to the present time we find:

$$\frac{\eta(t_0, t_{CMB})}{\eta(t_{CMB}, 0)} \approx 2\frac{\sqrt{1+z_{CMB}}-1}{2-\sqrt{\frac{1+z_{CMB}}{1+z_{EQ}}}},$$

where $z_{CMB} = \frac{a_0}{a_{CMB}} - 1$ and $z_{EQ} = \frac{a_0}{a_{EQ}} - 1$ are appropriate redshifts.
With measured values $z_{EQ} = 3365 \pm 44$ and $z_{CMB} = 1089.9 \pm 0.4$, we get:

$$\frac{\eta(t_0, t_{CMB})}{\eta(t_{CMB}, 0)} \approx 45.$$

Thus, the standard cosmology predicts that the observed CMB radiation was emitted from regions of the Universe which have never been causally connected.

Flatness problem

From Friedman equations we find that the difference of the total energy density ρ_{tot} to the critical energy density evolve as:

$$\frac{\rho_{tot} - \rho_c}{\rho_c} = \frac{kc^2}{\frac{da}{dt}}.$$

In standard cosmological scenario the expansion of the Universe decelerate, thus the difference grows in time.
However, measured present value is equal to $-0.005^{+0.016}_{-0.017}$. This means that the Universe had to be very flat in the past.

Inflation

The alternative to the fine-tuning of the energy density of the early Universe is the existence of the epoch of accelerated expansion called inflationary epoch. Moreover inflation can solve the horizon problem.
For $a \propto t^p$ $(-1 < w < -\frac{1}{3})$:

$$\eta(t_e, t_b) = \frac{1}{p-1}\frac{1}{2H_e a_e}\left[\left(\frac{a_e}{a_b}\right)^{1-\frac{1}{p}} - 1\right]$$

or for $a \propto \exp t$ $(w = -1)$:

$$\eta(t_e, t_b) = \frac{1}{H_e a_e}\left(\frac{a_e}{a_b} - 1\right),$$

where subscript b correspond to the beginning of inflation and e to the end of inflation and beginning of the radiation domination epoch.

Inflaton field

The usual source of inflation is a scalar field. Assuming canonical kinetic term in Lagrangian density:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} - V(\phi),$$

which gives (in FRW background) the EOM in the form:

$$\frac{\partial^2 \phi}{\partial t^2} + 3H \frac{\partial \phi}{\partial t} - \frac{1}{a^2} \Delta \phi + \frac{\partial V}{\partial \phi}(\phi) = 0,$$

and energy-momentum tensor:

$$T^\alpha_\beta = g^{\alpha\nu} \frac{\partial \phi}{\partial x^\nu} \frac{\partial \phi}{\partial x^\beta} - g^\alpha_\beta \left[\frac{1}{2} g^{\mu\nu} \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} + V(\phi) \right].$$

Slow-roll inflation

Neglecting spacial inhomogeneities we get:

$$\rho_I = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + V(\phi) \qquad\qquad p_I = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - V(\phi)$$

Thus, for $\frac{d^2 a}{dt^2} > 0$ we need $p < \frac{1}{3}\rho$ what implicate $V(\phi) > \left(\frac{\partial \phi}{\partial t}\right)^2$.
In order to get as fast expansion as it can be we want the Hubble parameter H to be constant what means $\left(\frac{\partial \phi}{\partial t}\right)^2 \ll V(\phi)$, so slowly varying field.
During slow-roll phase the Univese expands nearly exponentially $a(t) \propto e^{Ht}$. For solving the horizon problem needed expansion is $\frac{a_e}{a_b} \sim e^{50} - e^{60}$ $(50 - 60$ e-folds).
Unfortunately, slow-roll requires extremely flat potentials.

Slow-roll parameters

The deviation from purely de Sitter exponential expansion $a(t) = e^{Ht}$ is usually parametrized by slow-roll parameters:

$$\epsilon_1 = -H^{-2}\frac{\partial H}{\partial t}, \qquad \epsilon_2 = H^{-1}\epsilon_1^{-1}\frac{\partial \epsilon_1}{\partial t}.$$

Condition of accelerated expansion leads to $\epsilon_1 < 1$. $\epsilon_2 \ll 1$ is required for inflation to be long enough to solve horizon problem.
In slow-roll regime the following expressions in terms of inflaton's potential V are valid

$$\epsilon_1 \approx \frac{M_{Pl}^2}{2}\frac{V'}{V}, \qquad \epsilon_2 \approx M_{Pl}^2\frac{V''}{V}$$

up to higher terms in slow-roll parameters.

Linear perturbations

Linear perturbations are described in terms of gauge invariant Mukhanov-Sasaki variables:

$$Q_\phi := \delta\phi + \frac{\partial\phi}{\partial t}\frac{\Psi}{H},$$

which obey the following equations of motion

$$\frac{\partial^2 Q_\phi}{\partial t^2} + 3H\frac{\partial Q_\phi}{\partial t} + \left(\frac{k^2}{a^2} + m_\phi^2(\phi)\right)Q_\phi = 0.$$

where the effective masses m_ϕ^2 of perturbations are:

$$m_\phi^2 = V''(\phi).$$

Mode functions

For nearly de Sitter expansion $a(t) \propto e^{Ht}$, $a(\eta)$ goes as

$$a(\eta) = -\frac{1}{H\eta} + \mathcal{O}(\text{slow roll parameters}).$$

Rewriting the Q_ϕ eom in terms of η one gets:

$$\frac{\partial^2 Q_\phi}{\partial \eta^2} + \left[k^2 - \eta^{-2}\left(\nu^2 - \frac{1}{4}\right)\right] Q_\phi = 0, \quad \nu = \frac{3}{2} + \mathcal{O}(\text{slow roll parameters}).$$

For ν real, the general solution of this equation is:

$$Q_\phi(\eta) = \sqrt{-\eta}\left(c_1(k) H_\nu^{(1)}(-k\eta) + c_2(k) H_\nu^{(2)}(-k\eta)\right)$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are Hankel functions of the first and second kind.

Horizon exit

For UV subhorizon modes $k/aH \gg 1$ we should recover the Mikowski limit:

$$Q_\phi \to \frac{1}{\sqrt{2k}} e^{-ik\eta},$$

thus

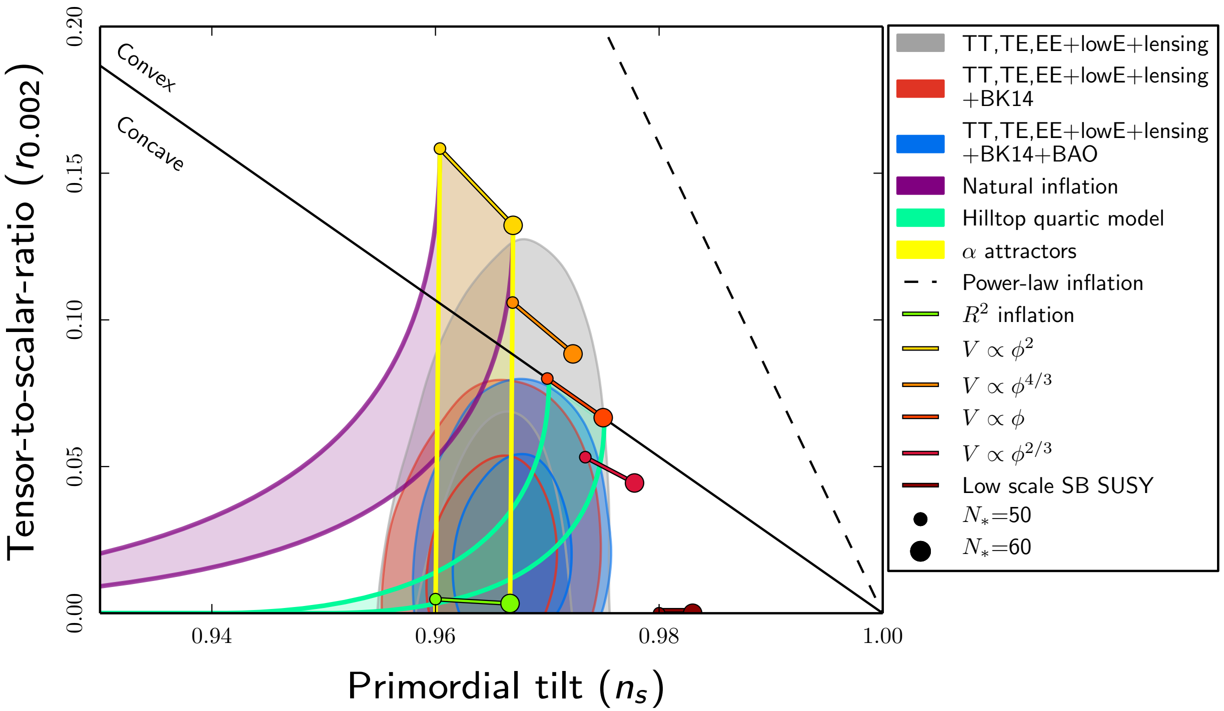
$$c_2(k) = 0 \qquad\qquad c_1(k) = \frac{\sqrt{\pi}}{2} e^{i(\nu+1/2)\pi/2}$$

In superhorizon limit $k/aH \ll 1$ one gets

$$Q_\phi(\eta) \to e^{i(\nu-1/2)\pi/2} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\eta)^{1/2-\nu}.$$

CMB fluctuations measurement

Predictions of some multi-field inflationary models (especially α-attractor ones[9]) fits very well to current measurements of CMB radiation inhomogeneities.



Constraints on inflationary models from Planck2018[10].

