Supernovæ and shocks

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Outline

- ⋆ Method of characteristics,
- ⋆ Riemann problem and Godunov solution,
- ⋆ Rankine-Hugoniot shock conditions,
- ⋆ Bondi accretion,
- ⋆ Supernovæ Ia - deflagration and detonation,
- ⋆ Core-collapse supernovæ- collapse, bounce and shock revival.

What is a shock?

Shock is a thin ($\sim \lambda$) transition layer in which

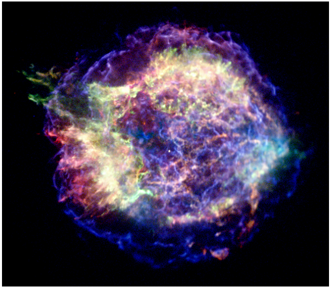
- ⋆ bulk flow energy is dissipated as heat,
- ⋆ the entropy of the system is increased.

Shocks can be

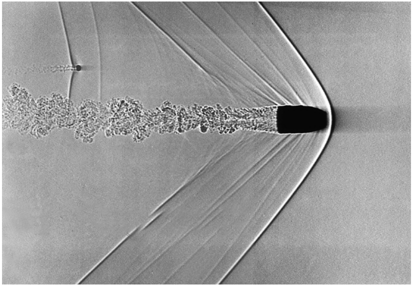
- ⋆ collisional, if molecular/Coulomb dissipation operates,
- ⋆ collisionless, for interaction with electromagnetic waves, particle/wave scattering.

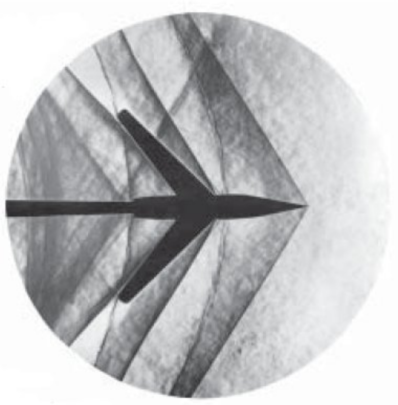
Shocks form by

- ⋆ fast flow around an obstacle,
- ⋆ accretion onto compact objects,
- ⋆ explosions (supernova, GRB),
- ⋆ non-linear steeping of waves.



What is a shock?





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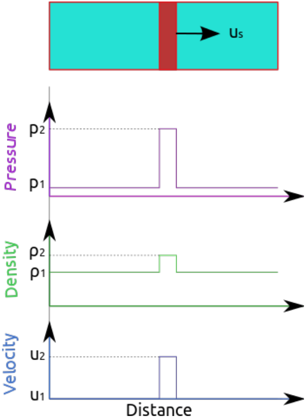
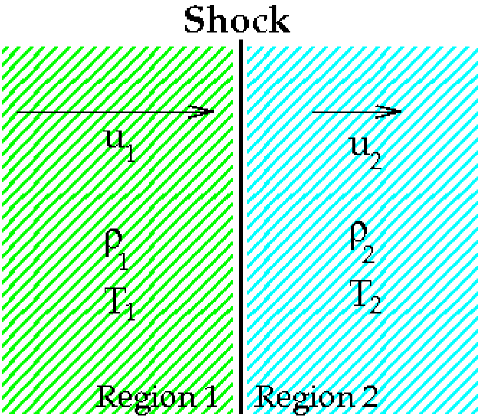




Upstream (region 1, pre-shock) and downstream (region 2, post-shock) parts of the system (from wikipedia).

Shock - a region of discontinuity

Assuming that we have a conserved quantity ϕ that undergoes an discontinuity at $x_s(t)$, and is governed by

$$\frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial x} f(\phi).$$

Integrating the above in the range of $x_1 < x_s < x_2$:

$$\frac{d}{dt}\left(\int_{x_1}^{x_s(t)} \phi dx + \int_{x_s(t)}^{x_2} \phi dx\right) = \phi_1 \frac{dx_s}{dt} - \phi_2 \frac{dx_s}{dt} + \int_{x_1}^{x_s(t)} \frac{\partial \phi}{\partial t} dx + \int_{x_s(t)}^{x_2} \frac{\partial \phi}{\partial t} dx$$
$$= \int_{x_1}^{x_2} \frac{\partial}{\partial x} f(\phi)\, dx = -f(\phi)|_{x_1}^{x_2}.$$

with $dx_1/dt = dx_2/dt = 0$. For the limit $x_1 \to x_s(t)$ and $x_2 \to x_s(t)$ one obtains u_s, the system characteristic speed (shock speed):

$$u_s = \frac{f(\phi_1) - f(\phi_2)}{\phi_1 - \phi_2}.$$

Method of characteristics

- A shock arises in a system where its characteristics intersect,
- To find out the real (single-valued) solution, the admissibility (entropy) condition is used. For physically real applications this means that the solution should satisfy the Lax entropy condition

$$f'(\phi_1) > u_s > f'(\phi_2).$$

where $f'(\phi_1)$ and $f'(\phi_2)$ represent characteristic speeds at upstream and downstream conditions (characteristics always enter a shock, but never leave it).

