COSMIC MAGNETIC FIELDS

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fluid dynamics

CONSERVATION OF PARTICLE NUMBER AND MASS

- Consider a (Lagrangian) volume element δV containing δN particles. The particle number density is $n = \dfrac{\delta N}{\delta V}$.

- Unless particles can be created or destroyed, the particle number is conserved $\mathrm{d}(\delta N) = 0$.

- For particles of mass m the mass element $\delta M = m\,\delta N$ is conserved $\mathrm{d}(\delta M) = 0$. Mass density is $\rho = \dfrac{\delta M}{\delta V} = mn$.

- A volume element evolves according to the local velocity field $\vec{\mathrm{v}}(\vec{r})$. During a time interval $\mathrm{d}t$ it will be displaced by $\mathrm{d}\vec{r} = \vec{\mathrm{v}}\,\mathrm{d}t$ and expand by $\mathrm{d}(\delta V) = \delta V\,(\vec{\nabla}\cdot\vec{\mathrm{v}})\,\mathrm{d}t$.

- With the material derivative: $\dfrac{\partial n}{\partial t} + \left(\vec{\mathrm{v}}\cdot\vec{\nabla}\right) n = \dfrac{\mathrm{d}n}{\mathrm{d}t} = \dfrac{\mathrm{d}(\delta N/\delta V)}{\mathrm{d}t} = -\,n\left(\vec{\nabla}\cdot\vec{\mathrm{v}}\right)$

- Continuity equation: $\dfrac{\partial n}{\partial t} + \vec{\nabla}\cdot\left(n\vec{\mathrm{v}}\right) = 0, \quad \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{\mathrm{v}}\right) = 0$

CONSERVATION OF MOMENTUM

- Consider a force $\delta\vec{F}$ acting on a fluid element of volume δV, mass δM, velocity $\vec{\mathrm{v}}$, and momentum $\delta\vec{p} = \vec{\mathrm{v}}\,\delta M = \rho\vec{\mathrm{v}}\,\delta V$.

- Newton's second law: $\delta\vec{F} = \dfrac{\mathrm{d}(\delta\vec{p})}{\mathrm{d}t} = \dfrac{\mathrm{d}\vec{\mathrm{v}}}{\mathrm{d}t}\delta M$

- Euler equation: $\dfrac{\partial\vec{\mathrm{v}}}{\partial t} + (\vec{\mathrm{v}}\cdot\vec{\nabla})\vec{\mathrm{v}} = \dfrac{\mathrm{d}\vec{\mathrm{v}}}{\mathrm{d}t} = \dfrac{\delta\vec{F}}{\delta M} = \dfrac{\vec{f}}{\rho}$,

  introducing the force density $\vec{f} = \dfrac{\delta\vec{F}}{\delta V}$

- Combined with the continuity equation, this can be written in the tensor form: $\partial_t(\rho\mathrm{v}^i) + \partial_j(\rho\mathrm{v}^i\mathrm{v}^j) = f^i$

KINETIC ENERGY DENSITY

- $u_{\rm kin} = \dfrac{\delta \mathcal{E}_{\rm kin}}{\delta V} = \rho \mathrm{v}^2/2$

- Continuity and Euler equations lead to:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{\mathrm{v}} \right) = 0$$

$$\vec{\mathrm{v}} \cdot \left[\frac{\partial \vec{\mathrm{v}}}{\partial t} + (\vec{\mathrm{v}} \cdot \vec{\nabla})\vec{\mathrm{v}} \right] = \vec{\mathrm{v}} \cdot \frac{\vec{f}}{\rho}$$

$$\frac{\partial u_{\rm kin}}{\partial t} + \vec{\nabla} \cdot \left(u_{\rm kin} \vec{\mathrm{v}} \right) = \frac{\mathrm{d} u_{\rm kin}}{\mathrm{d} t} + u_{\rm kin} \left(\vec{\nabla} \cdot \vec{\mathrm{v}} \right) = \vec{f} \cdot \vec{\mathrm{v}}$$

The last term represents work done on the fluid element in unit time.