9. Carrasco, J. J. M. *et al.* *Phys. Rev. D* **92,** 063519. arXiv: 1506.00936 [hep-th] (2015).
10. Akrami, Y. *et al.* arXiv: 1807.06211 [astro-ph.CO] (2018).

Reheating

- The product of inflation is the large, flat and empty (cold) Universe. In order to fill the Universe with all the staff that we observe (for example us) the energy of inflaton had to be transferred to other degrees of freedom.
- In majority of models inflation ends when inflaton leaves the slow-roll regime and evolves toward minimum of the potential around which if oscillate producing particles (reheating the Universe).
- The reheating is not completely understood, because it is model dependent process.
- During the reheating some of unwanted, heavy, pre-inflationary relics (magnetic-monopoles, dragons, politicians, etc.) which were diluted by inflation, could be reintroduced.

One-field α-attractor T-model

In the literature one-field simplification is considered with the potential of the form:

$$V(\phi, 0) = M^4 \tanh^{2n}\left(\frac{\beta|\phi|}{2}\right).$$

After inflation inflaton oscillates around minimum of the potential loosing its energy in favour of other degrees of freedom.

Perturbation around oscillating inflaton

Assuming that the transfer of the energy from homogenous inflaton beakground to fluctuations is slow one may approximate it as evolution of (non)harmonic oscillations with slowly decaying amplitude.
The equations of motion for Mukhanov-Sasaki variables can be written in Fourier space as first order ordinary differential equations:

$$\begin{pmatrix} \dot{Q}_{\phi,k} \\ \dot{\Pi}_{\phi,k} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\left(\frac{k^2}{a^2} + m_\phi^2\right) & -3H \end{pmatrix} \begin{pmatrix} Q_{\phi,k} \\ \Pi_{\phi,k} \end{pmatrix} =: \Lambda_\phi(t) \begin{pmatrix} Q_{\phi,k} \\ \Pi_{\phi,k} \end{pmatrix},$$

with $\Lambda_\phi(t)$ being (nearly) periodic matrices.

Floquet Theorem

Let

$$\dot{x}(t) = U(t)x(t) \tag{1}$$

be a linear first order differential equation, where:

- $x(t)$ is a column vector of dim N,
- $U(t)$ is a periodic $N \times N$ matrix valued function with period T.

The fundamental solution $O(t, t_0)$ of (1) can be expressed as:

$$O(t, t_0) = P(t, t_0) \exp\left((t - t_0)V\right)$$

where:

- $P(t, t_0)$ is a periodic matrix valued function with period T,
- V is constant $N \times N$ matrix satisfying $O(t_0 + T, t_0) = \exp\left(TV\right)$.

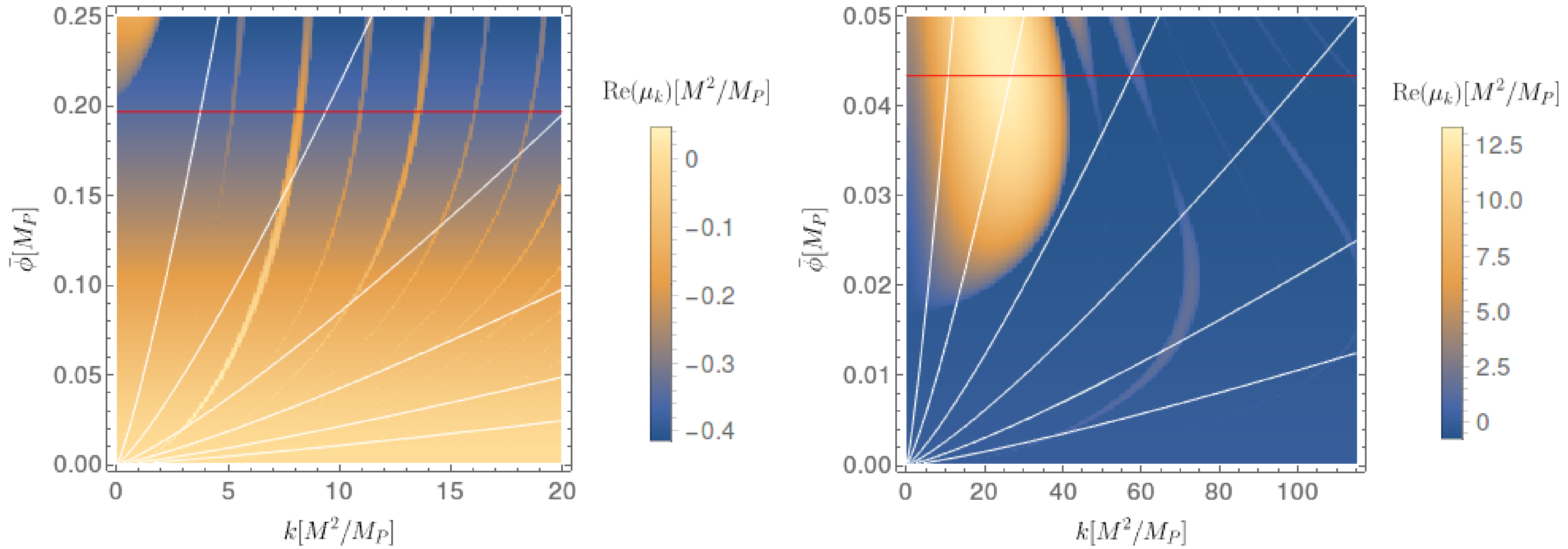
Amplification of linear perturbations

In virtue of Floquet theorem one deduces that linear perturbations can be approximated by:

$$Q\varphi, k(t) = \sum_{\psi=\phi,\chi} Q^{\psi}_{\varphi,k}(t, t_0) \exp\left(\mu^{\psi}_{\varphi,k}(t - t_0)\right)$$

during few oscillations of inflaton field.
Large positive real parts of Floquet exponents $\mu^{\psi}_{\varphi,k}$ indicate amplification of perturbations, thus instability.

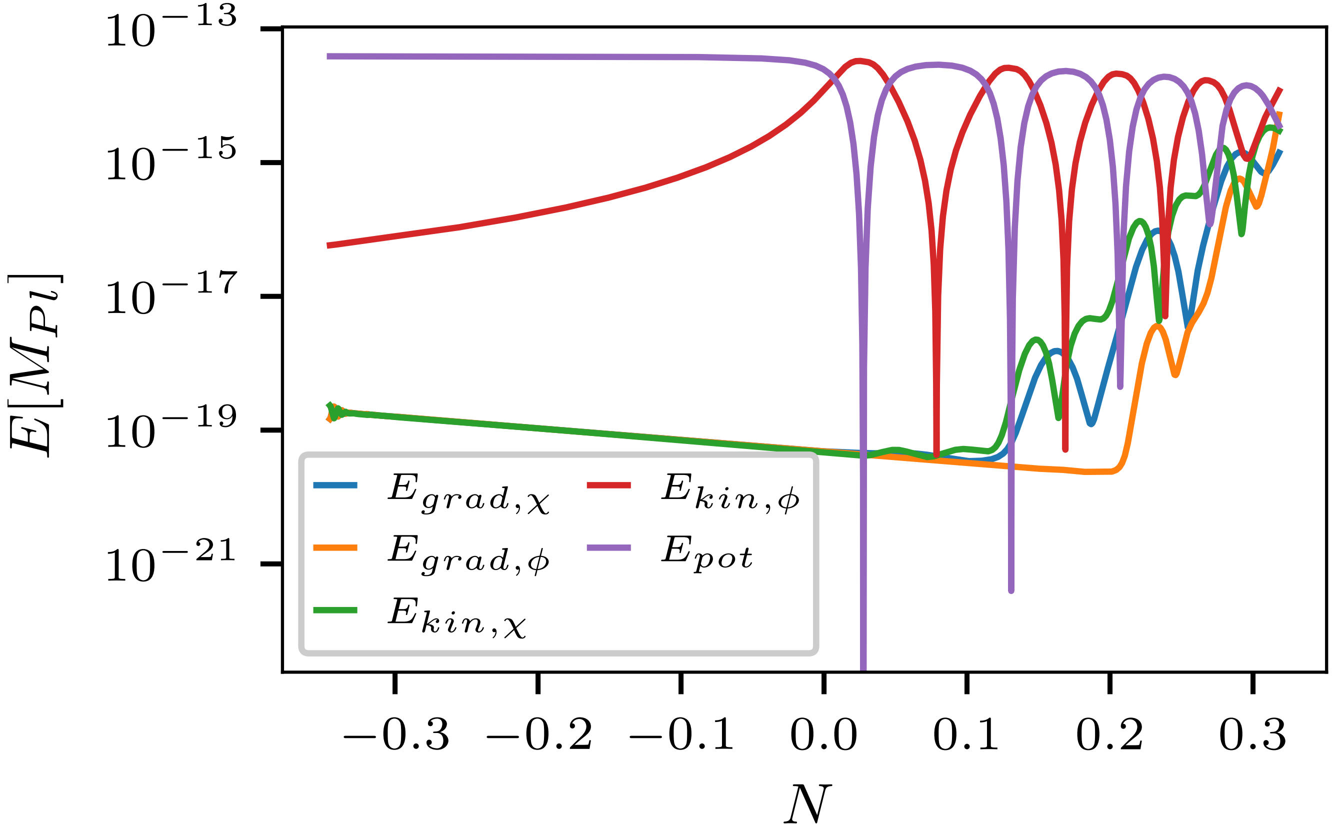
Floquet charts in α-attractors



Floquet exponents for the inflaton perturbations with $n = 3/2$ and $\alpha = 10^{-2}$ (left panel) and $\alpha = 10^{-4}$ (right panel).[11]

