COSMIC MAGNETIC FIELDS

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Electromagnetism

LORENTZ FORCE

- Consider a particle of mass m, electric charge q, velocity $\overrightarrow{\mathrm{v}} = \overrightarrow{\beta} c$, Lorentz factor $\gamma = (1-\beta^2)^{-1/2}$, energy $\mathscr{E} = \gamma m c^2$, and momentum $\overrightarrow{p} = \gamma \overrightarrow{\beta} m c$.

- The Lorentz force $\overrightarrow{F_{\mathrm{L}}} = \dfrac{\mathrm{d}\overrightarrow{p}}{\mathrm{d}t} = q\left(\overrightarrow{E} + \overrightarrow{\beta} \times \overrightarrow{B}\right)$ can be used to define the magnetic field $\overrightarrow{B}$ and electric field $\overrightarrow{E}$.

- We adopt Gaussian cgs (centimetre-gram-second) units throughout this lecture. The unit of B is gauss (G), 1 G $= 10^{-4}$ T (tesla - the SI unit). The unit of E is statvolt/cm (statV/cm), the unit of q is statcoulomb (statC).

ENERGIZATION OF CHARGED PARTICLES

- $\vec{p} \cdot \vec{F_{\mathrm{L}}} = \frac{1}{2}\frac{\mathrm{d}p^2}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}(\gamma^2 - 1)}{\mathrm{d}t}m^2c^2 = \gamma\frac{\mathrm{d}\gamma}{\mathrm{d}t}m^2c^2 = \gamma m\,\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t}$

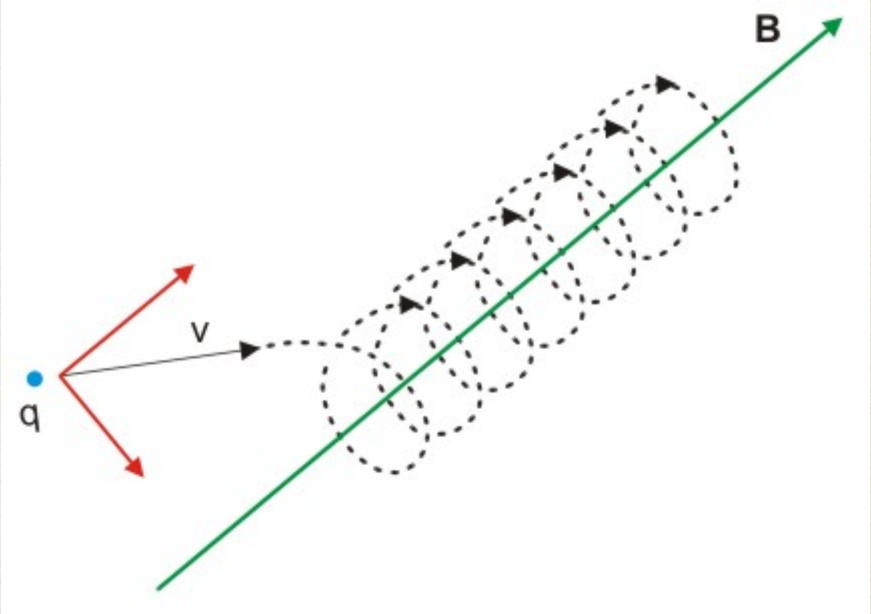
- $\vec{p} \cdot \vec{F_{\mathrm{L}}} = \gamma\vec{\beta}mc \cdot q\left(\vec{E} + \vec{\beta} \times \vec{B}\right) = \gamma m\, q\vec{\mathrm{v}} \cdot \vec{E}$

- particle energy change $\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = \frac{\mathrm{d}\gamma}{\mathrm{d}t}mc^2 = q\vec{\mathrm{v}} \cdot \vec{E}$

- magnetic fields do not contribute directly to particle energization

CHARGED PARTICLES IN UNIFORM MAGNETIC FIELD

- Consider a uniform magnetic field $\overrightarrow{B} = \text{const}$ without any electric field $\overrightarrow{E} = 0$, and a charged particle of constant energy $\gamma = \text{const}$ and velocity components $\overrightarrow{\beta}_{\parallel} \parallel \overrightarrow{B}$ and $\overrightarrow{\beta}_{\perp} \perp \overrightarrow{B}$.