The method of characteristics

Consider the following constant coefficient PDE for a function $u(x, y)$:

$$au_x + bu_y = 0, \quad \text{where} \quad a^2 + b^2 \neq 0.$$

This can be rewritten as a product of a vector $\mathbf{v}$ and operator ∇:

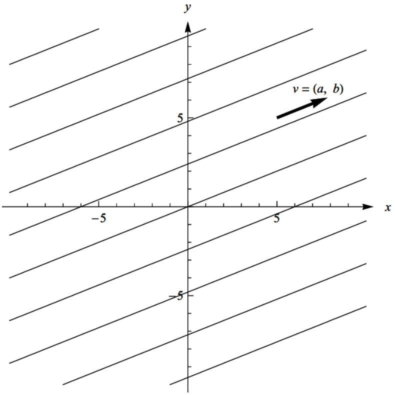
$$(a, b) \cdot \nabla u = \mathbf{v} \cdot \nabla u = 0.$$

→ The solution $u(x, y)$ is constant in the direction of $\mathbf{v}$. The lines

$$bx - ay = c$$

are called the *characteristics* of the problem, 'labeled' by an unique value of c. From the above

$$u(x, y) = f(bx - ay).$$



Characteristic lines $bx - ay = c$

The method of characteristics

More formally, in order to reduce the complexity of the problem (to make ODE out of PDE), one can change the coordinates to such that one of the axes is parallel to **v**:

$$(\xi, \eta) = (ax + by, bx - ay), \quad \text{so that}$$

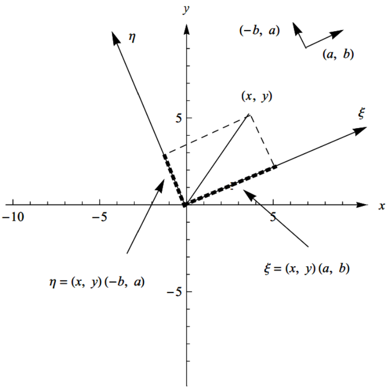
$$u_x = au_\xi + bu_\eta,$$
$$u_y = bu_\xi - au_\eta.$$

One has

$$au_x + bu_y \equiv 0 = (a^2 + b^2)u_\xi \to u_\xi = 0.$$

that is

$$u(\xi, \eta) = f(\eta) = f(bx - ay).$$



Change of coordinates $(x, y) \to (\xi, \eta)$

The method of characteristics: Burgers' equation

Consider a 1D diffusive PDE, which can serve as a very simple model of shock propagation:

$$u_t + uu_x = \nu u_{xx}.$$

A much simplified version of the above is the advection equation

$$u_t + cu_x = 0 \quad \rightarrow \quad (1, c) \cdot \nabla u = 0.$$

This means that the characteristic lines (characteristic speeds) are given by

$$\frac{dx}{dt} = c \rightarrow x = ct + x(0)$$

and the solution is $u(x, t) = f(x - ct)$, an initially defined shape moving from left to right on the $x-$axis.

Inviscid Burgers' equation: rarefaction

We now have the simplest nonlinear equation

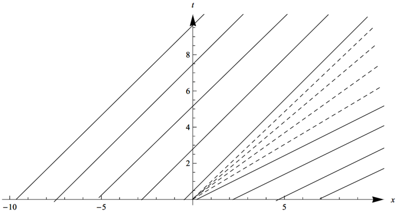
$$u_t + uu_x = 0 \rightarrow (1, u) \cdot \nabla u = 0$$

(the tangent to characteristics **v** depends on u). Suppose the following initial conditions,

$$u(0, x) = \begin{cases} 1 & x < 0, \\ 2 & x > 0, \end{cases}$$

and the slope of characteristics is

$$\frac{dx}{dt} = u(t, x(t)) = u(0, x(0)), \quad \text{which gives} \quad x(t) = \begin{cases} t + x(0) & x(0) < 0, \\ 2t + x(0) & x(0) > 0. \end{cases}$$



Rarefying characteristics.

The waves originating at $x(0) > 0$ move to the right faster than the waves originating at points $x(0) < 0$ → rarefaction region.

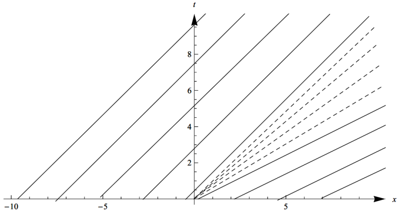
Inviscid Burgers' equation: rarefaction

The point $x = 0$ corresponds to the discontinuity in the initial conditions (a bundle of dashed characteristics emanating from $x = 0$). For $x(0) = 0$

$$x(t) = u(0, x(0))t + x(0)$$

implies

$$u = x/t \quad \text{for} \quad t < x < 2t.$$



Rarefying characteristics.

Final solution is

$$u(t,x) = \begin{cases} 1 & x < t, \\ x/t & t < x < 2t, \\ 2 & x > 2t. \end{cases}$$

Inviscid Burgers' equation: shock

The opposite initial conditions

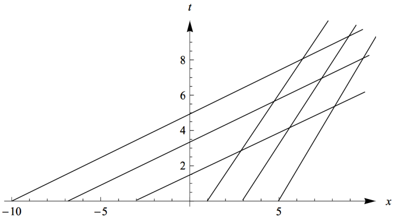
$$u(0,x) = \begin{cases} 2 & x < 0, \\ 1 & x > 0, \end{cases}$$

give the following characteristic lines

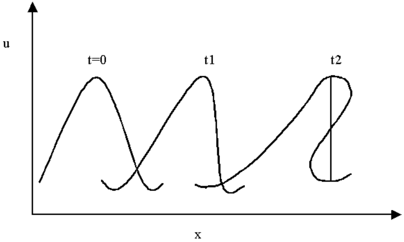
$$x(t) = \begin{cases} 2t + x(0) & x(0) < 0, \\ t + x(0) & x(0) > 0, \end{cases}$$

e.g., the slopes are opposite - the waves originating at $x(0) > 0$ move slower than the waves originating at points $x(0) < 0$.

Solution u becomes multi-valued at the crossing of characteristics $\rightarrow$ the shock is formed.

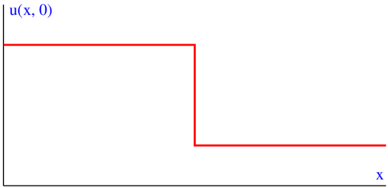


Characteristic lines showing a shock wave formation.