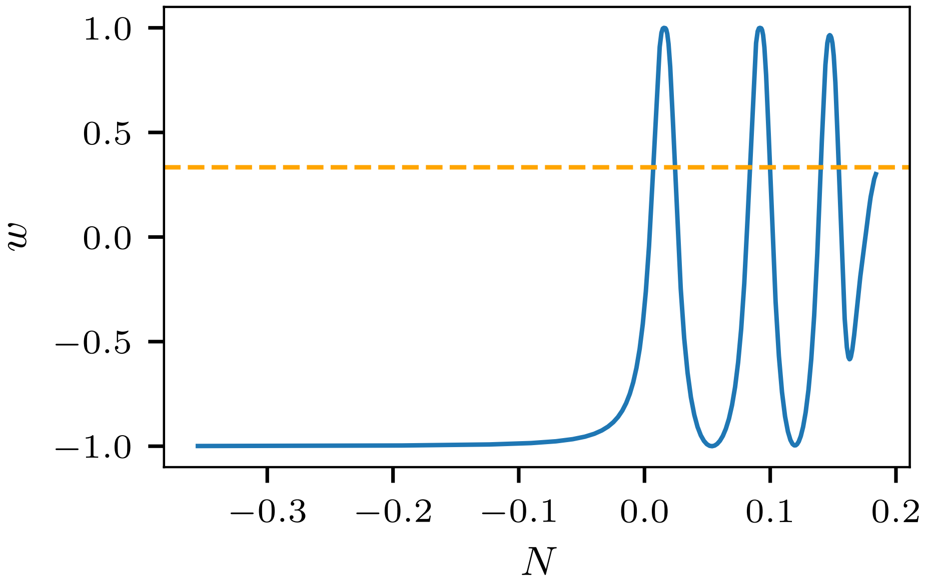
11. Krajewski, T. *et al. Eur. Phys. J. C* **79,** 654. arXiv: 1801.01786 [astro-ph.CO] (2019).

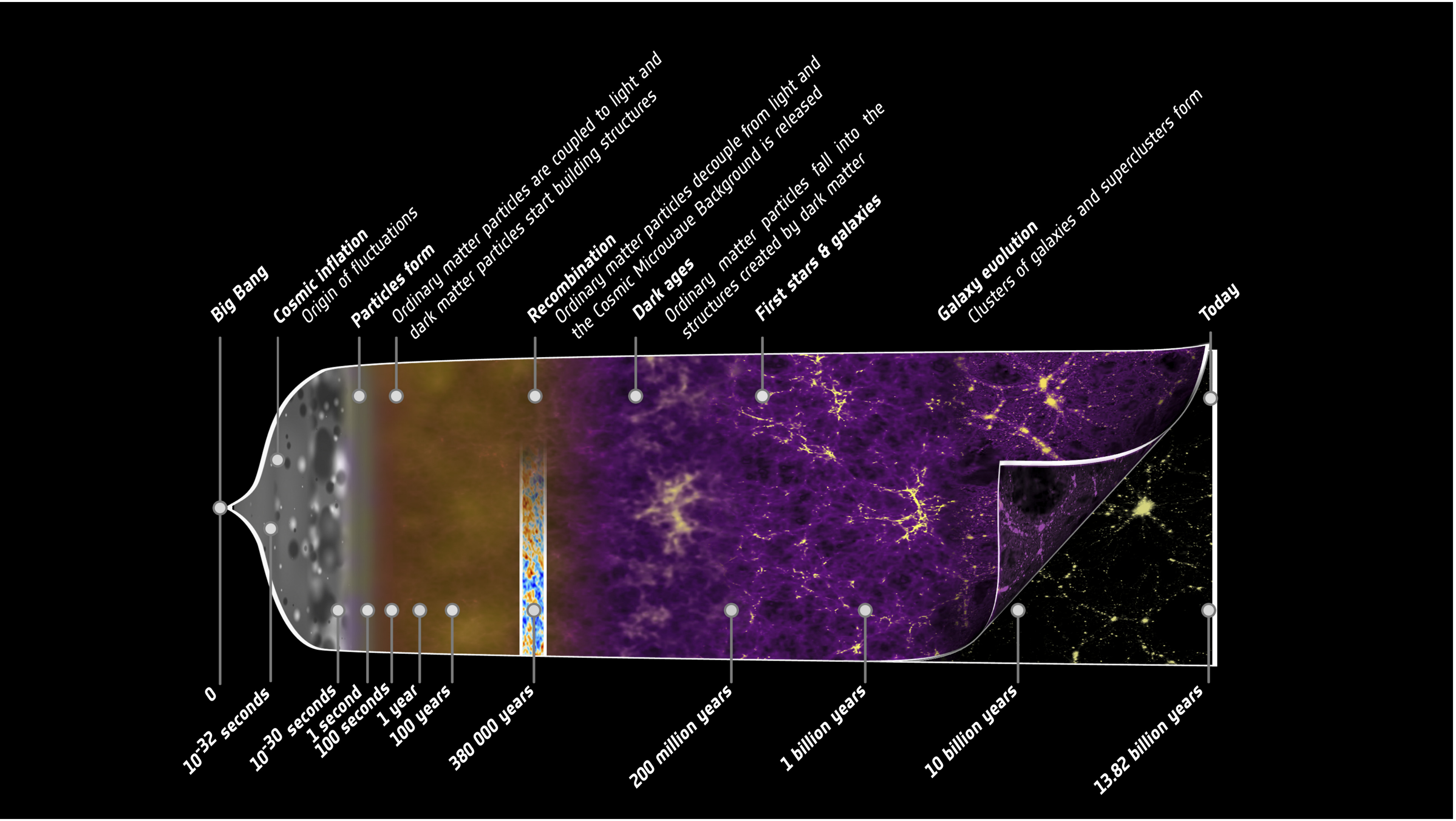
Preheating in two-fields α-attractors



Barotropic parameter

$$w = \frac{p}{\rho} = \frac{\langle \frac{1}{2}\left(\mathrm{e}^{2b(\chi)}\dot{\phi}^2 + \dot{\chi}^2\right) - \frac{1}{6a^2}\left(\mathrm{e}^{2b(\chi)}(\nabla\phi)^2 + (\nabla\chi)^2\right) - V(\phi,\chi)\rangle}{\langle \frac{1}{2}\left(\mathrm{e}^{2b(\chi)}\dot{\phi}^2 + \dot{\chi}^2\right) + \frac{1}{2a^2}\left(\mathrm{e}^{2b(\chi)}(\nabla\phi)^2 + (\nabla\chi)^2\right) + V(\phi,\chi)\rangle},$$





Big Bang
Cosmic inflation
Origin of fluctuations
Particles form
Ordinary matter particles are coupled to light and dark matter particles start building structures
Recombination
Ordinary matter particles decouple from light and the Cosmic Microwave Background is released
Dark ages
Ordinary matter particles fall into the structures created by dark matter
First stars & galaxies
Galaxy evolution
Clusters of galaxies and superclusters form
Today
0
10-32 seconds
10-30 seconds
1 second
100 seconds
1 year
100 years
380 000 years
200 million years
1 billion years
10 billion years
13.82 billion years

What are domain walls?

- Domain walls (DWs) are sheet-like topological defects.
- A potential with two (or more) local minima is necessary for the existence of DWs.
- Cosmological DWs could be produced in the early Universe during spontaneous symmetry breaking.
- DWs are formed at boundaries of regions (domains) where symmetry breaking field has different vacuum expectation values (VEVs).

Example

Let us consider the model given by the potential of the form:

$$V(\phi) = V_0 \left(\left(\frac{\phi^2}{{\phi_0}^2} \right) - 1 \right)^2 .$$

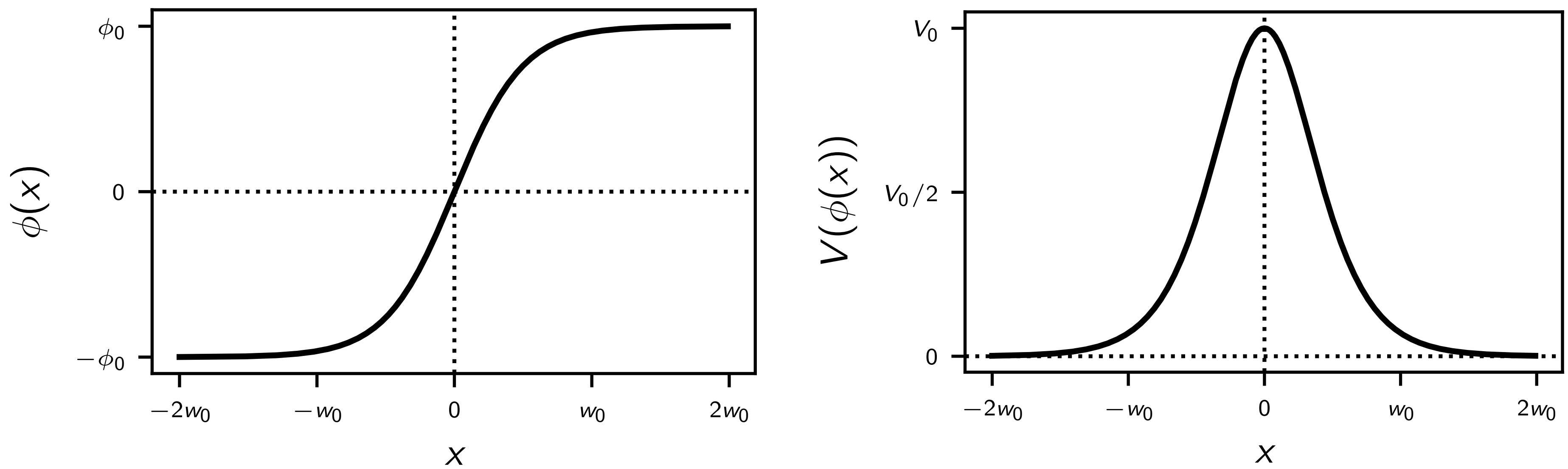
The EOM (in Minkowski background):

$$\frac{\partial^2 \phi}{\partial t^2} - \Delta \phi = 4 V_0 \left(\frac{\phi^2}{{\phi_0}^2} - 1 \right) \frac{\phi}{{\phi_0}^2}$$

has the time independent, planar solution

$$\phi(x) = \phi_0 \tanh \left(\frac{\pi x}{w_0} \right) ,$$

where $w_0 = \frac{\phi_0}{\sqrt{2 V_0}}$ is called width of the domain wall.