INTERNAL ENERGY DENSITY

- $u_{\text{int}} = \dfrac{\delta \mathscr{E}_{\text{int}}}{\delta V}$

- First law of thermodynamics: $\mathrm{d}(\delta \mathscr{E}_{\text{int}}) = T\,\mathrm{d}(\delta S) - P\,\mathrm{d}(\delta V)$
  with temperature T, entropy δS, pressure P.

- In an adiabatic process, entropy is conserved, $\mathrm{d}(\delta S) = 0$, hence $\mathrm{d}(\delta \mathscr{E}_{\text{int}}) = -P\,\mathrm{d}(\delta V)$.

- $\dfrac{\mathrm{d}u_{\text{int}}}{\mathrm{d}t} = -(u_{\text{int}} + P)\left(\vec{\nabla} \cdot \vec{\mathrm{v}}\right)$

- $w = u_{\text{int}} + P = \kappa u_{\text{int}}$ is the enthalpy density, with κ the adiabatic index (specific heats ratio)

- For ideal non-relativistic monoatomic gas: $\kappa = 5/3$, $u_{\text{int}} = (3/2)P$ and $w = (5/2)P$.

- $\dfrac{\partial P}{\partial t} + \left(\vec{\mathrm{v}} \cdot \vec{\nabla}\right) P = \dfrac{\mathrm{d}P}{\mathrm{d}t} = -\kappa P \left(\vec{\nabla} \cdot \vec{\mathrm{v}}\right)$

CONSERVATION OF ENERGY

- Pressure gradient as a force density: $\vec{f}_{\rm P} = -\,\vec{\nabla} P$.

- Work done by pressure gradient changes the kinetic energy:
  $\mathrm{d}(\delta\mathscr{E}_{\rm kin}) = \mathrm{d}(\delta W_{\rm P}) = \delta V \left(\vec{f}_{\rm P} \cdot \vec{\mathrm{v}}\right) \mathrm{d}t = -\,\delta V\,\mathrm{d}P$.

- Internal energy change: $\mathrm{d}(\delta\mathscr{E}_{\rm int}) = -\,P\,\mathrm{d}(\delta V)$.

- Total energy change: $\mathrm{d}(\delta\mathscr{E}_{\rm tot}) \equiv \mathrm{d}(\delta\mathscr{E}_{\rm kin} + \delta\mathscr{E}_{\rm int}) = -\,\mathrm{d}(P\,\delta V)$.

- Conservation of total energy: $\mathrm{d}(\delta\mathscr{E}_{\rm tot} + P\,\delta V) = 0$.

- Bernoulli's equation: $\dfrac{\mathrm{d}(\delta\mathscr{E}_{\rm tot} + P\,\delta V)}{\delta M} = \mathrm{d}\left(\dfrac{\mathrm{v}^2}{2} + \dfrac{w}{\rho}\right) = 0$.

- equivalent tensor form: $\dfrac{\partial u_{\rm tot}}{\partial t} + \vec{\nabla} \cdot \left[\left(u_{\rm kin} + w\right) \vec{\mathrm{v}}\right] = 0$.

CONSERVATION OF ENTROPY

- Relation between density and pressure variations:
$$\vec{\nabla} \cdot \vec{\mathrm{v}} = -\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{1}{\kappa P}\frac{\mathrm{d}P}{\mathrm{d}t}$$

- Adiabatic invariant: $\mathrm{d}[\ln(P/\rho^{\kappa})] = 0$.

- Adiabatic equation of state: $P \propto \rho^{\kappa}$ for a given Lagrangian fluid element.

- Specific entropy: $s = \dfrac{\delta S}{\delta M} = c_V \ln(P/\rho^{\kappa})$
with $c_V = (3/2)(k_\mathrm{B}/m)$: the specific heat capacity at constant volume.

- $\mathrm{d}s = 0$: equivalent to energy equation for adiabatic fluid.