- $\dfrac{\mathrm{d}\overrightarrow{\mathrm{v}}}{\mathrm{d}t} = \dfrac{q}{\gamma mc}\left(\overrightarrow{\mathrm{v}} \times \overrightarrow{B}\right) = \Omega\left(\overrightarrow{\mathrm{v}} \times \hat{B}\right)$,

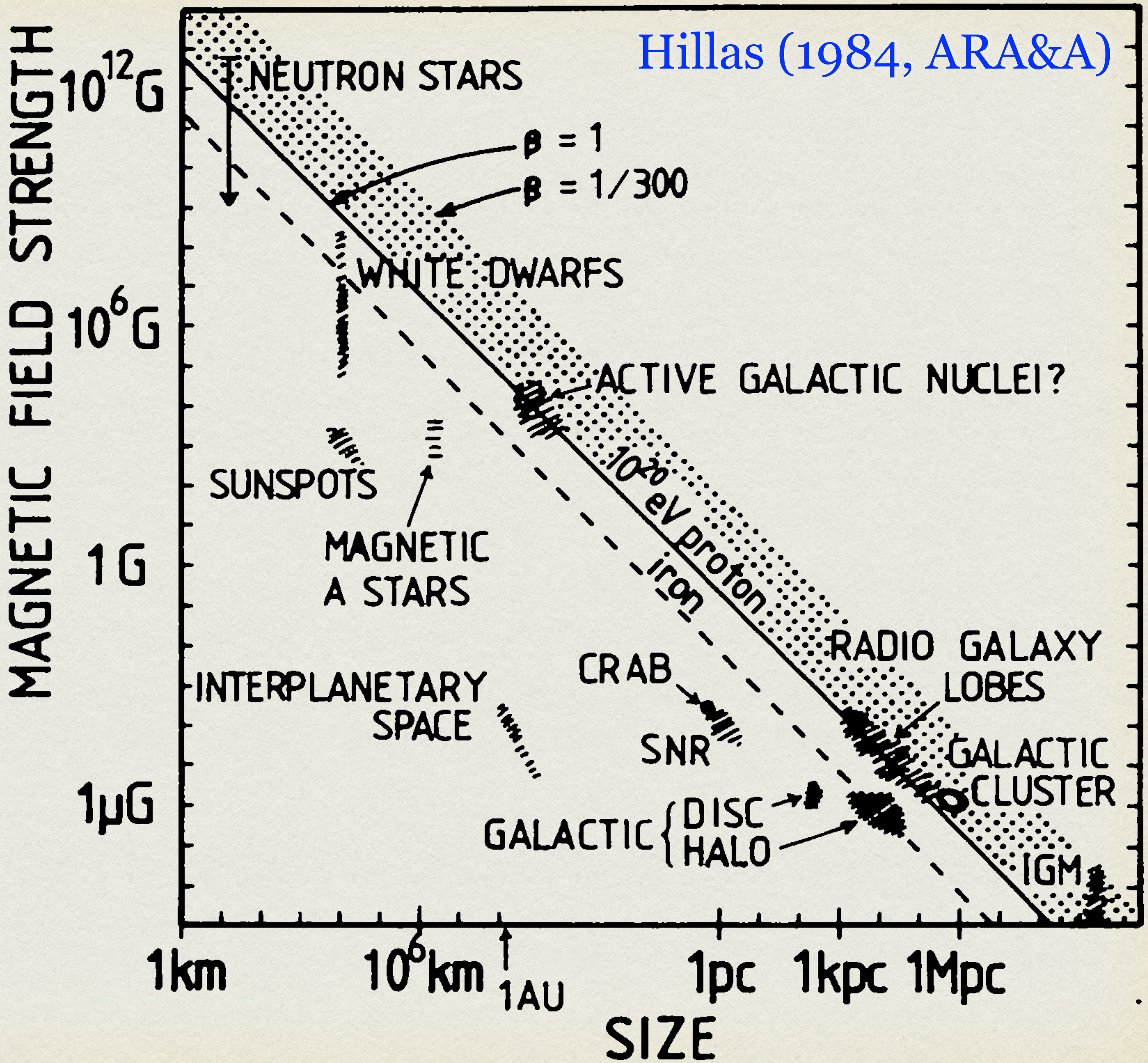
  where $\Omega = \dfrac{qB}{\gamma mc}$ is the gyrofrequency and $\hat{B} = \overrightarrow{B}/B$.

- Consider that $\hat{B} = (1,0,0)$, hence $\overrightarrow{\mathrm{v}} \times \hat{B} = (0, \mathrm{v}_z, -\mathrm{v}_y) = (0, \dot{z}, -\dot{y})$, where $\dot{z} \equiv \mathrm{d}z/\mathrm{d}t$, etc. The equations of motion are $\ddot{x} = 0$, $\ddot{y} = \Omega\dot{z}$, $\ddot{z} = -\Omega\dot{y} = -\Omega^2 z$. We thus have a combination of a uniform motion along $\hat{x} = \hat{B}$, and circular oscillation in the (y, z) plane.