Riemann problem and the Godunov scheme

In general, step function initial conditions constitute the *Riemann problem*. Consider the conservative form of the advection equation



$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial t}f(x,t) = 0,$$

where $f(x,t)$ is the flux of field $u(x,t)$ ($f = cu$, say) - physically, it can represent a moving shock front. The above is just a conservation equation for a vector (u, f):

$$\partial_t u + \partial_x f = \nabla \cdot \begin{pmatrix} u \\ f \end{pmatrix} = 0,$$

that is

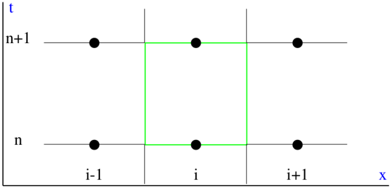
$$\int \nabla \cdot \begin{pmatrix} u \\ f \end{pmatrix} dxdt = \oint \begin{pmatrix} u \\ f \end{pmatrix} \cdot \mathbf{n} dl = 0.$$

Riemann problem and the Godunov scheme

In numerical settings, the Gauss integral formula is written as a line integral,

$$\left(u_i^{n+1} - u_i^n\right) dx + \left(f_{i+1/2} - f_{i-1/2}\right) dt = 0.$$

It requires to define the spatial integral averages of $u(x,t)$ (lower and upper side of a cell):



Space-time grid $x = ih,\ t = n\tau$ $(dx = h,\ dt = \tau)$.

$$u_i^n = \frac{1}{h}\int_{i-1/2}^{i+1/2} u(x,t_n)dx, \quad u_i^{n+1} = \frac{1}{h}\int_{i-1/2}^{i+1/2} u(x,t_{n+1})dx,$$

and time integral averages of the fluxes (left and right side of the cell):

$$f_{i-1/2} = \frac{1}{\tau}\int_n^{n+1} f(u(x_{i-1/2},t))dt, \quad f_{i+1/2} = \frac{1}{\tau}\int_n^{n+1} f(u(x_{i+1/2},t))dt.$$

so

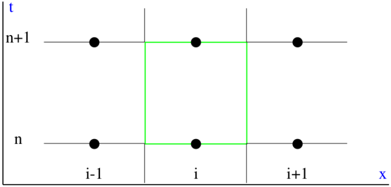
$$u_i^{n+1} = u_i^n - \frac{\tau}{h}\left(f_{i+1/2} - f_{i-1/2}\right).$$

Riemann problem and the Godunov scheme

To solve

$$u_i^{n+1} = u_i^n - \frac{\tau}{h}\left(f_{i+1/2} - f_{i-1/2}\right).$$

one needs the fluxes at each interface sides, $f_{i-1/2}$ and $f_{i+1/2}$. *Godunov's first order upwind method* treats every cell interface as a Riemann problem with piecewise constant initial data, e.g., at $i + 1/2$



Space-time grid $x = ih, \; t = n\tau$ $(dx = h, \; dt = \tau)$.

$$u_{i+1/2}^n = \begin{cases} u_i^n & x < x_{i+1/2}, \\ u_{i+1}^n & x > x_{i+1/2}, \end{cases} \rightarrow \text{ used to get the Godunov fluxes } f_{i-1/2} \text{ and } f_{i+1/2}.$$

For $f = cu$, the upwind scheme gives

$$u_i^{n+1} = u_i^n - \frac{\tau}{h}\left(f_{i+1/2} - f_{i-1/2}\right) = u_i^n - \begin{cases} \lambda_{cfl}\left(u_i^n - u_{i-1}^n\right) & c > 0, \\ \lambda_{cfl}\left(u_{i+1}^n - u_i^n\right) & c < 0, \end{cases}$$

with $\lambda_{cfl} = c\tau/h$ is the Courant-Friedrichs-Lewy parameter.

The Mach number

Consider a flow from the jet of the crossection A. From the Bernoulli and mass conservation,

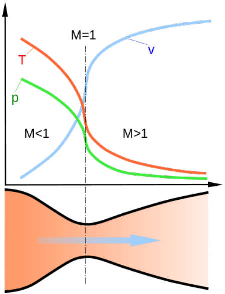
$$\frac{1}{2}u^2 + h = const., \quad \rho u A = const.$$

with enthalpy $h = \int dp/\rho$. Changes of A induce variations in other quantities:

$$udu + \frac{c^2}{\rho}d\rho = 0, \quad \rightarrow \quad \frac{d\rho}{\rho} = -\mathcal{M}^2\frac{du}{u}.$$

with the sound speed $c = \sqrt{dp/d\rho}$ and $\mathcal{M} = u/c$ is the Mach number.

- ⋆ For $\mathcal{M} \ll 1$ flow generally incompressible,
- ⋆ $\mathcal{M} < 1$ - subsonic flow, $\mathcal{M} > 1$ - supersonic flow.



The de Laval nozzle - Change of A makes the flow supersonic (in astrophysical situations acceleration due to e.g., gravity):

$$(1 - \mathcal{M}^2)\frac{du}{u} = -\frac{dA}{A}.$$

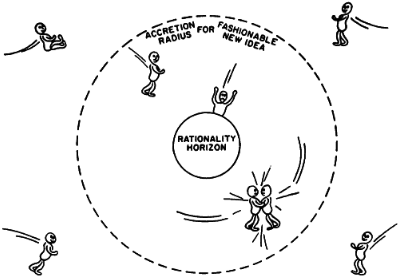
Bondi accretion

Consider steady, spherically symmetric accretion of gas onto mass M, assuming

- ⋆ barotropic flow, $p = p(\rho)$,
- ⋆ gas is not self-gravitating, $m_{gas} \ll M$.

