Optimal transient growth in keplerian discs via variational technique

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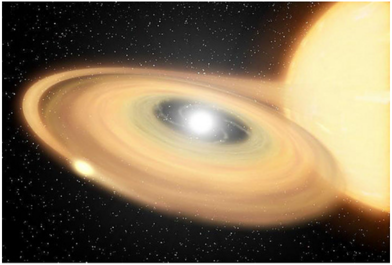

Modal analysis

The linear theory of perturbation dynamics — wave and oscillations in discs, stability and transition.

The basic and traditional framework

- make an assumption of an exponential time dependence
- initial-value problem transforms to an eigenvalue problem for underlying linear dynamical operator
- usually is referred to as a modal approach or as the spectral problem assosiated with the corresponding basic flow
- gives adequate description of perturbation dynamics in case of normal operators only when eigenfuctions are orthogonal to each other
- otherwise, only describes the asymptotic ($t \to \infty$) fate of perturbations

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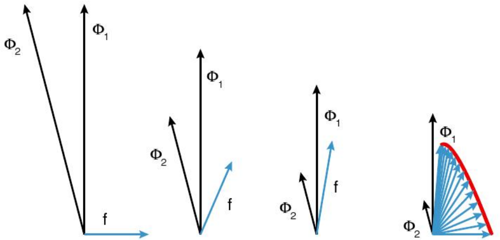
$$\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A}, \quad \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

Non-modal analysis

The existence of a non-zero shear in the flow makes dynamical operator non-normal and eigenfuctions become non-orthogonal making possible the transient growth of perturbations at finite time intervals.

Schmid (2007)



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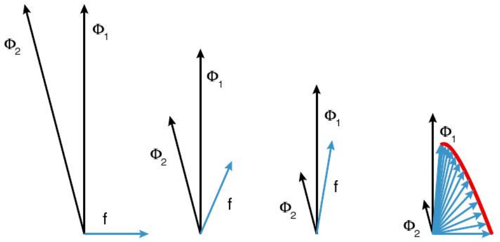
Schmid (2007)

$$\mathbf{A}\mathbf{A}^{\dagger} \neq \mathbf{A}^{\dagger}\mathbf{A},$$

$$\mathbf{A}\mathbf{v} = \sigma\mathbf{u}, \quad \mathbf{A}^{\dagger}\mathbf{u} = \sigma\mathbf{v}$$

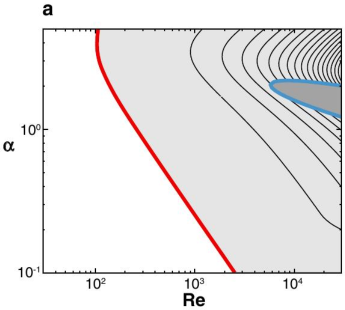
$Max\{\sigma_i\}$ is a maximum stretch that **A** can give



Non-modal analysis

An example: the plane Poiseuille flow

Schmid (2007)



Non-modal analysis

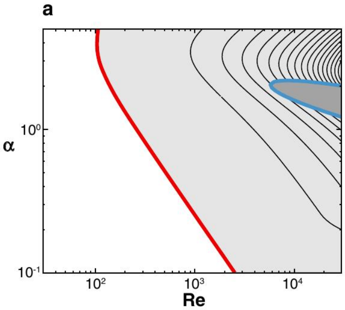
An example: the plane Poiseuille flow

Schmid (2007)

$$\mathbf{A}^{\dagger}\mathbf{A}\,\mathbf{v} = \sigma^2\,\mathbf{v}$$

$$\mathbf{A}\mathbf{A}^{\dagger}\,\mathbf{u} = \sigma^2\,\mathbf{u}$$

The singular values of **A**
are
the eigenvalues of $\mathbf{A}\mathbf{A}^{\dagger}$ and $\mathbf{A}^{\dagger}\mathbf{A}$



Non-modal analysis

Modal analysis $\Longrightarrow$ arrange eigenmodes according to their eigenvalues // timespan independent

Non-modal analysis $\Longrightarrow$ arrange singular vectors according to their singular values // for each timespan

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Modal analysis $\Longrightarrow$ arrange eigenmodes according to their eigenvalues // timespan independent

Non-modal analysis $\Longrightarrow$ arrange singular vectors according to their singular values // for each timespan

The system of singular vectors of non-normal operator
is *orthogonal and complete*
just like the system of eigen-vectors of the hermitian operator

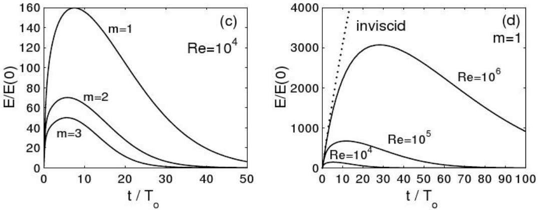
To find optimal perturbations means to find the highest singular value and corresponding singular vector.