- The particle follows a helical trajectory with the gyroradius $R = \dfrac{\mathrm{v}_\perp}{\Omega} = \dfrac{\gamma\beta_\perp mc^2}{qB} = \dfrac{p_\perp c}{qB}$.

HILLAS PLOT

$$R = \frac{p_\perp c}{qB} \simeq \frac{E}{qB}$$



CHARGED PARTICLES IN NON-UNIFORM MAGNETIC FIELD

- gradient drift for $\vec{\nabla} B \perp \vec{B}$
- curvature drift for bent field lines $\vec{\nabla} B \parallel \left(\vec{B} \times \hat{z} \right)$
- magnetic mirror for $\vec{\nabla} B \parallel \vec{B}$
- $\vec{E} \times \vec{B}$ drift for uniform $\vec{E}$

PROBLEM 1: MAGNETIC MIRROR

- Consider the case of $\overrightarrow{\nabla} B \parallel \overrightarrow{B}$.
  What may happen to a charged particle?

- Suggestion 1: solve particle motion along $\overrightarrow{B}$ by linearizing about uniform gyration.

- Suggestion 2: prove that $\mu = \mathrm{v}_\perp^2 / B$ is invariant.
  What is the implication?

This problem is worth 5 points. Solutions should be sent as 1-page PDF files to knalew@camk.edu.pl before the next lecture.

MAXWELL'S EQUATIONS

- $\vec{\nabla} \cdot \vec{E} = 4\pi\rho_{\mathrm{e}}$ Gauss's law for electric fields
- $\vec{\nabla} \cdot \vec{B} = 0$ Gauss's law for magnetic fields
- $\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ Maxwell-Faraday equation
- $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ Ampère-Maxwell equation

ELECTRIC CHARGE CONSERVATION

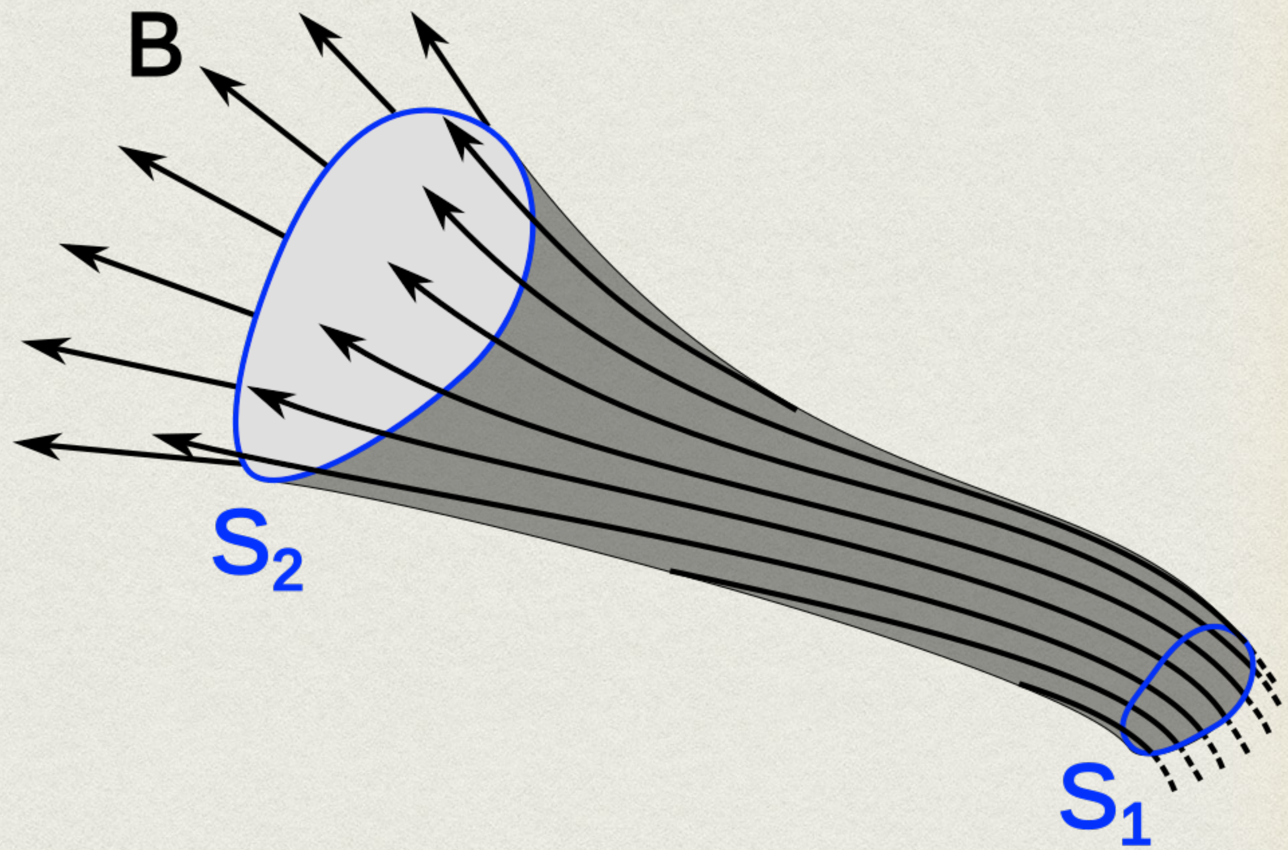
- electric charge density $\rho_{\mathrm{e}} = \sum_i q_i n_i$
- electric current density $\vec{j} = \sum_i q_i n_i \vec{\mathrm{v}}_i$
- electric four-current density $j^\mu = (\rho_{\mathrm{e}} c, \vec{j})$ for $\mu \in 0,1,2,3$
- $\partial_\mu j^\mu = \dfrac{\partial \rho_{\mathrm{e}}}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$