Profiles of the field strength (left panel) and of the potential energy density (right panel) of the soliton solution in the simple model.

First integral method

For time independent, planar solutions $\phi(t,x,y,z)=\varphi(x)$ the conserved quantity associated with the translational symmetry in x is

$$E=\frac{1}{2}(\partial_x\varphi)^2-V(\varphi)=\frac{1}{2}(\partial_x\varphi)^2-V_0\left(\frac{\varphi^2}{{\phi_0}^2}-1\right)^2.$$

The conservation of E leads to integral representation of solution:

$$x(\varphi_2)-x(\varphi_1)=\pm\int_{\varphi_1}^{\varphi_2}\frac{d\varphi}{\sqrt{2\left(E+V(\varphi)\right)}}.$$

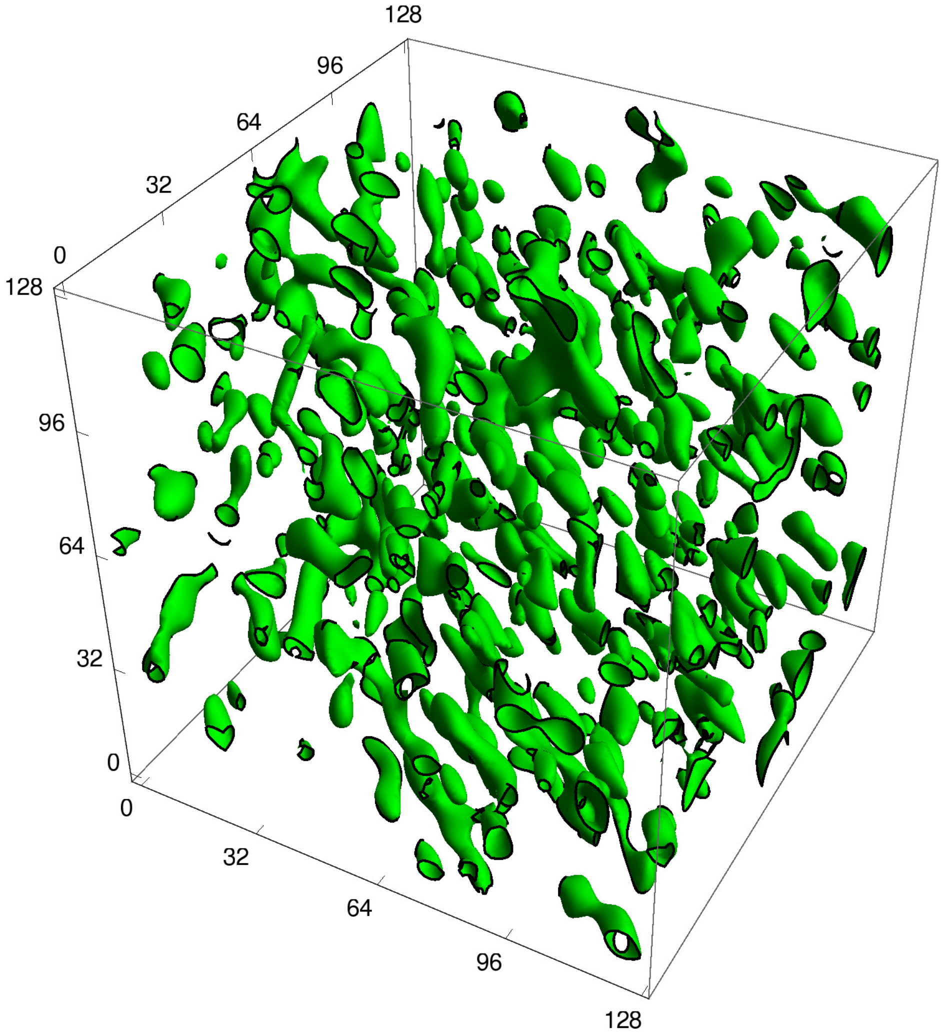
We can calculate surface potential energy, using:

$$\sigma(x_1,x_2):=\int_{x_1}^{x_2}V(\varphi(x))dx=\int_{\varphi(x_1)}^{\varphi(x_2)}\frac{V(\varphi)d\varphi}{\sqrt{2\left(E+V(\varphi)\right)}}.$$

Networks of domain walls

Networks of cosmological domain walls could have twofold topologies:

- finite bubbles of one vacuum in a sea of the other,
- an infinite network spreading through whole Universe.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96
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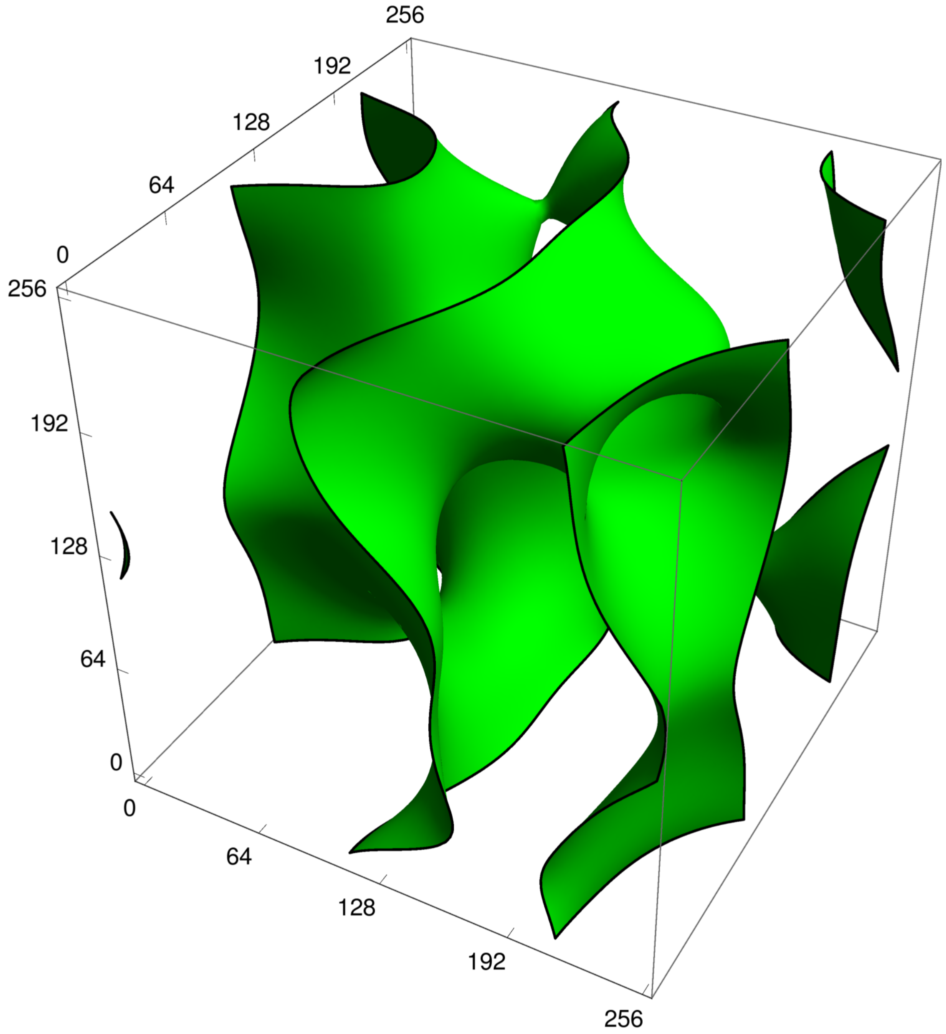
128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0
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192
256

VOS

The velocity depended one scale model (VOS) derived from Nambu-Goto action describes evolution of average length scale:

$$L := \frac{\sigma_{wall}}{\rho_{wall}}, \tag{2}$$

and the average velocity norm of the walls v.

$$\frac{dL}{dt} = (1+3v^2)HL + F(v), \qquad \frac{dv}{dt} = (1-v^2)\left(\frac{k(v)}{L} - 3Hv\right), \tag{3}$$