Transient activity of keplerian accretion flow

Optimal global perturbations.

Ioannou & Kakouris (2001) — studied 2D rotating flow with a keplerian rotation constructed of a viscous incompressible fluid. Calculated initial optimal perturbations and the statistical steady state of the flow under an external stochastic forcing.

Optimal vortical perturbations

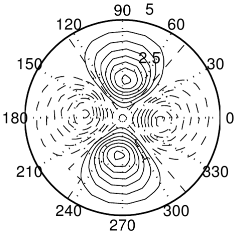


Transient activity of keplerian accretion flow

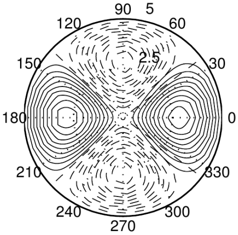
Optimal global perturbations.

Ioannou & Kakouris (2001)

The shape of optimal perturbation



The shape of the leading component of the flow response to an external stochastic forcing



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But what will get if consider an almost keplerian hypersonic thin disc?

What will be the picture when involve a sonic wave-like perturbations?

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The determination of optimal perturbation in such a complex flow can be addressed using two different mathematical descriptions

Optimization: using the spectral solutions

Henningson & Schmid (2001)

Consider a linear subspace S_N of the solutions

$$q = \sum_{n=1}^{N} \kappa_n \tilde{q}_n$$

$\Longrightarrow$

with the basis

$$\tilde{q}_n = \{\bar{v}_r,\, \bar{v}_\varphi,\, \bar{h}\}_n \times e^{im\varphi} = \{\tilde{q}^1,\, \tilde{q}^2,\, \tilde{q}^3\}_n \times e^{im\varphi}$$

where

$$\{\bar{v}_r, \bar{v}_\varphi, \bar{h}\}_n$$

are the profiles constituting the Fourier-amplitudes of n eigen-mode.

(We assume the baratropic case)

In that way the coordinates of q in S_N are

$$\kappa_n(t) = \kappa_n^0 e^{-i\omega_n t}$$

$\Longrightarrow$

$$\boldsymbol{\kappa} = e^{\boldsymbol{\Lambda} t} \boldsymbol{\kappa}^0$$

$$\boldsymbol{\kappa}^0 = (\kappa_1^0, \kappa_2^0, ..., \kappa_N^0)^T$$

$$\boldsymbol{\kappa} = (\kappa_1, \kappa_2, ..., \kappa_N)^T$$

$$\boldsymbol{\Lambda} = diag\{-i\omega_1, -i\omega_2, ..., -i\omega_N\}$$

Optimization: using the spectral solutions

The Metrics

- Let us measure perturbation by its total acoustic energy

$$E_a = \frac{1}{2} \int \rho \left(\delta v_r^2 + \delta v_\varphi^2 + n \frac{\delta h^2}{h_{eq}} \right) rdr\, dz\, d\varphi$$

- So introduce the inner product in the following way

$$(f, g) = \pi \int_{r_1}^{r_2} \rho \left[f^{1*} g^1 + f^{2*} g^2 + \frac{n}{h_{eq}} f^{3*} g^3 \right] rdr\, dz$$

- And the norm of the arbitrary perturbation is

$$(q, q) = E_a = \|\boldsymbol{\kappa}\|^2 = \sum_{i,j=1}^{N} \kappa_i^* \kappa_j M_{ij} = \boldsymbol{\kappa}^\dagger \mathbf{M} \boldsymbol{\kappa}$$

with metrics in S_N defined as $M_{ij} = (\tilde{q}_i, \tilde{q}_j)$

Optimization: using the spectral solutions

The optimal growth

- The growth factor is

$$g(t) = \frac{\|\boldsymbol{\kappa}(t)\|^2}{\|\boldsymbol{\kappa}^0\|^2} = \frac{\|e^{\mathbf{\Lambda} t}\boldsymbol{\kappa}^0\|^2}{\|\boldsymbol{\kappa}^0\|^2}$$