MAGNETIC MONOPOLES

- $\vec{\nabla} \cdot \vec{E} = 4\pi\rho_{\mathrm{e}}$ Gauss's law for electric fields
- $\vec{\nabla} \cdot \vec{B} = 4\pi\rho_{\mathrm{m}}$ Gauss's law for magnetic fields
- $\vec{\nabla} \times \vec{E} = -\frac{4\pi}{c}\vec{j}_{\mathrm{m}} - \frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ Maxwell-Faraday equation
- $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ Ampère-Maxwell equation
- magnetic charge density $\rho_{\mathrm{m}} = \sum_i q_{\mathrm{m},i}\, n_i$ and current density $\vec{j}_{\mathrm{m}} = \sum_i q_{\mathrm{m},i}\, n_i \vec{\mathrm{v}}_i$
- For $q_{\mathrm{m}} \neq 0$ magnetic fields would be screened and limited to short ranges.

MAGNETIC FLUX TUBE

- magnetic flux: $\Phi = \iint \vec{B} \cdot \mathrm{d}\vec{S}$
- magnetic flux tube: a volume element bounded by a surface parallel to local magnetic fields (no magnetic flux through the sides)
- divergence theorem:
  $\Phi_2 - \Phi_1 = \iiint \left(\vec{\nabla} \cdot \vec{B} \right) \mathrm{d}V$
- The Gauss law for magnetism $\vec{\nabla} \cdot \vec{B} = 0$ implies that $\Phi_2 = \Phi_1$



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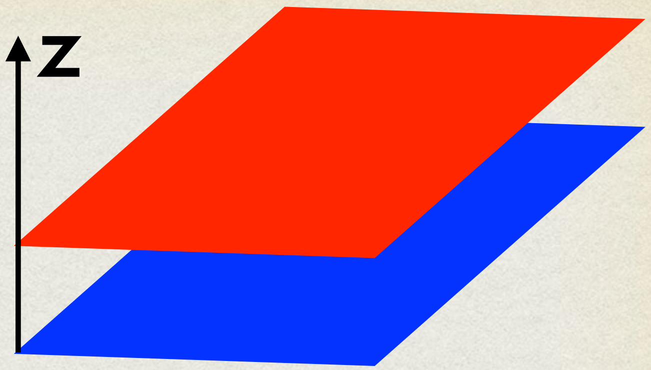
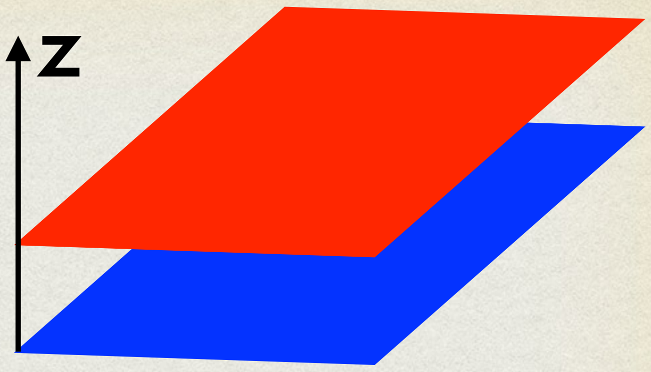
ELECTRIC ENERGY DENSITY



