Comparison of solutions with different pseudo-Newtonian potentials in numerical simulations of accretion disk around compact objects

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We present hydrodynamic simulations of a thin accretion disk around compact objects using three different pseudo-Newtonian potentials: the Paczyński–Wiita and Kluźniak–Lee for a Schwarzschild black hole, and the newly devised potential for a Reissner–Nordström (RN) naked singularity. We study the differences between the properties of disks surrounding Schwarzschild black holes in two related pseudo-Newtonian potentials and compare the results with those for the RN naked singularity. The Paczyński–Wiita solution accurately reproduces the location of stable and bound orbits, as well as the form of the Keplerian angular momentum, while the Kluźniak–Lee potential reproduces the ratio of the orbital and epicyclic frequencies. The radial dependence of angular momentum, angular velocity, and epicyclic frequency from simulations are compared with those predicted by each pseudo-Newtonian potential. For the RN naked singularity, We consider different values of the charge-to-mass ratio, which affects the localization of orbits for test particles. Accretion around the naked singularity stops at the zero-gravity sphere, where angular velocity vanishes. Due to the effective potential, gravity inside this sphere is repulsive, causing the disk to become constricted at the edge of the sphere. In the case of black holes, matter falls toward the event horizon, with the disk either remaining thin for the rest of its length or thickening, depending on the chosen pseudo-potential. Distinct geometry of the sources may provide a valuable tool for distinguishing these compact objects in observations with facilities such as the Event Horizon Telescope.

Introduction:

Naked singularities (NkS) appear as the Reissner–Nordström spacetime solution when the charge exceeds the mass Q > M. To find potential observational distinctions between NkSs and black holes (BHs), we perform numerical simulations of an accretion disk using pseudo-Newtonian potentials, which approximate general relativistic effects by reproducing relevant properties.

We apply two pseudo-potentials for the Schwarzschild BH: the Paczyński–Wiita (PW) *(Paczyńsky & Wiita 1980*) and the Kluźniak–Lee (KL) (*Kluźniak & Lee 2002*) potentials, as well as a newly devised pseudo-potential for the Reissner–Nordström (RN) NkS *(Čemeljić & Kluźniak 2025)*. All formulas are shown below.

$$V_{PW} = -\frac{GM}{r-r_g}, \quad V_{KL} = \frac{GM}{3r_g}\left(1-e^{\frac{3r_g}{r}}\right), \quad V_{RN} = -\frac{M}{r}+\frac{Q^2}{2r^2}.$$

We evaluate their accuracy by comparing numerical results with analytical solutions. For BHs, we analyze the profile of Keplerian angular momentum derived in Schwarzschild spacetime $L(r)^2 = Mr^3/(r-r_g)^2$ and the ratio of radial epicyclic frequency to orbital angular frequency $\kappa^2/\Omega^2 = 1-3r_g/r$, reproduced by the PW and the KL potentials, respectively. The RN potential replicates the Keplerian formula of orbital angular frequency $\Omega(r)^2 = (1-r_o/r)M/r^3$, along with the radii: $r_{max} = 4r_o/3$ and $r_o = Q^2/M$, with the latter defining zero-gravity sphere (*Vieira & Kluźniak 2023*).

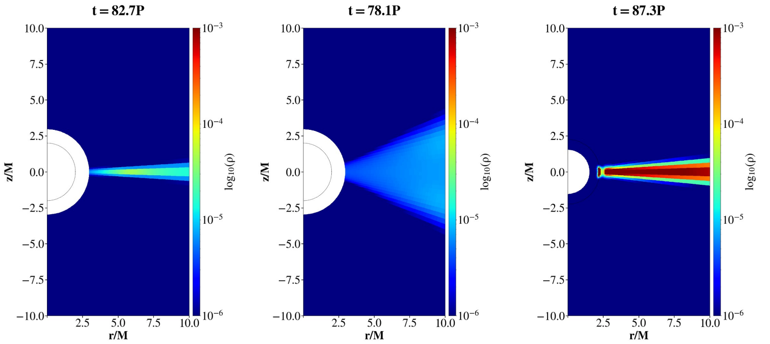

Figure 4: *Gas density obtained for stable state of the disk for $\alpha_v = 0.5$ and Q = 1.5 (NkS). Black dashed lines mark the event horizon (BH) and the zero-gravity sphere (NkS). Left panel: Result for the PW potential. Middle panel: Result for the KL potential. Right panel: Result for the RN potential.*