- Conservation of mass implies the conservation of entropy $\mathrm{d}(\delta S) = 0$.

EQUATIONS FOR FLUID DYNAMICS

- continuity equation (conservation of mass):

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{\mathrm{v}} \right) = 0$$

- Euler equation (conservation of momentum):

$$\frac{\partial \vec{\mathrm{v}}}{\partial t} + \left(\vec{\mathrm{v}} \cdot \vec{\nabla} \right) \vec{\mathrm{v}} = \frac{\vec{f}}{\rho}$$

- pressure equation (conservation of energy and entropy):

$$\frac{\partial P}{\partial t} + \left(\vec{\mathrm{v}} \cdot \vec{\nabla} \right) P + \kappa P \left(\vec{\nabla} \cdot \vec{\mathrm{v}} \right) = 0$$

LINEARIZATION

- $\rho = \rho_0 + \rho_1$ with $|\rho_1| \ll |\rho_0|$, etc.

- continuity equation:
$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{\mathrm{v}}) = 0$$
$$\frac{\partial \rho_1}{\partial t} + \vec{\nabla} \cdot (\rho_1 \vec{\mathrm{v}}_0) + \vec{\nabla} \cdot (\rho_0 \vec{\mathrm{v}}_1) = 0$$

- Euler equation:
$$\frac{\partial \vec{\mathrm{v}}}{\partial t} + \left(\vec{\mathrm{v}} \cdot \vec{\nabla}\right) \vec{\mathrm{v}} = \frac{\vec{f}}{\rho}$$
$$\frac{\partial \vec{\mathrm{v}}_1}{\partial t} + \left(\vec{\mathrm{v}}_1 \cdot \vec{\nabla}\right) \vec{\mathrm{v}}_0 + \left(\vec{\mathrm{v}}_0 \cdot \vec{\nabla}\right) \vec{\mathrm{v}}_1 = \frac{\vec{f}_1}{\rho_0} - \frac{\vec{f}_0}{\rho_0^2} \rho_1$$

- pressure equation:
$$\frac{\partial P}{\partial t} + \left(\vec{\mathrm{v}} \cdot \vec{\nabla}\right) P + \kappa P \left(\vec{\nabla} \cdot \vec{\mathrm{v}}\right) = 0$$
$$\frac{\partial P_1}{\partial t} + \left(\vec{\mathrm{v}}_1 \cdot \vec{\nabla}\right) P_0 + \left(\vec{\mathrm{v}}_0 \cdot \vec{\nabla}\right) P_1 + \kappa P_1 \left(\vec{\nabla} \cdot \vec{\mathrm{v}}_0\right) + \kappa P_0 \left(\vec{\nabla} \cdot \vec{\mathrm{v}}_1\right) = 0$$

LINEARIZATION: UNIFORM STATIC BACKGROUND

- Assume static ($\vec{v}_0 = 0$) and uniform (constant ρ_0, P_0) background. This implies that $\vec{f}_0 = -\vec{\nabla} P_0 = 0$.

- continuity equation:

$$\frac{\partial \rho_1}{\partial t} + \vec{\nabla} \cdot \left(\rho_1 \vec{v}_0\right) + \vec{\nabla} \cdot \left(\rho_0 \vec{v}_1\right) = 0$$

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \left(\vec{\nabla} \cdot \vec{v}_1\right)$$

- Euler equation:

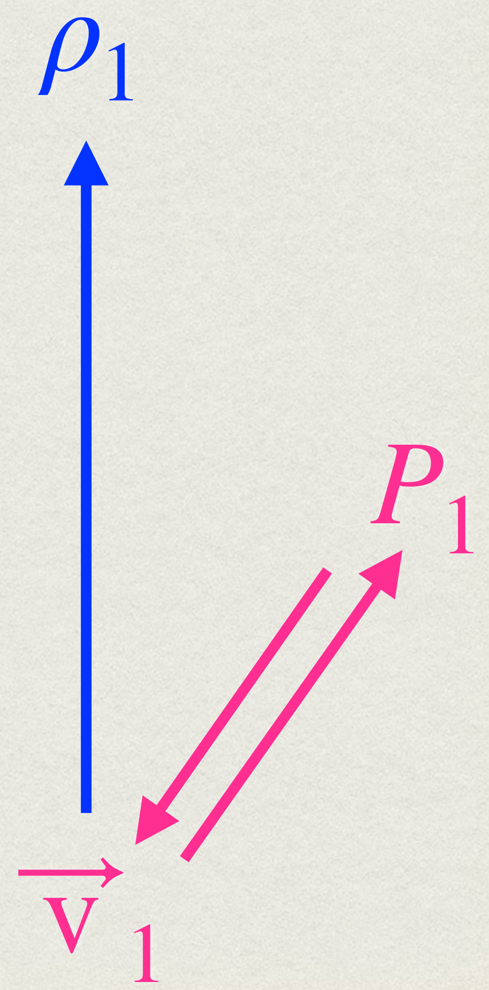
$$\frac{\partial \vec{v}_1}{\partial t} + \left(\vec{v}_1 \cdot \vec{\nabla}\right) \vec{v}_0 + \left(\vec{v}_0 \cdot \vec{\nabla}\right) \vec{v}_1 = \frac{\vec{f}_1}{\rho_0} - \frac{\vec{f}_0}{\rho_0^2} \rho_1$$

$$\frac{\partial \vec{v}_1}{\partial t} = -\frac{\vec{\nabla} P_1}{\rho_0}$$

- pressure equation:

$$\frac{\partial P_1}{\partial t} + \left(\vec{v}_1 \cdot \vec{\nabla}\right) P_0 + \left(\vec{v}_0 \cdot \vec{\nabla}\right) P_1 + \kappa P_1 \left(\vec{\nabla} \cdot \vec{v}_0\right) + \kappa P_0 \left(\vec{\nabla} \cdot \vec{v}_1\right) = 0$$

$$\frac{\partial P_1}{\partial t} = -\kappa P_0 \left(\vec{\nabla} \cdot \vec{v}_1\right)$$



LINEARIZATION: UNIFORM STATIC BACKGROUND

- Adopt oscillatory velocity perturbation $\vec{v}_1 \propto \exp\left(i\omega t + i\vec{k}\cdot\vec{r}\right)$

- pressure equation:
  $$\frac{\partial P_1}{\partial t} = -\kappa P_0\left(\vec{\nabla}\cdot\vec{v}_1\right)$$
  $$i\omega P_1 = -\kappa P_0\left(i\vec{k}\cdot\vec{v}_1\right) \quad P_1 \propto \exp\left(i\omega t + i\vec{k}\cdot\vec{r}\right)$$

- Euler equation:
  $$\frac{\partial \vec{v}_1}{\partial t} = -\frac{\vec{\nabla}P_1}{\rho_0}$$
  $i\omega\vec{v}_1 = -\frac{i\vec{k}P_1}{\rho_0} = \frac{i\vec{k}}{\rho_0}\frac{\kappa P_0}{\omega}\left(\vec{k}\cdot\vec{v}_1\right)$: longitudinal velocity perturbation ($\vec{v}_1 \parallel \vec{k}$), hence $P_1 \neq 0$ (compression).

- dot product with $-i\omega\vec{k}$ yields: $\omega^2\left(\vec{k}\cdot\vec{v}_1\right) = k^2\frac{\kappa P_0}{\rho_0}\left(\vec{k}\cdot\vec{v}_1\right)$