- Consider two large flat parallel plates of thickness δz with opposite electric charges of uniform surface densities $\pm(\Sigma_\mathrm{e} = \rho_\mathrm{e}\,\delta z)$.
- Using $\vec{\nabla} \cdot \vec{E} = 4\pi\rho_\mathrm{e}$, the electric field between the plates is $E_z \simeq 4\pi\rho_\mathrm{e}\,\delta z = 4\pi\Sigma_\mathrm{e}$.
- Each plate is attracted to the other by Lorentz force of surface density
$$\frac{\Delta F_{\mathrm{L},z}}{\Delta A} = \frac{E_z}{2}\Sigma_\mathrm{e} = \frac{E^2}{8\pi}$$
(the 1/2 factor accounts for the linear decay of E_z across each plate).
- Increasing separation between the plates by Δz requires a work $\Delta W = \Delta F_{\mathrm{L},z}\,\Delta z = \Delta E_\mathrm{E}$, increasing the electric energy for the same E_z.
- The electric energy density is thus $u_\mathrm{E} = \dfrac{\Delta E_\mathrm{E}}{\Delta A\,\Delta z} = \dfrac{\Delta F_{\mathrm{L},z}}{\Delta A} = \dfrac{E^2}{8\pi}$.

MAGNETIC ENERGY DENSITY



- An analogous argument can be made, considering large flat plates with opposite current densities, e.g. $\pm(\mathscr{I}_x = j_x\,\delta z)$.

- Using $\vec{\nabla} \times \vec{B} = \dfrac{4\pi}{c}\vec{j}$, the magnetic field between the plates is $B_y \simeq \dfrac{4\pi}{c}\mathscr{I}_x$.

- Each plate is attracted to the other by Lorentz force of surface density $\dfrac{\Delta F_{\mathrm{L},z}}{\Delta A} = \dfrac{B_y}{2}\dfrac{\mathscr{I}_x}{c} = \dfrac{B^2}{8\pi}$.

- The magnetic energy density is thus $u_{\mathrm{B}} = \dfrac{B^2}{8\pi}$. $(1\,\mathrm{G}^2 = 8\pi\,\dfrac{\mathrm{erg}}{\mathrm{cm}^3})$

- Magnetic fields can store energy, they can also release it. This is fundamentally important for magnetic reconnection, magnetoluminescence, etc.

MAGNETOLUMINESCENCE

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Magnetoluminescence

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Abstract Pulsar Wind Nebulae, Blazars, Gamma Ray Bursts and Magnetars all contain regions where the electromagnetic energy density greatly exceeds the plasma energy density. These sources exhibit dramatic flaring activity where the electromagnetic energy distributed over large volumes, appears to be converted efficiently into high energy particles and γ-rays. We call this general process magnetoluminescence. Global requirements on the underlying, extreme particle acceleration processes are described and the likely importance of relativistic beaming in enhancing the observed radiation from a flare is emphasized. Recent research on fluid descriptions of unstable electromagnetic configurations are summarized and progress on the associated kinetic simulations that are needed to account for the acceleration and radiation is discussed. Future observational, simulation and experimental opportunities are briefly summarized.

ELECTROMAGNETIC ENERGY CHANGE

- electric and magnetic energy densities: $u_{\mathrm{E}} \equiv \dfrac{E^2}{8\pi}, u_{\mathrm{B}} \equiv \dfrac{B^2}{8\pi}$

- electric energy change: $\dfrac{\partial u_{\mathrm{E}}}{\partial t} = \dfrac{c}{4\pi}\vec{E} \cdot \left(\vec{\nabla} \times \vec{B}\right) - \vec{E} \cdot \vec{j}$

Ohmic dissipation

- magnetic energy change: $\dfrac{\partial u_{\mathrm{B}}}{\partial t} = -\dfrac{c}{4\pi}\vec{B} \cdot \left(\vec{\nabla} \times \vec{E}\right)$

- electromagnetic energy change: $\dfrac{\partial u_{\mathrm{EM}}}{\partial t} \equiv \dfrac{\partial (u_{\mathrm{E}} + u_{\mathrm{B}})}{\partial t} = -\dfrac{c}{4\pi}\vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right) - \vec{E} \cdot \vec{j}$

- introducing the Poynting flux $\vec{S} \equiv \dfrac{c}{4\pi}\vec{E} \times \vec{B}$:

$$\frac{\partial u_{\mathrm{EM}}}{\partial t} = -\vec{\nabla} \cdot \vec{S} - \vec{E} \cdot \vec{j}$$

POYNTING FLUX CHANGE

- $\overrightarrow{S} \equiv \frac{c}{4\pi}\overrightarrow{E} \times \overrightarrow{B}$

- $\partial_t S^i = \frac{c}{4\pi}\epsilon_{ijk}(B^k \partial_t E^j + E^j \partial_t B^k)$

- $\frac{\partial_t S^i}{c^2} = -\,\partial_i \frac{E^2 + B^2}{8\pi} + \partial_j \frac{E^i E^j + B^i B^j}{4\pi} - \rho_e E^i - \frac{1}{c}\left(\vec{j} \times \overrightarrow{B}\right)^i$