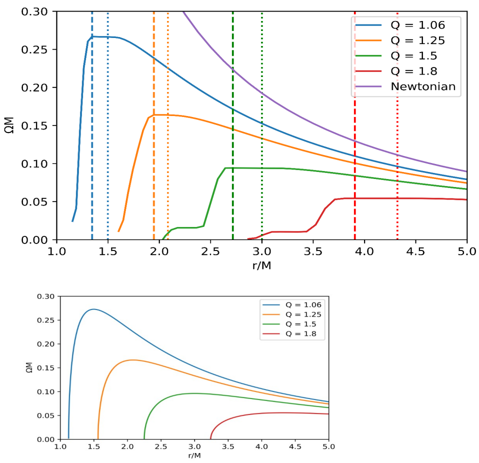

Figure 1: *Orbital angular frequency ΩM as a function of r/M for four values of charge Q. Upper panel: Numerical results. Dashed lines indicate r_{max} determined numerically; dotted lines represent analytical expectations. Bottom panel: Analytical solutions for the corresponding numerical results.*

Numerical setup:

Initial state of the accretion disk was set by the analytic solution (*Kluźniak & Kita 2000*), surrounded by non-rotating corona in hydrostatic equilibrium. We performed simulations using PLUTO code (*Mignone et al. 2007*) in 2D-axisymmetric viscous hydrodynamic mode. Inner boundary conditions were defined as reflective for NkS and outflow for BH. The disk aspect ratio was fixed at ε = h/R = 0.1 and the polytropic index in disk was set to ɣ = 4/3. Using second-order piecewise linear reconstruction the hydrodynamic equations were solved with approximate Harten-Lax-van Leer Riemann solver (hll) combined with second-order Runge–Kutta time stepping (RK2). Each setup was run for two values of anomalous viscosity parameter $\alpha_v = 0.5$ and 0.8.

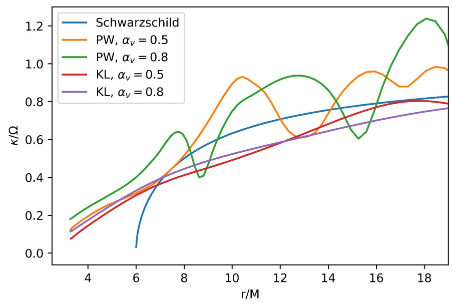

Figure 2: *Ratio of radial epicyclic frequency κ to orbital frequency Ω as a function of r/M. Results are shown for the KL potential with $\alpha_v = 0.5$ (red) and $\alpha_v = 0.8$ (purple) and for the PW potential with $\alpha_v = 0.5$ (yellow) and $\alpha_v = 0.8$ (green). Blue line represents analytical ratio κ/Ω derived in Schwarzschild spacetime.*

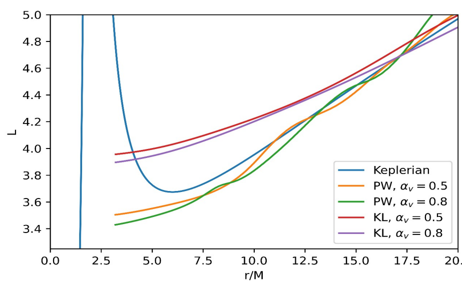

Figure 3: *Keplerian angular momentum L(r) as a function of r/M. Results are shown for the PW potential with $\alpha_v = 0.5$ (yellow) and $\alpha_v = 0.8$ (green) and for the KL potential with $\alpha_v = 0.5$ (red) and $\alpha_v = 0.8$ (purple). Blue line represents analytical Keplerian formula L(r) derived in Schwarzschild spacetime.*

Results:

We find that the accretion disk exhibits different behaviour for each pseudo-Newtonian potential near the compact object. In the case of the NkS, matter visibly accumulates at a certain distance from the singularity, with a drop in density, occurring near the zero-gravity sphere (Fig.4). Furthermore, the density of the disk is considerably higher than for BHs. We notice differences between BHs potentials: the disk modeled with the KL potential becomes significantly thicker and less dense.

As shown in Fig.1, the shape of the orbital frequency function compares well with analytical expectations, although the accuracy in reproducing r_{max} decreases as the Q/M ratio increases. In Fig.3, the numerical results obtained for the KL potential, that reproduces ratio κ/Ω, show limited agreement with analytical formula. More accurate fit is observed at larger radii. Nevertheless, the deviations in the results for the PW potential are significantly greater. The profile of the Keplerian angular momentum in Fig.4, is closely followed by the PW potential, as expected, with agreement enhancing with radius. The results obtained from the KL potential assume higher values across the entire domain. In both cases, variations of viscosity parameter does not affect the numerical results.

Conclusions and future work:

In our numerical simulations with PLUTO code, we compared the analytical prediction with the results from numerical simulations for the properties reproduced by each pseudo-potential. The disk structure and density vary significantly depending on the central object, which may help in distinguishing compact objects by observations.

In the future work we will perform numerical simulations with a magnetic field.

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