Integrating the continuity equation on shells surrounding the central mass gives (with velocity u directed inwards), one get the following:

$$\dot{M} = -4\pi r^2 \rho u$$



McCray 1979 (at the Cambridge AGN meeting)

For a given radial streamline, the Bernoulli theorem is

$$\frac{1}{2}u^2 - \frac{GM}{r} + \underbrace{\int_{\rho_\infty}^{\rho} \frac{dp}{\rho}}_{enthalpy\ \ h} = 0,$$

with ρ_∞ the density at infinity.

Bondi accretion

There are particular cases of flows:

- ⋆ Isothermal flow, $p = c_\infty^2 \rho$, with $c_\infty = const.$ being the "thermal speed" and
$$h = c_\infty^2 \ln(\rho/\rho_\infty).$$
- ⋆ polytropic flow, $p = p_\infty(\rho/\rho_\infty)^\gamma$ with
$$h = \frac{\gamma c_\infty^2}{\gamma - 1}\left(\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} - 1\right),$$
where $c_\infty = p_\infty/\rho_\infty$.

For the polytropic flow, the "thermal speed" is related to the "acoustic speed" c
$$c^2 = \frac{dp}{d\rho} = \gamma c_\infty^2 \left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1}.$$
This can be used to define the characteristic length (Bondi radius):
$$r_B = \frac{GM}{c_\infty^2}.$$

Bondi accretion

In the dimensionless setup one has

$$x = \frac{r}{r_B}, \quad \bar{u} = \frac{|u|}{c_\infty}, \quad \bar{\rho} = \frac{\rho}{\rho_\infty}, \quad \dot{m} = \frac{\dot{M}}{4\pi\rho_\infty (GM)^2/c_\infty^3}.$$

where $\dot{m}$ is the dimensionless accretion rate $\dot{m}$ (unit of mass flux $\rho_\infty c_\infty$ across $4\pi r_B^2$): The conservation and Bernoulli equations are

$$x^2\bar{\rho}\bar{u} = \dot{m}, \quad \frac{1}{2}\bar{u}^2 + H(\bar{\rho}) - \frac{1}{x} = 0.$$

For isothermal flow ($\gamma = 1$) $H(\bar{\rho}) = \ln\bar{\rho}$, otherwise $H(\bar{\rho}) = (\bar{\rho}^{\gamma-1} - 1)\gamma/(\gamma - 1)$. From the above, for isothermal flow:

$$\left(\bar{u} - \frac{1}{\bar{u}}\right) d\bar{u} = \left(\frac{2}{x} - \frac{1}{x^2}\right) dx.$$

Bondi accretion

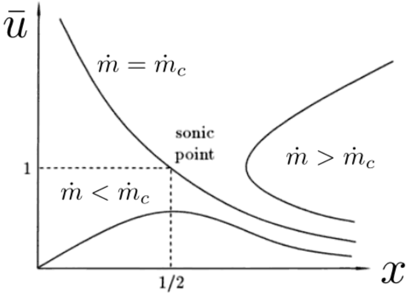
The solutions of the previous equation can be categorized according to a critical value $\dot{m}_c$:

- ⋆ a sonic transition (at the Mach number $\bar{u} = 1$) occurs at $x = 1/2$. This gives

$$\bar{\rho} = \exp(\frac{3}{2}) \quad \text{and} \quad \dot{m}_c = \frac{1}{4}\exp(\frac{3}{2}).$$

- ⋆ Flows with $\dot{m} < \dot{m}_c$ subsonic everywhere,
- ⋆ Flows with $\dot{m} = \dot{m}_c$ subsonic for $x > 1/2$, supersonic for $x < 1/2$.

→ near accreting body there may be a shock transition downstream of the shock.



from Shu (1992)

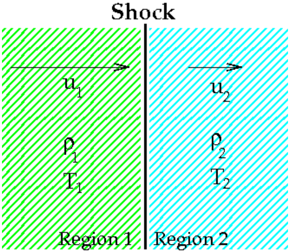
The maximum accretion rate for steady spherical isothermal flow is, from the condition $\dot{m} = \dot{m}_c$

$$\dot{M} = 4\pi\rho_\infty \dot{m}_c \frac{(GM)^2}{c_\infty^3}.$$

Rankine–Hugoniot conditions

A general 1D set of equations of motion with viscosity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0,$$
$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho u^2 + p - \frac{4}{3}\rho\nu\frac{\partial u}{\partial x}\right) = 0,$$
$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x}\left(\rho u\left(e + \frac{1}{2}u^2 + p/\rho - \frac{4}{3}\nu\frac{\partial u}{\partial x}\right)\right) = 0.$$



where e is the specific internal energy of the fluid and E is the total specific energy,

$$E = e + \frac{1}{2}u^2.$$

The system is described by the *state vector* $(\rho, u\rho, E\rho)$.

Rankine–Hugoniot conditions: steady shock

The jump conditions corresponding to the conservation of mass, momentum and energy are

$$u_s\,(\rho_2 - \rho_1) = \rho_2 u_2 - \rho_1 u_1$$

$$u_s\,(\rho_2 u_2 - \rho_1 u_1) = \left(\rho_2 u_2^2 + p_2\right) - \left(\rho_1 u_1^2 + p_1\right)$$

$$u_s\,(\rho_2 E_2 - \rho_1 E_1) = \rho_2 u_2 \left(e_2 + \frac{1}{2}u_2^2 + \frac{p_2}{\rho_2}\right) - \rho_1 u_1 \left(e_1 + \frac{1}{2}u_1^2 + \frac{p_1}{\rho_1}\right)$$

with u_s, for a polytropic equation of state ($p \propto \rho^\gamma$),

$$u_s = u_1 + \underbrace{\sqrt{\frac{\gamma p_1}{\rho_1}}}_{sound\ \ speed} \left(1 + \frac{\gamma+1}{2\gamma}\left(\frac{p_2}{p_1} - 1\right)\right)^{1/2}.$$