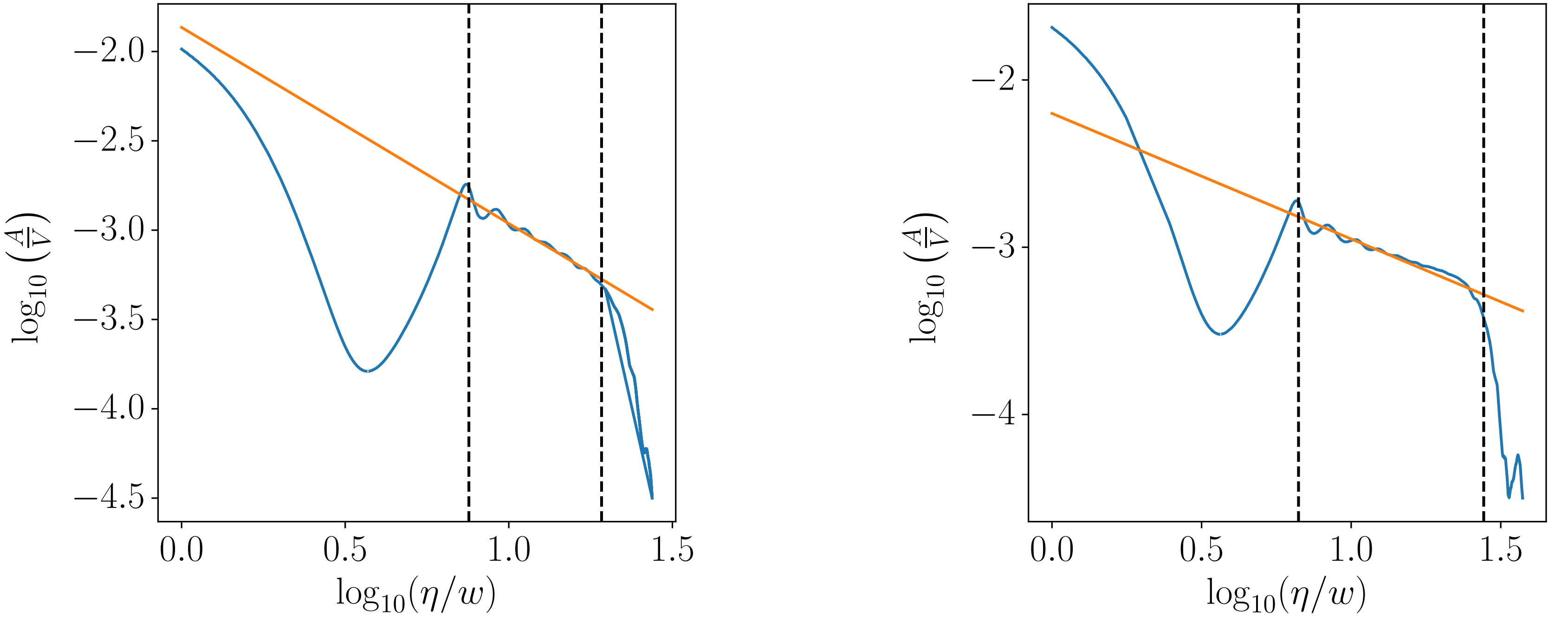
VOS has a simple scaling solution

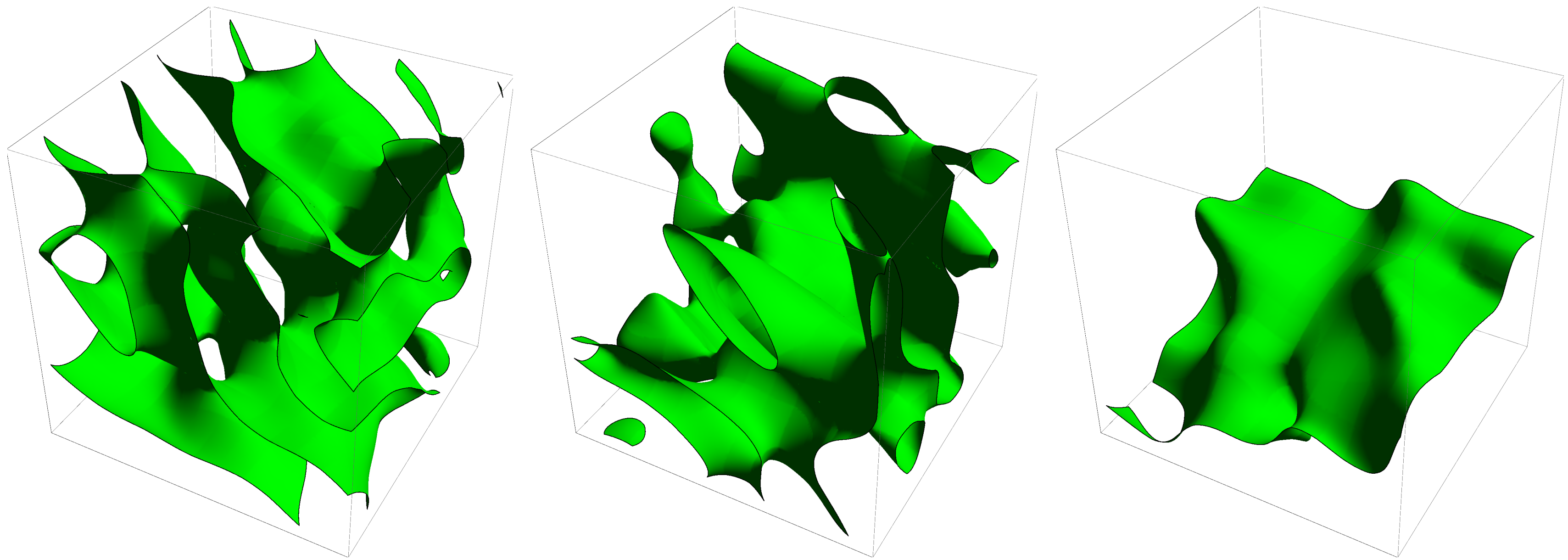
$$L = \epsilon t, \qquad v = const, \tag{4}$$

for the scale factor with power law dependence on time $a \propto t^{\lambda}$.

Scaling regime

So called scaling regime when $\frac{\sigma_{wall}}{\rho_{wall}} \propto t$ is an attractor of evolution of metastable networks.





The visualization of the network of SM domain walls obtained during a simulation at tree different times: $\eta = 1.7 \times 10^{-10}\ \mathrm{GeV}^{-1}$ (left panel), $\eta = 4.0 \times 10^{-10}\ \mathrm{GeV}^{-1}$ (middle panel) and $\eta = 5.5 \times 10^{-10}\ \mathrm{GeV}^{-1}$ (right panel).

Algorithm for computing the spectrum of GWs

We consider the linear perturbation expansion around the Friedman-Robertson-Walker background solution:

$$g = dt^2 - a^2(t)\,(\delta_{ij} + h_{ij})\,dx^i dx^j = a^2(\eta)\left(d\eta^2 - (\delta_{ij} + h_{ij})dx^i dx^j\right).$$

Linearised Einstein equation leads to equation

$$\partial_\eta{}^2 h_{ij} + 2\frac{\partial_\eta a}{a}\partial_\eta h_{ij} - \Delta h_{ij} = \frac{2}{M_{Pl}{}^2} T_{ij}^{TT},$$

where T^{TT} is the transverse-traceless part of the stress tensor T of the Higgs field approximated by

$$T_{ij} = \partial_{x^i}\phi\partial_{x^j}\phi.$$

Emission of gravitational waves

The solution of the EOM for perturbation in radiation dominated era can be easily solved in Fourier space using retarded Green's function:

$$\widehat{\bar{h}}_{ij}(\eta, k) = \frac{2}{{M_{Pl}}^2} \int_{\eta_i}^{\eta} d\eta' \frac{\sin\left(|k|\left(\eta - \eta'\right)\right)}{|k|} a(\eta') \widehat{T^{TT}}_{ij}(\eta', k),$$

where η_i is the value of conformal time before which the source $\widehat{T^{TT}}_{ij}$ appeared and

$$\bar{h}_{ij}(\eta, x) := a(\eta) h_{ij}(\eta, x).$$

It can be easily verified that after the disappearance of the source $\widehat{T^{TT}}_{ij}$ the perturbations are just red-shifted.

Sources

The traverse-traceless part of energy-momentum tensor can be expressed in Fourier space in compact form

$$\widehat{T^{TT}}_{ij}(k) = \mathcal{O}_{ijlm}(k)\widehat{T}_{ij}(k) = \mathcal{O}_{ijlm}(k)(\widehat{\partial_{x^i}\phi\partial_{x^j}\phi})(k),$$

using projection operators

$$\mathcal{O}_{ijlm}(k) := P_{il}(k)P_{jm}(k) - \frac{1}{2}P_{ij}(k)P_{lm}(k),$$

where

$$P_{ij}(k) := \delta_{ij} - \frac{k_i k_j}{|k|^2}.$$

Spectrum of GWs

The spectrum of gravitational waves' can be calculated as

$$\frac{d\rho_{gw}}{d\log|k|}(\eta,k)=|k|^3\int_{S^2}d\Omega_k\varrho_{gw}(\eta,k)=:\frac{1}{(2\pi)^2{M_{Pl}}^2a(\eta)^4V}S(\eta,k),$$

where the function S is defined as:

$$S(\eta,k)=\frac{|k|^3}{4\pi}\int_{S^2}d\Omega_k\sum_{i,j,l,m}\left[\left|\int_{\eta_i}^{\eta_f}d\eta'\cos\left(|k|\left(\eta-\eta'\right)\right)a(\eta')\mathcal{O}_{ijlm}\widehat{T}_{lm}(\eta',k)\right|^2\right.$$
$$\left.+\left|\int_{\eta_i}^{\eta_f}d\eta'\sin\left(|k|\left(\eta-\eta'\right)\right)a(\eta')\mathcal{O}_{ijlm}\widehat{T}_{lm}(\eta',k)\right|^2\right].$$