- $\frac{\partial_t S^i}{c^2} = -\,\partial_j T^{ij}_{\mathrm{EM}} - f^i_{\mathrm{L}}$

with the (-)Maxwell stress tensor $T^{ij}_{\mathrm{EM}} = \underbrace{\frac{E^2 + B^2}{8\pi}\delta^{ij}}_{\text{pressure}} - \underbrace{\frac{E^i E^j + B^i B^j}{4\pi}}_{\text{tension}}$ (symmetric)

and the Lorentz force density $\overrightarrow{f_{\mathrm{L}}} = \rho_e \overrightarrow{E} + \frac{1}{c}\left(\vec{j} \times \overrightarrow{B}\right)$

STRESS TENSOR GRADIENT

- $T^{ij}_{\mathrm{EM}} = \frac{E^2 + B^2}{8\pi}\delta^{ij} - \frac{E^i E^j + B^i B^j}{4\pi}$

- $\partial_j T^{ij}_{\mathrm{EM}} = -\frac{1}{4\pi}\left(\vec{\nabla}\cdot\vec{E}\right)E^i - \frac{1}{4\pi}\left[\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}\right]^i - \frac{1}{4\pi}\left[\left(\vec{\nabla}\times\vec{E}\right)\times\vec{E}\right]^i$

- $\vec{f}_{\mathrm{L}} = \rho_{\mathrm{e}}\vec{E} + \frac{1}{c}\left(\vec{j}\times\vec{B}\right)$

  $\vec{j} = \frac{c}{4\pi}\left(\vec{\nabla}\times\vec{B}\right) - \frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}$

- $f^i_{\mathrm{L}} = \frac{1}{4\pi}\left(\vec{\nabla}\cdot\vec{E}\right)E^i + \frac{1}{4\pi}\left[\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}\right]^i - \frac{1}{4\pi}\left[\frac{\partial\vec{E}}{\partial ct}\times\vec{B}\right]^i$

- Very close correspondence: $f^i_{\mathrm{L}} \simeq -\partial_j T^{ij}_{\mathrm{EM}}$

ENERGY-MOMENTUM EQUATIONS

- energy equation: $\dfrac{\partial u_{\mathrm{EM}}}{\partial t} = -\vec{\nabla}\cdot\vec{S} - \vec{E}\cdot\vec{j}$

- momentum equation: $\dfrac{1}{c^2}\dfrac{\partial\vec{S}}{\partial t} = -\vec{\nabla}\cdot\mathbf{T}_{\mathrm{EM}} - \vec{f}_{\mathrm{L}}$

- let $T^{00}_{\mathrm{EM}} = u_{\mathrm{EM}}$ and $T^{i0}_{\mathrm{EM}} = \dfrac{S^i}{c}$

- this leads to $\partial_\mu T^{\mu 0}_{\mathrm{EM}} = -\dfrac{\vec{E}\cdot\vec{j}}{c}$ and $\partial_\mu T^{\mu i}_{\mathrm{EM}} = -f^i_{\mathrm{L}}$

  where $\partial_\mu \equiv \partial/\partial x^\mu$ for $\mu \in \{0,1,2,3\}$ with $x^0 = ct$

- the Poynting flux $\vec{S}/c$ can thus be identified as the electromagnetic momentum density

ELECTROMAGNETIC STRESS TENSOR

- $T^{00}_{\rm EM} = u_{\rm EM} = \dfrac{E^2 + B^2}{8\pi}$

- $T^{0i}_{\rm EM} = \dfrac{S^i}{c} = \dfrac{1}{4\pi}\left(\overrightarrow{E} \times \overrightarrow{B}\right)^i$

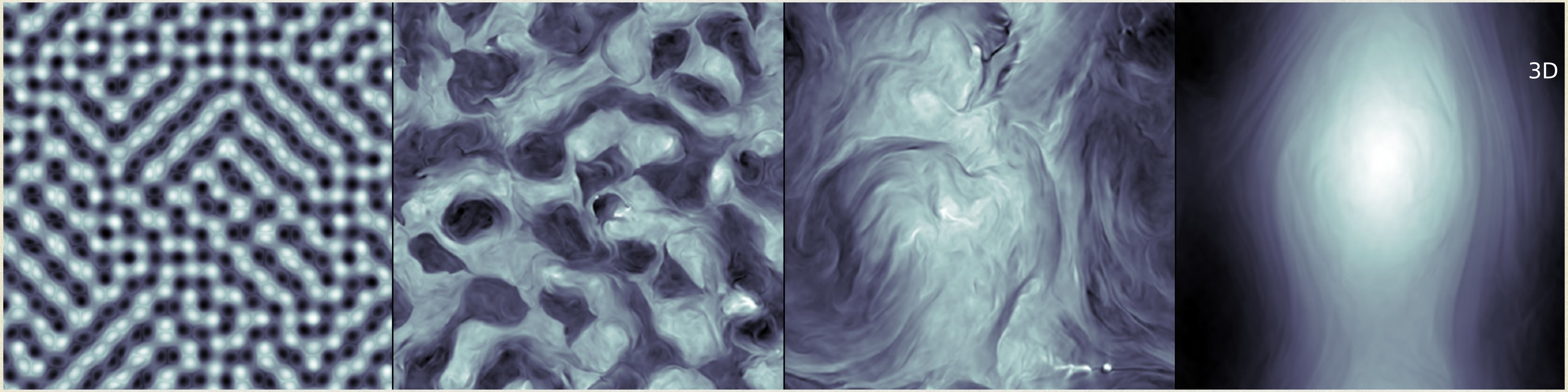
- $T^{ij}_{\rm EM} = \dfrac{E^2 + B^2}{8\pi}\delta^{ij} - \dfrac{E^i E^j + B^i B^j}{4\pi}$