- The optimal growth is

$$G(T) = \max_{\boldsymbol{\kappa}^0 \neq 0} g(T) \equiv \|e^{\mathbf{\Lambda} T}\|^2 = \|\mathbf{F} e^{\mathbf{\Lambda} T}\mathbf{F}^{-1}\|_2^2 = \sigma_1^2\left(\mathbf{F} e^{\mathbf{\Lambda} T}\mathbf{F}^{-1}\right)$$

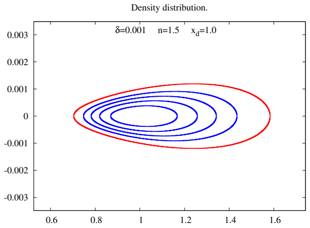
where $\mathbf{M} = \mathbf{F}^\dagger \mathbf{F}$ is Cholesky decomposition and
σ_1 is the first singular value of the corresponding matrix

- SVD decomposition gives σ_1 & $\mathbf{v}_1$

$\boldsymbol{\kappa}^0 = \mathbf{F}^{-1}\mathbf{v}_1$ is the initial unit vector that corresponds to perturbation that attains the largest possible energy growth G at the moment T

Approach discs: the thin ring

- Let us take the baratropic torus constructed of perfect fluid rotating with a $\Omega \sim r^{-q}$ in an external Newtonian potential.
- This model case was extensively studied in 1980's as a subject to Papaloizou-Pringle instability.
- Consider the case $q \to 3/2$, i.e. the thin quasi-keplerian torus with $\delta = H/r \ll 1$.
- Take the most trivial part of the spectrum —
  using WKBJ method calculate modes $\sim exp(-i\omega t + im\varphi)$,
  (i) with no dependence on z,
  (ii) with corotation radius beyond the outer boundary of torus.



The result of optimization

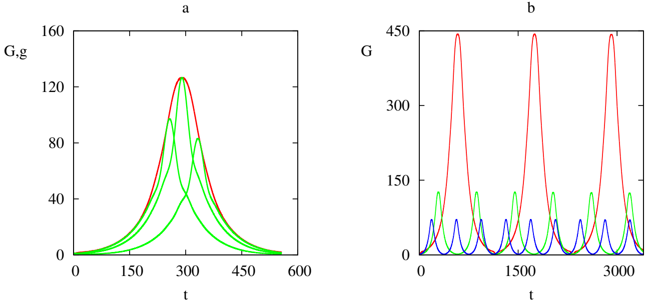
Optimal growth curves for the linear combination of neutral acoustic modes with dimension $N = 20$ and parameters $x_d = 1.0$, $m = 25$, $n = 3/2$.

Left panel

$\delta = 0.002$, growth factors for optimal combinations at $t = 250, 290, 390$

Right panel

$\delta = 0.001, 0.002, 0.003$



- Sonic time scale $t_s \sim (\delta\Omega_0)^{-1}$
- Magnitude of $G \sim \delta^{-(\dots)}$

The result of optimization

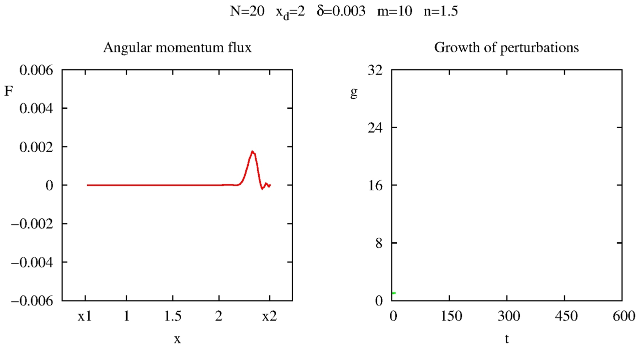
Evolution of the particular optimal combination of neutral acoustic modes.