Redshifting the spectrum

Assuming that the energy density of GWs scales as a^{-4} we can write:

$$\frac{d\rho_{gw}}{d\log|k|}(\eta_0, k) = (1+z_{EQ})^{-4}\frac{a(\eta_{dec})^4}{a(\eta_{EQ})^4}\frac{d\rho_{gw}}{d\log|k|}(\eta_{dec}, k),$$

where η_0 is the present time and z_{EQ} is the red-shift to the epoch of matter-radiation equality.
We estimate the redshift factor $\frac{a(\eta_{dec})}{a(\eta_{EQ})}$ as:

$$\frac{a(\eta_{dec})}{a(\eta_{EQ})} = \sqrt{\frac{H_{EQ}}{H_{dec}}}.$$

Modelling of spectrum

For nearly degenerate minima the energy density of GWs at the peak can be estimated using:

$$\Omega_{GW}(\eta_{dec})|_{peak} = \frac{\tilde{\epsilon}_{GW}\mathcal{A}^2{\sigma_{wall}}^2}{24\pi {H_{dec}}^2 {M_{Pl}}^4},$$

where $\tilde{\epsilon}_{GW} \simeq 0.7$ and

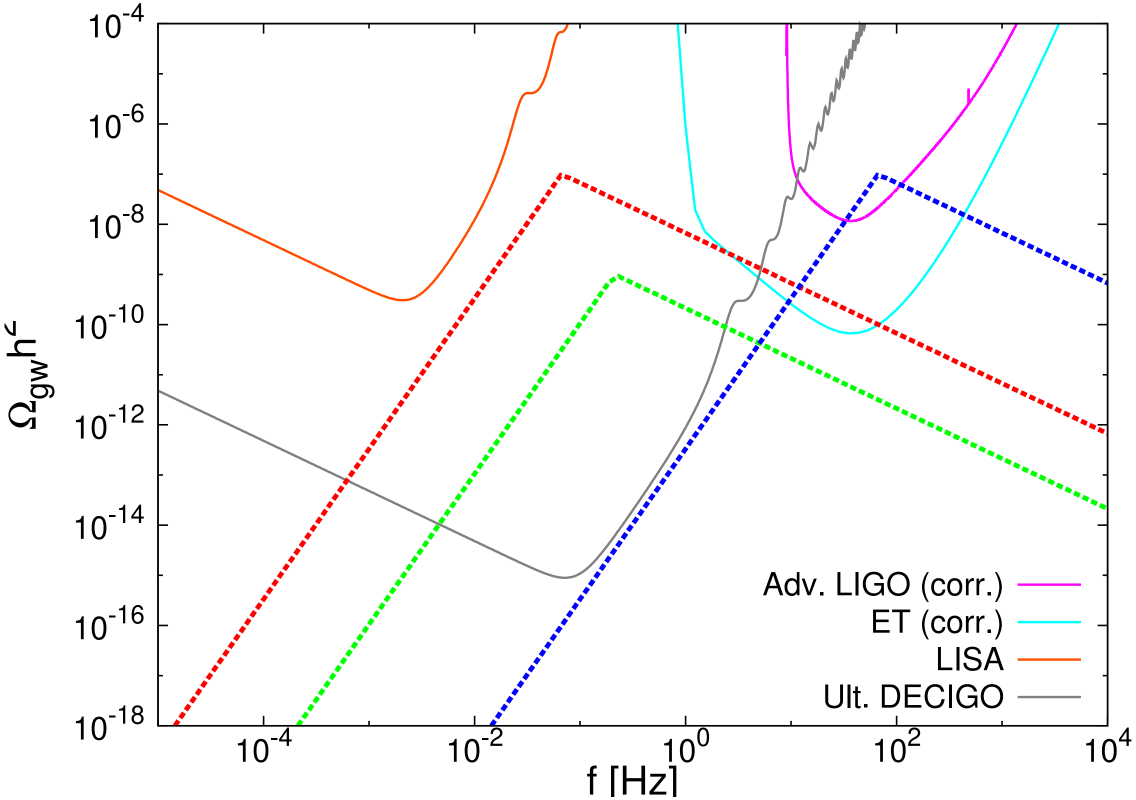
$$\mathcal{A} = \frac{\rho_{wall}}{\sigma_{wall}} t$$

is a scaling parameter $\mathcal{A} \sim 0.8$ (extracted from numerical simulations[13]).
Value of the Hubble parameter at the time of the decay H_{dec} determines the peak frequency:

$$f_0|_{peak} \sim H_{dec}^{-1}$$

13. Hiramatsu, T. *et al.* *JCAP* **1402,** 031. arXiv: `1309.5001 [astro-ph.CO]` (2014).

Spectrum of emitted GWs



13. Hiramatsu, T. *et al. JCAP* **1402,** 031. arXiv: 1309.5001 [astro-ph.CO] (2014).

Cosmological first order phase transitions

Let us consider theory of scalar order parameter given by Lagrangian density:

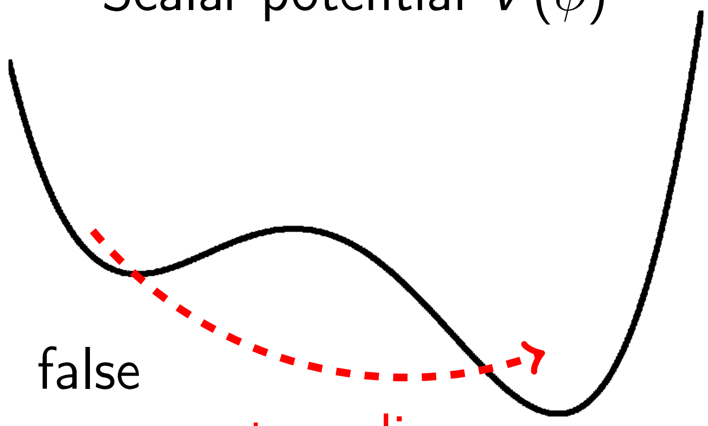
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi, T),$$

leading to the eom in the form:

$$\frac{\partial^2 \phi}{\partial t^2} - \Delta \phi = \frac{dV}{d\phi}(\phi, T),$$

where T is temperature.

Scalar potential $V(\phi)$



Tunnelling bubbles

Nucleation rate:

$$\Gamma(T) = A(T) \cdot \exp(-S)$$

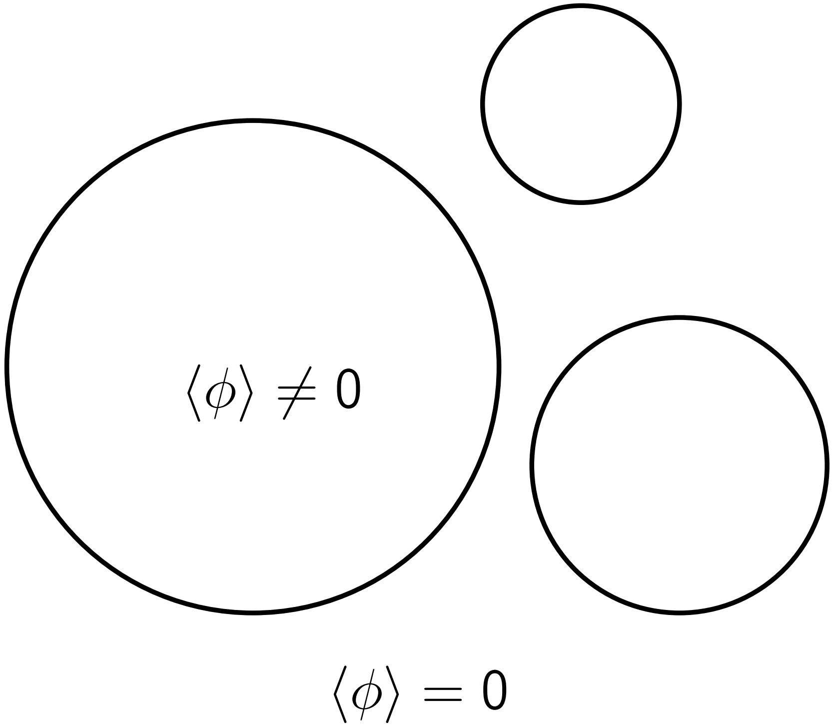
For tunnelling in finite temperatures:

$$S = \frac{S_3}{T} \qquad A(T) = T^4 \left(\frac{S_3}{2\pi T}\right)^{\frac{3}{2}}$$

where S_3 is an action of $\mathrm{O}(3)$-symmetric solution of the eom.
Nucleation condition:

$$\frac{\Gamma(T_n)}{H^4} \approx 1$$

Nucleation of bubbles

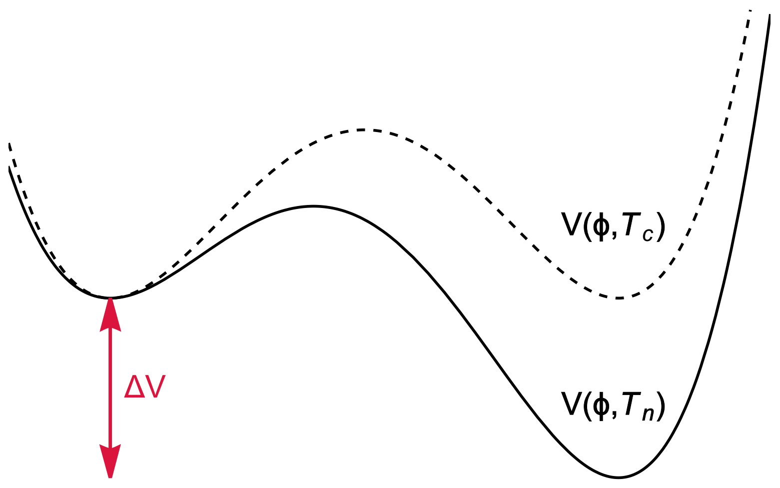


Phase transition parameters

- Critical and nucleation temperatures: T_c, T_n
- Level of supercooling: T_n/T_c
- Transition strength: $\alpha \sim \Delta V/\rho_r$
  In this work:

$$\alpha_{\bar{\theta}} = \frac{\Delta\bar{\theta}}{3w_s}, \qquad \text{with} \qquad \bar{\theta} = \epsilon - \frac{p}{c_b^2}$$

  with the speed of sound in the broken phase c_b and model-dependent energy e, pressure p and enthalpy w.
- Bubble-wall velocity: v_w