For a stationary shock $u_s = 0$:

$$\rho_1 u_1 = \rho_2 u_2,$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2,$$

$$\rho_1 u_1 \left(e_1 + \frac{u_1^2}{2} + \frac{p_1}{\rho_1}\right) = \rho_2 u_2 \left(e_2 + \frac{u_2^2}{2} + \frac{p_2}{\rho_2}\right).$$

Rankine–Hugoniot conditions: steady shock

Combining the conservation of mass and energy we recover the Bernoulli theorem,

$$e_1 + \frac{u_1^2}{2} + \frac{p_1}{\rho_1} = e_2 + \frac{u_2^2}{2} + \frac{p_2}{\rho_2}.$$

For a polytropic ($p \propto \rho^\gamma$) ideal gas equation of state ($p/\rho = kT/m$ written as $p = (\gamma - 1)\rho e$) the solution for density, pressures ratios and hence temperature ratios, is:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2}, \qquad \frac{p_2}{p_1} = \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{\gamma+1},$$

$$\frac{T_2}{T_1} = \frac{((\gamma-1)\mathcal{M}_1^2 + 2)(2\gamma\mathcal{M}_1^2 - (\gamma-1))}{(\gamma-1)^2\mathcal{M}_1^2},$$

where

$$\mathcal{M}_1 = \frac{u_1}{c_1} = \left(\frac{\rho_1 u_1^2}{\gamma p_1}\right)^{1/2}$$

is again the Mach number, characterizing the strength of the shock (also, ratio of the ”ram pressure” to thermal pressure in the pre-shock gas, or kinetic energy density-thermal energy density ratio).

Strong and weak shocks

Strong shock is $\mathcal{M}_1 \gg 1$, for $\gamma = 5/3$ one has

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} \approx \frac{\gamma+1}{\gamma-1} = 4,$$
$$p_2 \approx \frac{2\gamma}{\gamma+1}\mathcal{M}_1^2 p_1 = \frac{3}{4}\rho_1 u_1^2,$$
$$T_2 \approx \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} T_1 \mathcal{M}_1^2.$$

In the frame of the shock, post-shock kinetic and thermal energies are

$$\frac{1}{2}u_2^2 \approx \frac{1}{32}u_1^2, \quad \frac{3}{2}\frac{kT_2}{m} \approx \frac{9}{32}u_1^2.$$

→ half of the pre-shock kinetic energy converted to thermal energy.

A weak shock has $\mathcal{M}_1 - 1 = \epsilon \ll 1$:

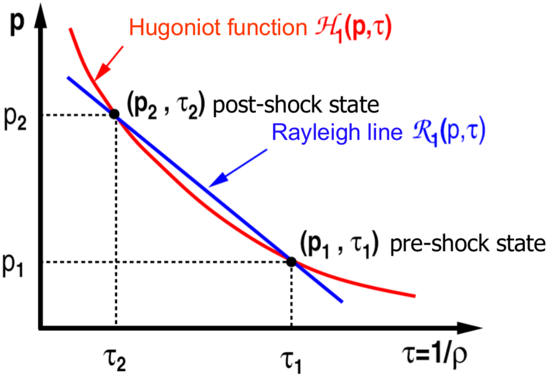
$$\frac{\rho_2}{\rho_1} = 1 + \frac{4}{\gamma+1}\epsilon = 1 + \frac{3}{2}\epsilon,$$
$$\frac{p_2}{p_1} = 1 + \frac{4\gamma}{\gamma+1}\epsilon = 1 + \frac{5}{2}\epsilon,$$
$$\frac{T_2}{T_1} = 1 + \frac{4(\gamma-1)}{\gamma+1}\epsilon = 1 + \epsilon.$$

A shock converts supersonic gas into denser, slower, higher pressure subsonic gas. It increases the specific entropy of the gas by

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - \frac{k}{m}\ln\left(\frac{p_2}{p_1}\right),$$

with $c_p = \gamma c_v$. Shock shifts gas to a higher adiabat.

Hugoniot curve and Rayleigh line



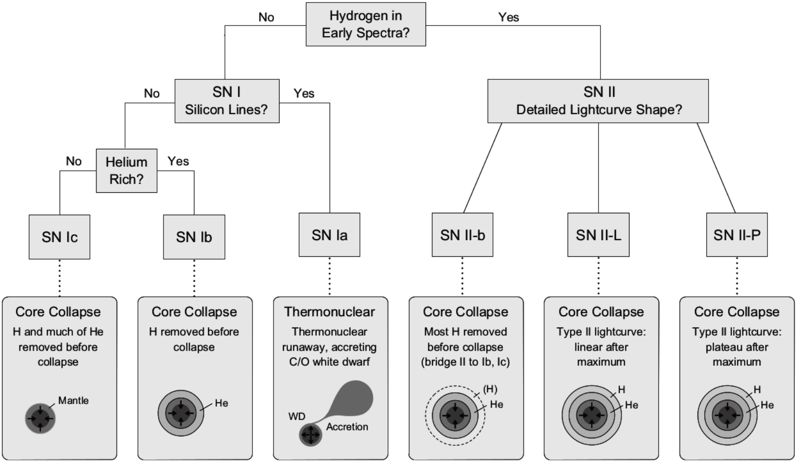
Jump conditions on the $p - 1/\rho$ plane. Rayleigh line

$$p_2 - p_1 = u_s^2 \left(\rho_1 - \frac{\rho_1^2}{\rho_2} \right)$$

has a slope proportional to F_m^2 (F_m - flux of mass over the jump region).