Left panel

The instant angular momentum flux density alternating with time

Right panel

The growth of the total acoustic energy



Optimization: method of Lagrange multipliers

- The goal is to determine the maximum of the following cost Lagrangian

$$\mathcal{J} = \frac{\|\mathbf{q}(\tau)\|^2}{\|\mathbf{q}(0)\|^2}$$

- Method of Lagrange multipliers gives an augmented Lagrangian
(the Lagrange multipliers here — $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{q}}_0$ which are the adjoint variables)

$$\mathcal{L}(\mathbf{q}, \tilde{\mathbf{q}}, \mathbf{q_0}, \tilde{\mathbf{q}}_0) = \mathcal{J} - \int_0^\tau (\, \tilde{\mathbf{q}} \,, \dot{\mathbf{q}} - \mathbf{A}\mathbf{q} \,)\, dt - (\, \tilde{\mathbf{q}}_0 \,, \mathbf{q}(\mathbf{0}) - \mathbf{q_0} \,)$$

- The zero variations of $\mathcal{L}$ over the corresponding quantities give
1) over the $\mathbf{q} \longmapsto$ the direct dynamical equations
2) over the $\tilde{\mathbf{q}} \longmapsto$ the adjoint equations
3) over the $\mathbf{q_0} \longmapsto$ the relationship between $\mathbf{q}(\tau)$ and $\tilde{\mathbf{q}}(\tau)$
4) over the $\tilde{\mathbf{q}}_0 \longmapsto$ the relationship between $\mathbf{q}(0)$ and $\tilde{\mathbf{q}}(0)$

Direct and adjoint system of equations

$$\mathbf{q} = \{\, \delta v_r, \delta v_\varphi, \delta v_z, \delta p \,\}$$

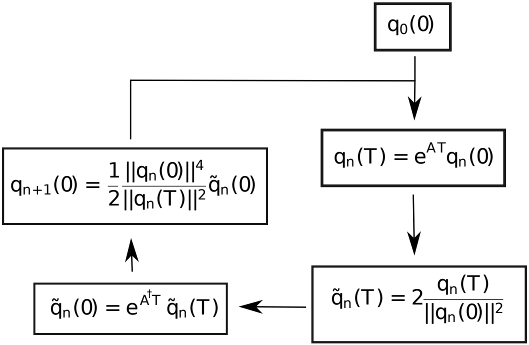
$$\dot{\mathbf{q}} = \mathbf{A}\,\mathbf{q}$$
$$\dot{\tilde{\mathbf{q}}} = -\mathbf{A}^\dagger\,\tilde{\mathbf{q}}$$

$$\mathbf{A} = \begin{pmatrix} -im\Omega & 2\Omega & 0 & \frac{1}{\rho}\left(\frac{\rho_{,r}}{\rho} - \frac{\partial}{\partial r}\right) \\ -\frac{\kappa^2}{2\Omega} & -im\Omega & 0 & -\frac{im}{r\rho} \\ 0 & 0 & -im\Omega & \frac{1}{\rho}\left(\frac{\rho_{,z}}{\rho} - \frac{\partial}{\partial z}\right) \\ -\rho a^2\left(\frac{1}{r} + \frac{\rho_{,r}}{\rho} + \frac{\partial}{\partial r}\right) & -\frac{im}{r}\rho a^2 & -\rho a^2\left(\frac{\rho_{,z}}{\rho} + \frac{\partial}{\partial z}\right) & -im\Omega \end{pmatrix}$$

$$\mathbf{A}^\dagger = \begin{pmatrix} im\Omega & -\frac{\kappa^2}{2\Omega} & 0 & -\frac{1}{\rho}\left(\frac{\rho_{,r}}{\rho} - \frac{\partial}{\partial r}\right) \\ 2\Omega & im\Omega & 0 & \frac{im}{r\rho} \\ 0 & 0 & im\Omega & -\frac{1}{\rho}\left(\frac{\rho_{,z}}{\rho} - \frac{\partial}{\partial z}\right) \\ \rho a^2\left(\frac{1}{r} + \frac{\rho_{,r}}{\rho} + \frac{\partial}{\partial r}\right) & \frac{im}{r}\rho a^2 & \rho a^2\left(\frac{\rho_{,z}}{\rho} + \frac{\partial}{\partial z}\right) & im\Omega \end{pmatrix}$$

Iterative scheme

The straightforward way to find the solution to variational problem described above is the following iterative scheme