Bag model

Cosmic plasma coexist in two phases:

- Symmetric phase outside the bubble
- Broken phase inside the bubble

Equation of state

$$\epsilon_s = 3a_s T_s^4 + \theta_s \qquad\qquad \epsilon_b = 3a_b T_b^4 + \theta_b$$

$$p_s = a_s T_s^4 - \theta_s \qquad\qquad p_b = a_b T_b^4 - \theta_b$$

Strength of the transition is defined as

$$\alpha = \left.\frac{\theta_s - \theta_b}{\epsilon_r}\right|_{T=T_n}.$$

Hydrodynamics of bag model

Energy-momentum tensor for the plasma is given by

$$T^{\mu\nu} = wu^{\mu}u^{\nu} + pg^{\mu\nu}$$

Conservation of $T^{\mu\nu}$ along the flow and projection orthogonal to the flow lead to

$$\partial_{\mu}(u^{\mu}w) - u_{\mu}\partial^{\mu}p = 0,$$
$$\bar{u}^{\nu}u^{\mu}w\partial_{\mu}u_{\nu} - \bar{u}^{\nu}\partial_{\mu}p = 0.$$

Hydrodynamic equation

$$2\frac{v}{\xi} = \gamma^2(1 - v\xi)\left[\frac{\mu^2}{c_s^2} - 1\right]\partial_{\xi}v,$$

with Lorentz-transformed fluid velocity $\mu(\xi, v) = \frac{\xi - v}{1 - \xi v}$ and $\xi = r/t$.

Analytic methods for stationary profiles

Hydrodynamic equation

$$2\frac{v}{\xi} = \gamma^2(1 - v\xi)\left[\frac{\mu^2}{c_s^2} - 1\right]\partial_\xi v,$$

Matching equations

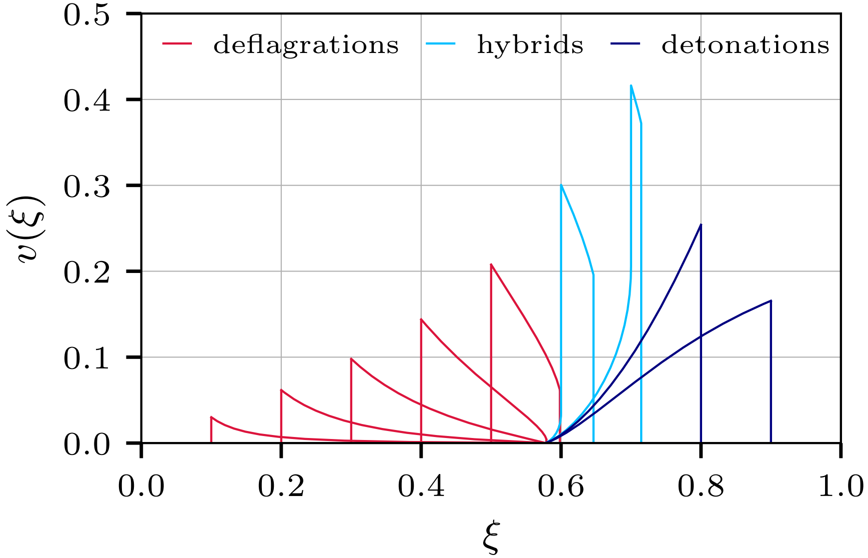
1. $\omega_-\gamma_-^2 v_- = \omega_+\gamma_+^2 v_+$
2. $\omega_-\gamma_-^2 v_-^2 + p_- = \omega_+\gamma_+^2 v_+^2 + p_+$

Bag equation of state

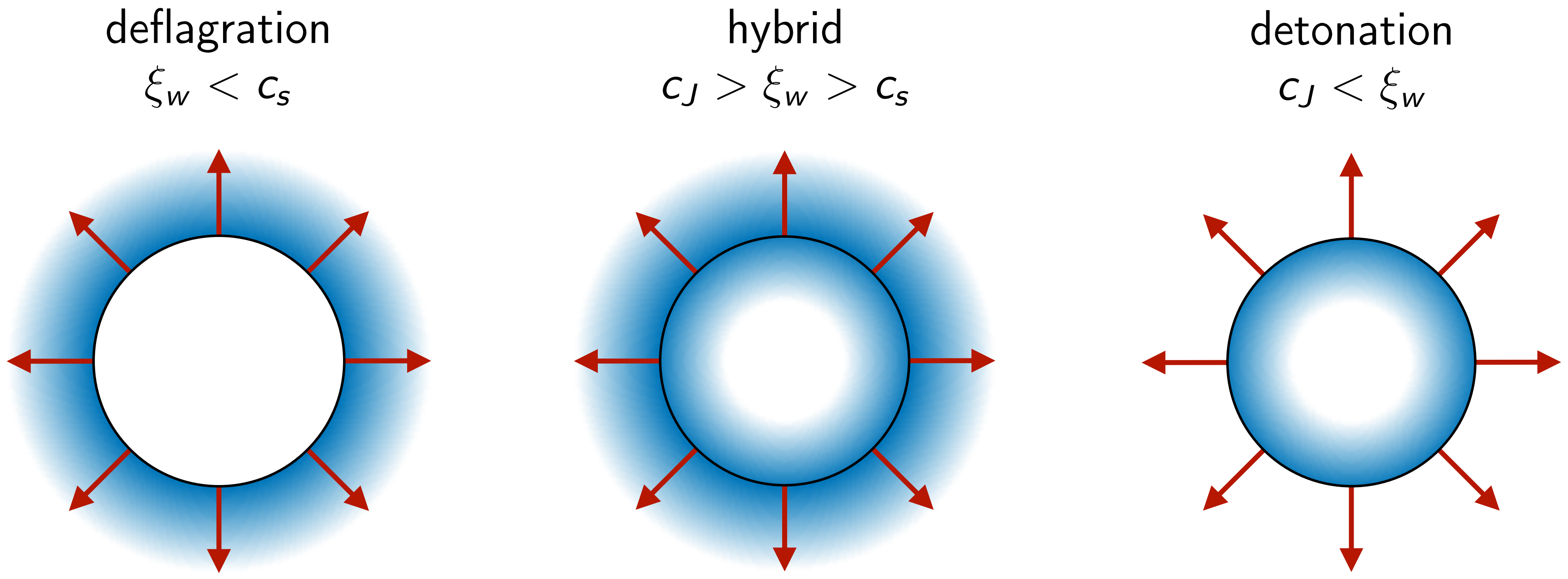
$$\epsilon_s = 3a_s T_s^4 + \theta_s \qquad \epsilon_b = 3a_b T_b^4 + \theta_b$$

$$p_s = a_s T_s^4 - \theta_s \qquad p_b = a_b T_b^4 - \theta_b$$

Solving hydrodynamic equation with proper boundary conditions and matching conditions (1 and 2), we get profiles $v(\xi)$ depending on ξ_w, α.



Bubble profiles



Where Jouget velocity is $c_J = \frac{1}{\sqrt{3}} \frac{1+\sqrt{1+3\alpha^2+2\alpha}}{1+\alpha}$.

Equations of motion

Total energy-momentum tensor in conserved, but both contributions are not:

$$\nabla_\mu T^{\mu\nu}_{\text{field}} = \frac{\partial V}{\partial \phi}\partial^\nu \phi + \eta u^\mu \partial_\mu \phi \partial^\nu \phi = -\nabla_\mu T^{\mu\nu}_{\text{fluid}}.$$

Equation of motion of scalar field

$$-\partial_t^2 \phi + \frac{1}{r^2}\partial_r(r^2 \partial_r \phi) - \frac{\partial V}{\partial \phi} = \eta\gamma(\partial_t \phi + v\partial_r \phi)$$