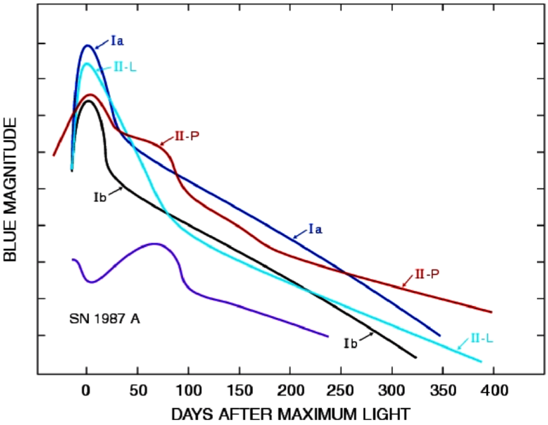
Supernovæ

Supernovæ classification: spectra



SN Ia - thermonuclear explosions, core-collapse SN - gravity bombs.

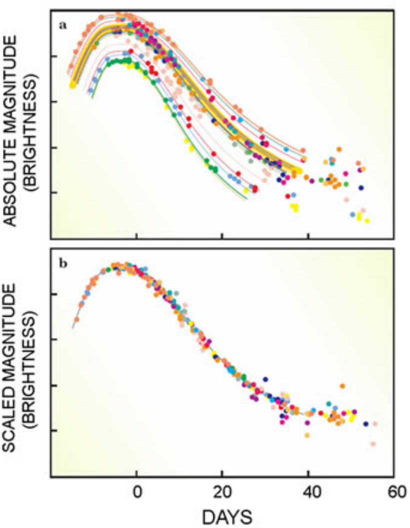
Supernovæ classification: lightcurves

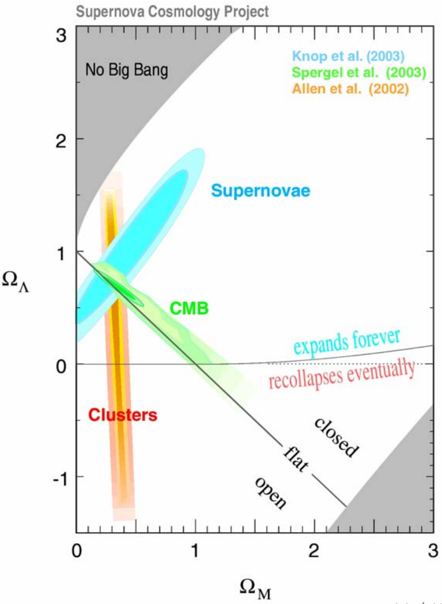


- ⋆ Ia,b,c - source of emission are photons from decaying radioactive elements: $^{56}Ni \rightarrow ^{56}Co \rightarrow ^{56}Fe$,
- ⋆ II - kinetic energy reservoir, hydrogen becomes heated and ionized, plateau due to hydrogen recombines, then radioactive decay tail,
- ⋆ Related: acceleration of cosmic rays.

SN Ia

SN Ia as standard(izable) candles



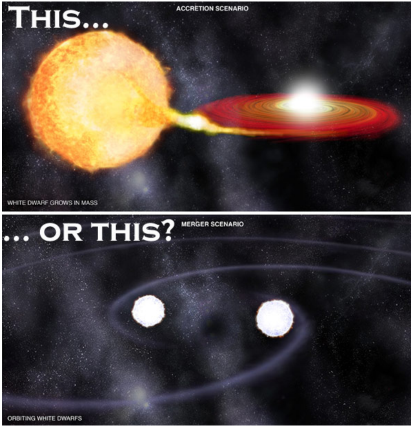


SN Ia: astrophysical model

- Energy source: thermonuclear explosion,
- kinetic energy of ejecta: 10^{51} *erg*,
- more-or-less the same brightness $M_{bol} \sim -19$,
- no H, He in spectra → exploding C+O WD star

Two main scenarios:

- Single-degenerate: accretion onto WD from companion, $\rightarrow M > M_{Ch} \rightarrow$ catastrophical collapse,
- Double-degenerate: merger of two WDs → collapse of an unstable object.



Explosion rate $\sim 10^{-2}/galaxy/yr$.

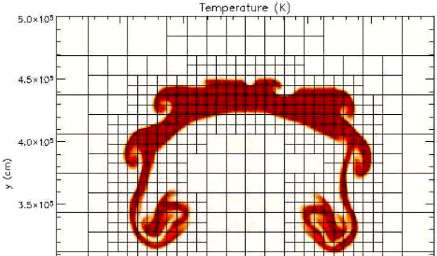
SN Ia: astrophysical model

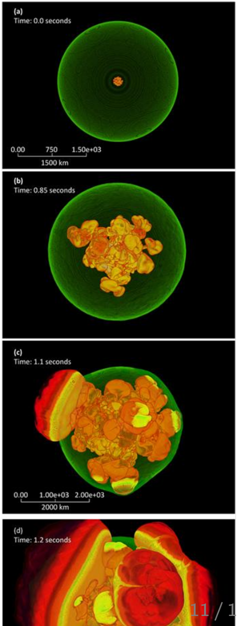
Initial condition before the collapse (lasts $\sim 100\ yr$):

- ⋆ convective carbon burning - convective energy transport,
- ⋆ cooling by plasma oscillation neutrino emission.