- example: $\overrightarrow{B} = (B_x, 0, 0)$ and $\overrightarrow{E} = 0$: $T^{\mu\nu}_{\rm EM} = \dfrac{B_x^2}{8\pi}\,\mathrm{diag}(1, -1, 1, 1)$

- Note the fundamental difference: pressure pushes, tension pulls. This is fundamentally important for magnetic reconnection, inverse cascade, etc.

INVERSE CASCADE

- Energy flow from small scales to large scales.
- If the field line topology can be simplified (by reconnection), tension will straighten the field lines.



Zrake & East (2016)

FORCE-FREE ELECTRODYNAMICS

- $\vec{f}_{\rm L} = \rho_{\rm e}\vec{E} + \frac{1}{c}\left(\vec{j}\times\vec{B}\right) = 0$

  e.g., in relativistically magnetized plasmas, neglecting particle pressure and inertia

- $\vec{f}_{\rm L}\cdot\vec{B} = \rho_{\rm e}\left(\vec{E}\cdot\vec{B}\right) = 0$

- $\frac{\vec{\nabla}\cdot\vec{E}}{4\pi}\vec{E}\times\vec{B} + \frac{1}{c}\left[\left(\vec{j}\cdot\vec{B}\right)\vec{B} - B^2\vec{j}\right] = 0$

- $\vec{B}\cdot\left(\vec{\nabla}\times\vec{B}\right) = \frac{4\pi}{c}\vec{B}\cdot\vec{j} + \vec{B}\cdot\frac{\partial\vec{E}}{c\,\partial t}$

- $0 = \frac{\partial}{c\,\partial t}\left(\vec{E}\cdot\vec{B}\right) = \vec{B}\cdot\frac{\partial\vec{E}}{c\,\partial t} + \vec{E}\cdot\frac{\partial\vec{B}}{c\,\partial t} = \vec{B}\cdot\frac{\partial\vec{E}}{c\,\partial t} - \vec{E}\cdot\left(\vec{\nabla}\times\vec{E}\right)$

- $\vec{j}\left(\vec{E},\vec{B}\right) = \frac{c}{4\pi B^2}\left\{\left(\vec{E}\times\vec{B}\right)\left(\vec{\nabla}\cdot\vec{E}\right) + \vec{B}\left[\vec{B}\cdot\left(\vec{\nabla}\times\vec{B}\right) - \vec{E}\cdot\left(\vec{\nabla}\times\vec{E}\right)\right]\right\}$

  current density as a function of instantaneous $\vec{E}$, $\vec{B}$ fields

ELECTROMAGNETIC POTENTIALS

- $\vec{B} = \vec{\nabla} \times \vec{A}$, introducing the magnetic vector potential $\vec{A}$
- $\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$, introducing the electric scalar potential ϕ
- designed to satisfy source-free Maxwell's equations $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$
- gauge function: $\vec{A} \to \vec{A} + \vec{\nabla}\psi$ and $\phi \to \phi - \partial_t \psi / c$
- Coulomb gauge: $\vec{\nabla} \cdot \vec{A} = 0$ (for slowly varying fields)
- Lorenz gauge: $\partial_\mu A^\mu = 0$ (for rapidly varying fields, radiation),
  introducing the electromagnetic four-potential $A^\mu = (\phi, \vec{A})$

MAGNETIC MOMENT AND DIPOLE

- Sources of magnetic field (e.g., magnets) can be characterized by magnetic moment $\vec{m}$.