Equations of motion of plasma

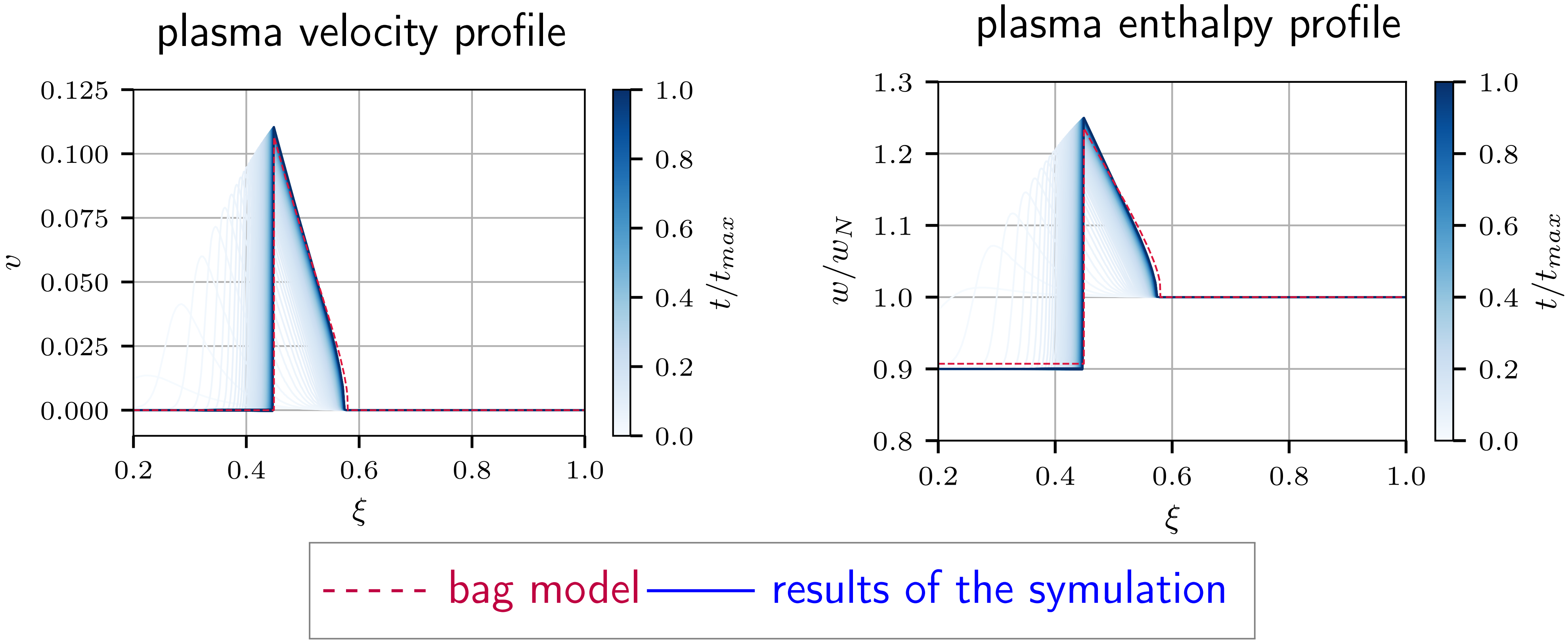
$$\partial_t \tau + \frac{1}{r^2}\partial_r(r^2(\tau + p)v) = \frac{\partial V}{\partial \phi}\partial_t \phi + \eta\gamma(\partial_t \phi + v\partial_r \phi)\partial_t \phi,$$

$$\partial_t Z + \frac{1}{r^2}\partial_r\left(r^2 Z v\right) + \partial_r p = -\frac{\partial V}{\partial \phi}\partial_r \phi - \eta(\partial_t \phi + v\partial_r \phi)\partial_r \phi.$$

where $Z := w\gamma^2 v$ and $\tau := w\gamma^2 - p$

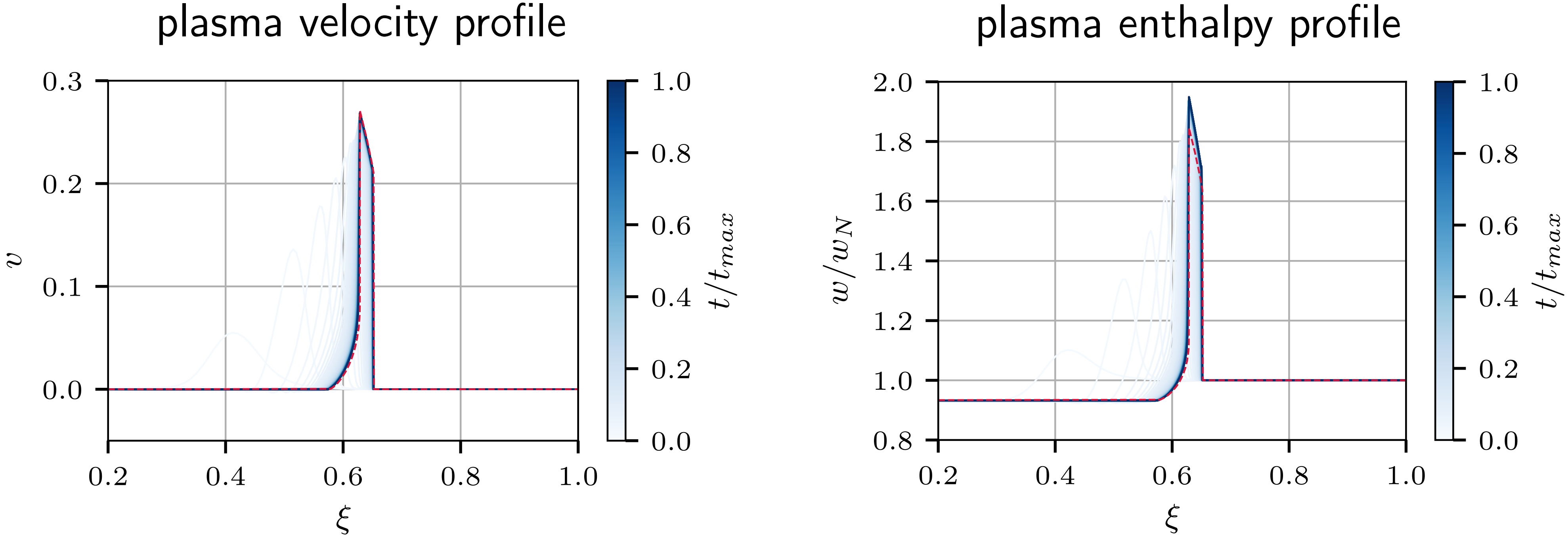
Stationary states

1. Deflagration ($\xi_w = 0.45$)



Stationary states

2. Hybrid ($\xi_w = 0.63$)

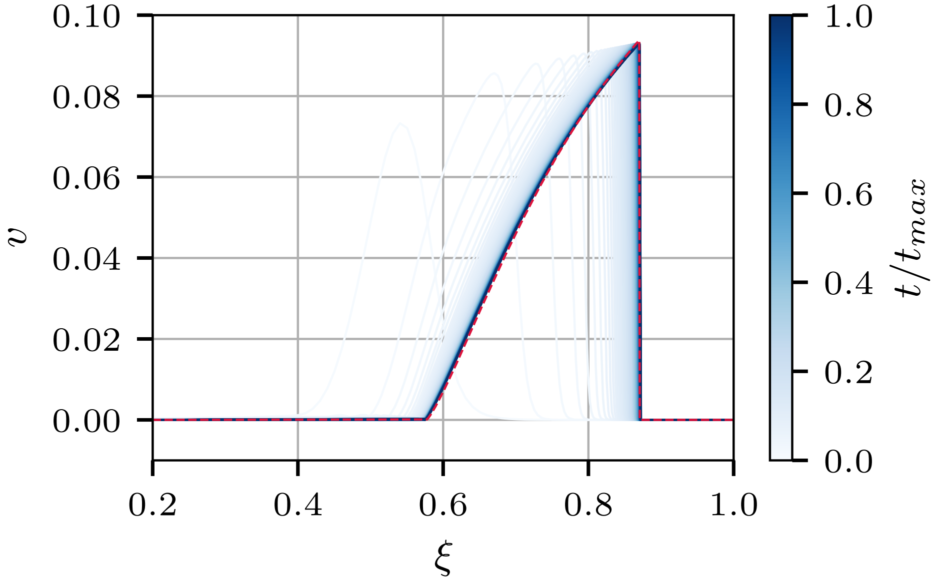


- - - - - bag model —— results of the symulation

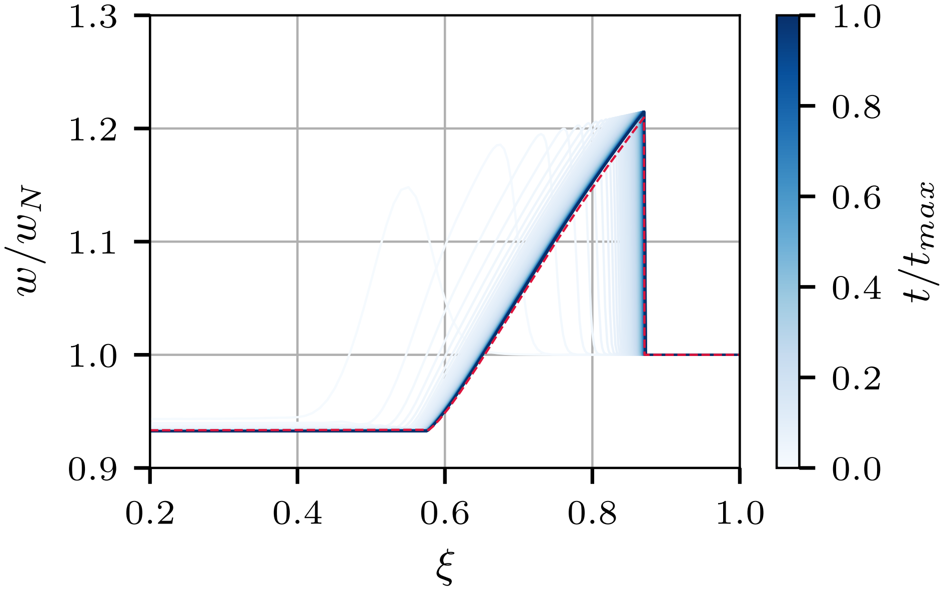
Stationary states

3. Detonation ($\xi_w = 0.87$)

plasma velocity profile



plasma enthalpy profile



- - - - - bag model ——— results of the symulation

Instead of summary

What have we not talked about, but is extremely fascinating and important:

- Baryogenesis realized on walls of tunnelling bubbles and others mechanisms,
- Cosmic strings as topological defects in models with continuous symmetry,
- Axions as dark matter candidates,
- Solitions and oscillons as metastable leftovers from the early Universe,
- Primordial black holes,
- Modified theories of gravity.