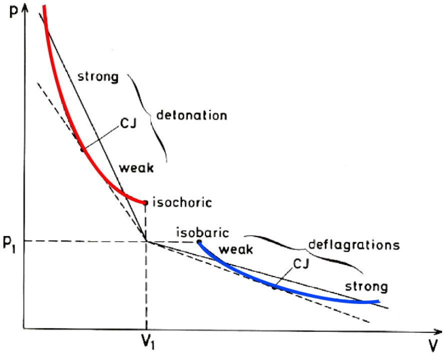
Explosion trigger:

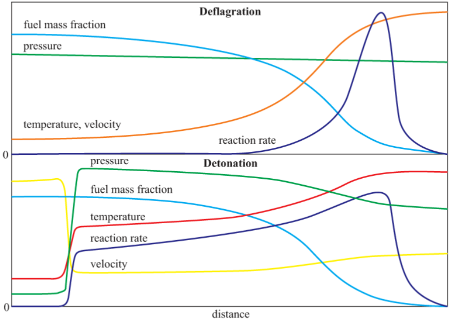
- ⋆ thermonuclear runaway near the center, in fully degenerate matter ($p = p(\rho)$) $\rightarrow$ temperature increase not cooled by expansion,
- ⋆ thermonuclear flame of 'burning' C+C (reaction rate scales as $\sim T^{20}$).





SN Ia: explosion regimes





Two branches on the Hugoniot curve:

- ⋆ detonation ($p_2 > p_2$, $\rho_2 > \rho_1$) - supersonic w.r.t fuel,
- ⋆ deflagration ($p_2 < p_2$, $\rho_2 < \rho_1$) - subsonic w.r.t fuel.

Many instabilities: R-T, K-H and those related to the turbulent combustion front (Landau-Darrieus flame cusp creation).

SN Ia: deflagration & delayed detonation

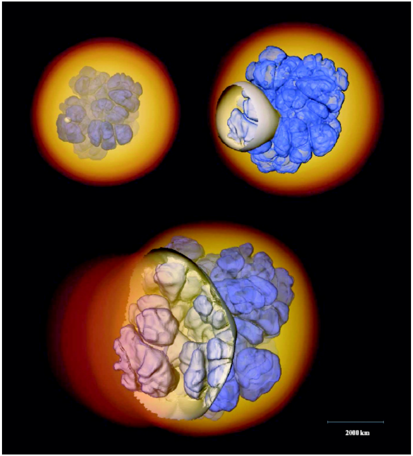
Deflagration model work quite well:

- ⋆ explodes without fine-tuning,
- ⋆ based on fundamental principles,

but has problems:

- ⋆ explosions usually not very bright,
- ⋆ outer layer composition inconsistent with bright SN Ia.

→ Delayed detonation (mechanism unknown, still an open question)



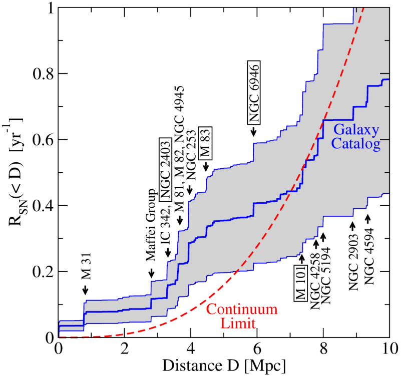
Onset of the detonation phase: 0.72 s (top left), 0.80 s (top right), and 0.90 s (Röpke and Bruckschen 2008)

Core-collapse supernovæ

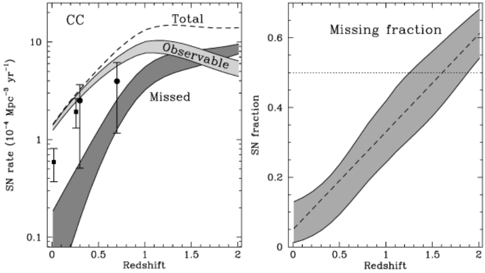
Core-collapse supernovæ rates

CCSN rate = star-formation rate times the fraction of stars in the in the proper mass range. In the Local Group of galaxies,

- ⋆ SN every 100 years, optimistically every 20 yr,
- ⋆ most local events at $< 100 kpc$.



from Ando et al. (2005)

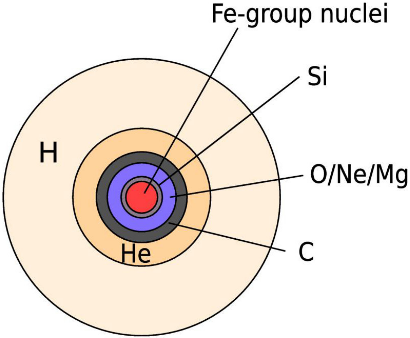


Evolution of the core-collapse supernova rate with redshift (from lrr-2011-1)

Core-collapse supernovæ progenitors

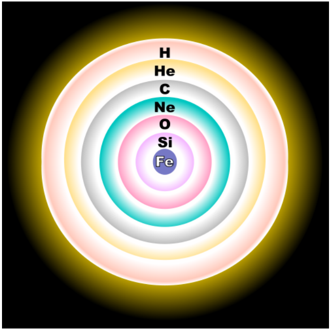
We are interested in massive stars, $7 - 10\ M_\odot \leqslant M \leqslant\sim 150 M_\odot$. Their evolution proceeds differently, according to mass, metalicity, rotation and the presence of companion:

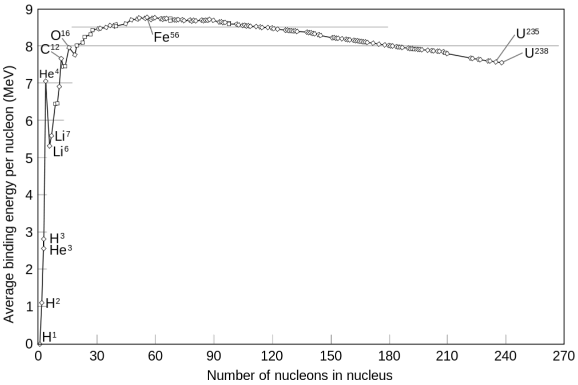
- ⋆ For masses $< 7\ M_\odot$, the core is mostly Carbon-Oxygen $\rightarrow$ envelope is ejected $\rightarrow$ CO WD,
- ⋆ slightly more massive produce O-Ne-Mg WDs,
- ⋆ $M > 10\ M_\odot$ burning to silicon and iron $\rightarrow$ instability.