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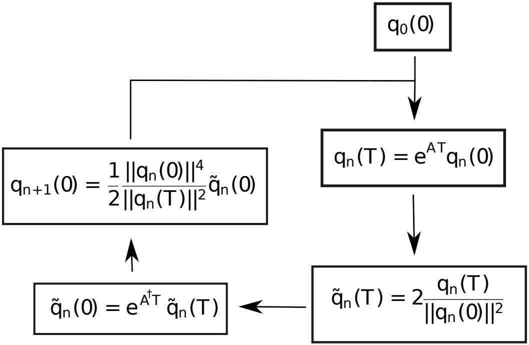


The iterative scheme above is equivalent to iteration $\mathbf{q}_{n+1} = \mathbf{U} \cdot \mathbf{q}_n$, where $\mathbf{U} = e^{(\mathbf{A}^\dagger + \mathbf{A})\, t}$ is a hermitian positive operator.

Convergence of iterative scheme

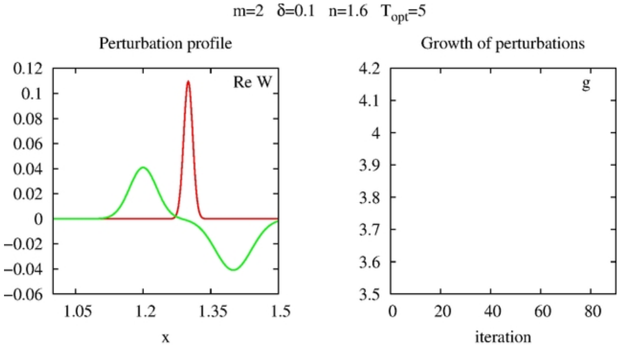
Conversion of two different arbirary initial perturbations into the unique optimal initial perturbation corresponding to a particular timespan.

Left panel

The instant shapes

Right panel

The instant values of energy growth



Evolution of optimal perturbation

Comparative evolution of any arbitrary initial profile and optimal initial profile seen on the previous slide

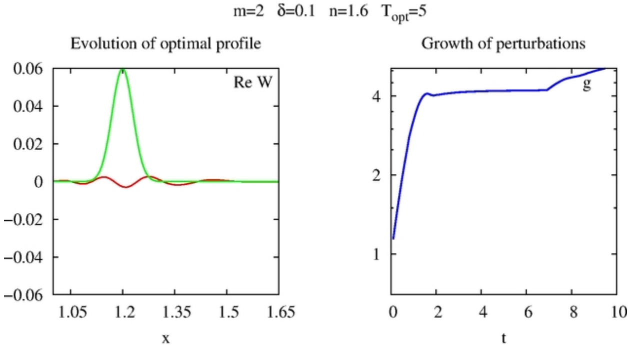
Left panel

The instant shapes

Right panel

The instant values of energy growth.

Blue curve represents the optimal growth which is the highest possible energy enhancement at each point.



Variational technique: the advantages

- Don't need spectral solutions anymore (no issues with corotational and Lindblad resonances of modes, continuous spectra, unbounded flows etc)
- Don't need any kind of matrix representation for dynamical operator (profit in numerical resourses)
- Viscous and other additional terms can be straightforwardly added into the underlying cauchy problem
- The determination of optimal perturbations can be easily extended to the non-stationary flows (i.e. transient dynamics in the non-stationary accretion discs)
- Finally, the variational technique can be employed in the domain of the non-linear perturbations

Conclusions

- A global non-modal approach to a hypersonic polytropic flow with free boundaries and a nearly keplerian rotation was attempted
- Two different methods of optimization were employed,
  i) the one involving the decomposition of perturbations over the spectral solutions (i.e. over the modes)
  ii) the one with variational formulation that requires the advance of direct and adjoint cauchy problems
- The advantages of the latter method were discussed briefly
- It was revealed that in quasikeplerian tori the certain (optimal) combinations of the neutral sonic modes with corotation radius beyond the outer boundary can exhibit a considerable transient growth at the sonic timescale.
- Keplerian disc also has a potential to drive a global transient dynamics.

Possible developments

- Include the vertical motions in the perturbed flow.
- Include the viscosity and consider the accreting flows.
- Solve the problem of the stochastic forcing of thin accretion disc.

Thank you for your attention

What astrophysical applications could it have?..