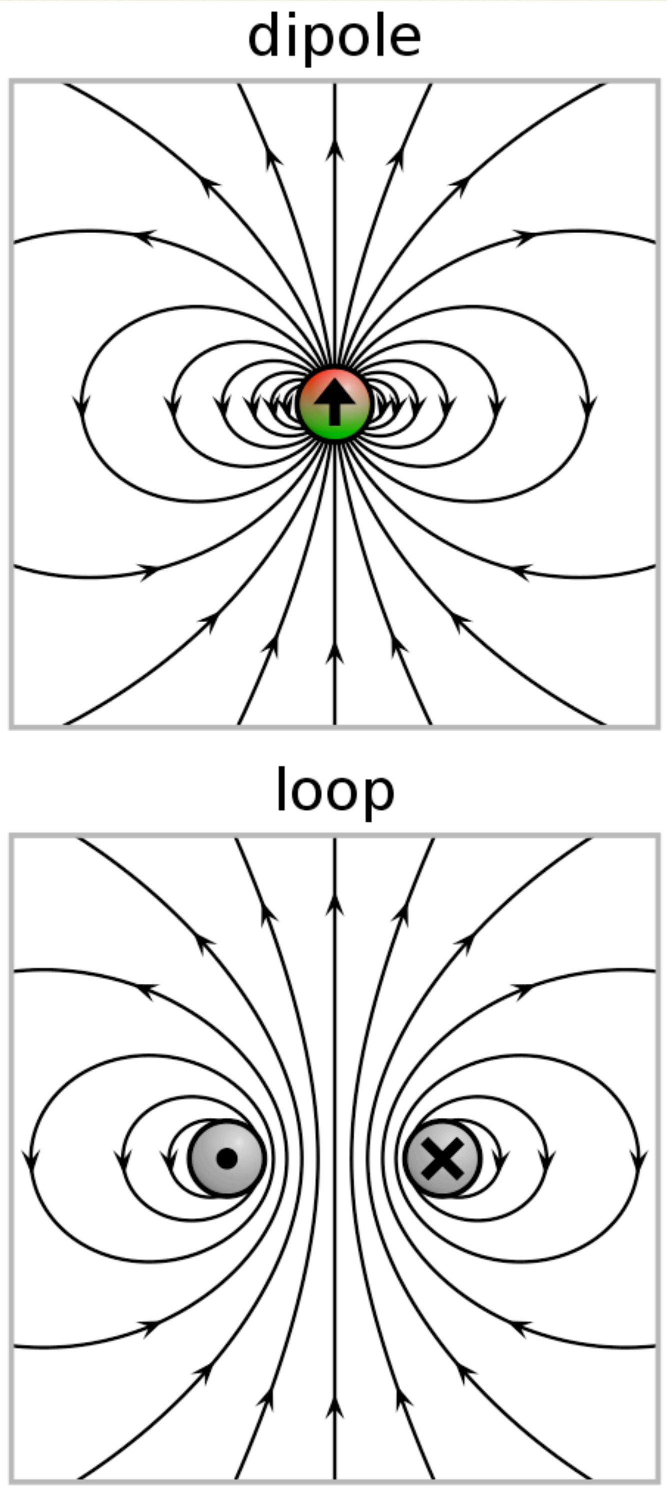
- When placed in external magnetic field $\vec{B}_0$, a torque is induced $\vec{\tau} = \vec{m} \times \vec{B}_0$.

- Magnetic moment generates a magnetic dipole described by vector potential $\vec{A} = \dfrac{\vec{m} \times \vec{R}}{R^3}$. In cylindrical coordinates (r, ϕ, z) where $\vec{m} = (0{,}0{,}m)$ we have $\vec{A} = \dfrac{mr}{R^3}\hat{\phi} = \dfrac{m}{R^2}\sin\theta\,\hat{\phi}$, where $\theta = \angle(\vec{m}, \vec{R})$.

- The corresponding dipole magnetic field is:

$$\vec{B} = \vec{\nabla} \times \vec{A} = -\frac{\partial A_\phi}{\partial z}\hat{r} + \left(\frac{A_\phi}{r} + \frac{\partial A_\phi}{\partial r}\right)\hat{z}$$

$$\vec{B} = \frac{m}{R^3}\left[3\sin\theta\cos\theta\,\hat{r} + (3\cos^2\theta - 1)\hat{z}\right] = \frac{1}{R^3}\left[3\left(\vec{m}\cdot\hat{R}\right)\hat{R} - \vec{m}\right]$$



SUMMARY

- Magnetic fields confine the motion of charged particles, forcing them to gyrate on helical paths.
- Magnetic fields do not directly energize charged particles.
- Magnetic fields are divergence-free and long-range, hard to screen.
- Magnetic field is a form of energy. Poynting flux is a measure of momentum density.
- Magnetic stress tensor is anisotropic: pressure across the field lines, tension along the field lines.