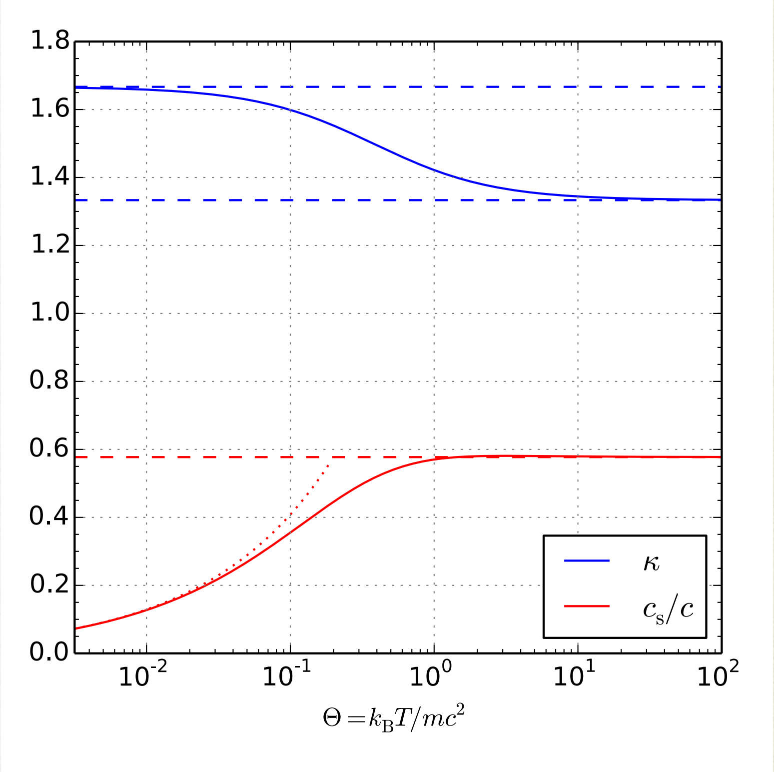
- dispersion relation $\frac{\omega^2}{k^2} = \frac{\kappa P_0}{\rho_0} \equiv c_{s,0}^2$: stable ($\omega^2 > 0$) wave – the sound wave.

  $c_{s,0}$ is both the phase and group speed (uniform and isotropic) – the speed of sound.

RELATIVISTIC SPEED OF SOUND

- Maxwell-Jüttner distribution $f(\gamma)\,\mathrm{d}\gamma = \dfrac{\gamma^2\beta\,\mathrm{d}\gamma}{\Theta\,K_2(1/\Theta)}\exp\left(-\dfrac{\gamma}{\Theta}\right)$

  where $\Theta = k_\mathrm{B}T/mc^2$ is the relativistic temperature, $\gamma = (1-\beta^2)^{-1/2}$ is the particle Lorentz factor, $K_2(x)$ is the modified Bessel function of the second kind.



- mean particle energy $\langle\gamma\rangle = 3\Theta + h(\Theta)$ with $h(\Theta) = \dfrac{K_1(1/\Theta)}{K_2(1/\Theta)}$

  pressure $P = nk_\mathrm{B}T = \Theta\rho c^2$

  relativistic enthalpy density $w = \rho c^2 + \dfrac{\kappa}{\kappa-1}P = [4\Theta + h(\Theta)]\rho c^2$

  adiabatic index $\kappa = 1 + \dfrac{\Theta}{\langle\gamma\rangle - 1}$

  speed of sound $c_\mathrm{s} = c\sqrt{\dfrac{\kappa P}{w}}$

- In the limit of non-relativistic temperatures $\Theta \ll 1$: $h(\Theta) \simeq 1 - \dfrac{3}{2}\Theta$, $\kappa \simeq \dfrac{5}{3}$, $w \simeq \rho c^2 + \dfrac{5}{2}P$,

  $c_\mathrm{s} \simeq c\sqrt{\dfrac{5\Theta}{3}} = \sqrt{\dfrac{5P}{3\rho}}$.

- In the limit of ultra-relativistic temperatures $\Theta \gg 1$: $h(\Theta) = \dfrac{1}{2\Theta}$, $\kappa \simeq \dfrac{4}{3}$, $w \simeq 4P$, $c_\mathrm{s} \simeq \dfrac{c}{\sqrt{3}}$.