CCSN red supergiant progenitor

Core-collapse supernovæ progenitors





Instability in previously stable "burning" because there is no gain in energy from combining two Fe nuclei (Fe are well bound; the boundary between fusion and fission regimes)

Onset of the collapse

The core is in hydrostatic equilibrium,

$$\frac{dp}{dr} = -\frac{GM\rho}{r^2}, \quad \text{with} \quad p = p_e + p_i + p_{rad},$$

where pressure comes from

- ⋆ degenerate and relativistic electrons:

$$p_e \simeq \frac{2\pi}{3} \frac{\mu_e^4}{c^3 h^3} \simeq 10^{28}\ dyne/cm^2,$$
$$\mu_e \simeq (\rho_7 Y_e)^{1/3}\ MeV,$$

- ⋆ ions (assuming iron for simplicity):

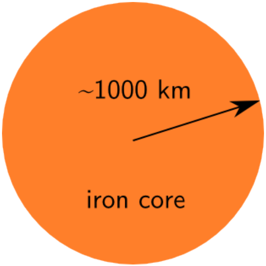
$$p_i \propto Y_{Fe} \rho k T \simeq 10^{26}\ dyne/cm^2,$$

- ⋆ radiation:

$$p_{rad} = \frac{a}{3} T^4 \simeq 10^{25}\ dyne/cm^2.$$

All these numbers assuming the iron core just before the collapse:

- ⋆ size ~ 1000 km,
- ⋆ temperature $1 MeV \simeq 10^{10}\ K$,
- ⋆ lepton fraction $Y_e \simeq 0.5$.



Onset of the collapse

The maximum mass of a star supported by degenerate relativistic electrons (Chandrasekhar mass):

$$M_{Ch} \simeq 5.8 Y_e^2 \ M_\odot .$$

General relativity and thermal corrections are of secondary importance; for $Y_e = 0.5$ $M_{Ch} \simeq 1.45$ $M_\odot$. How to destabilize the core?

- ⋆ gain mass by burning Si layers, so $M > M_{Ch} \rightarrow$ it becomes unstable w.r.t radial perturbations,
- ⋆ reduce $Y_e \rightarrow$ electron capture.

In the simplest case (electron capture on free protons):

$$e^- + p \rightarrow n + \nu_e, \quad \text{works when} \quad \mu_e > \mu_n - \mu_p \quad (\mu_\nu = 0).$$

For $T = 0$ with non-degenerate neutrons $\mu_e = m_n - m_p \simeq 1.2$ MeV. In realistic core-collapse, capture at $\mu_e \simeq 10$ MeV (neutrinos taking excess energy).

Onset of the collapse: photodissociation & capture rates

Electron capture rate is

$$\frac{\partial}{\partial t} Y_e \propto \mu_e^5 \propto \rho^{5/3}.$$

(Bethe et al., 1979). In realistic situation,

$$e^- + (Z, A) \rightarrow \nu_e + (Z-1, A),$$

nuclei instead of free protons (blockage due to neutrons filling shells, Pauli blocking of lower-energy states). In addition, pressure support is reduced by *photodissociation* of nuclei ($125 MeV$/nucleon):

$$\gamma +^{56}_{26} Fe \leftrightarrow 13\alpha + 4n.$$

As a result, ν escape freely for densities $< 10^{12}\ g/cm^3$.

- rapid deleptonization of the core,
- entropy changes are small - the collapse is almost adiabatic.

Collapse: neutrino trapping

The density is growing, and for $\rho > 3 \times 10^{12}\ g/cm^3$ neutrino diffusion time

$$\tau_{diff} \gg \tau_{col}.$$

is much larger than the timescale for collisions due to scattering on nuclei:

$$\nu + (Z, A) \leftrightarrow \nu + (Z, A), \quad \text{the mean-free path being}$$

$$\lambda_\nu \simeq 10^7 \left(\frac{10^{12}\ g/cm^3}{\rho}\right)\left(\frac{10\ MeV}{\epsilon_\nu^2}\right)\frac{A}{N^2},$$

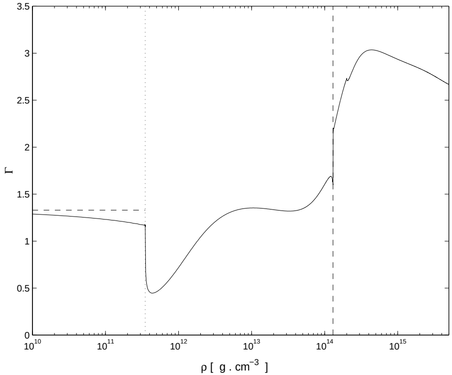
neutrinos are trapped in the collapsing core:

- ⋆ Deleptonization stops: $Y_l = Y_e + Y_\nu \simeq 0.32$,
- ⋆ $\beta-$equilibrium is reintroduced:

$$e^- + p \leftrightarrow n + \nu_e$$

$$\mu_e + \mu_p = \mu_n + \mu_\nu.$$

Nuclear EOS for $T = 0$



Stiffness of the equation of state measured by
$\Gamma = \mathrm{d}\ln P/\mathrm{d}\ln n_{\mathrm{b}} = (n+1)/n$

pressure $P = \kappa n_b^\Gamma$,

baryon density n_b,

energy density
$\mathcal{E} = P/(\Gamma - 1) + n_{\mathrm{b}} m_b c^2$,

...from the first law of thermodynamics,
$d\left(\frac{\mathcal{E}}{n_b}\right) = -Pd\left(\frac{1}{n_b}\right) + Tds$

Collapse: core bounce, nuclear EOS for $T \neq 0$

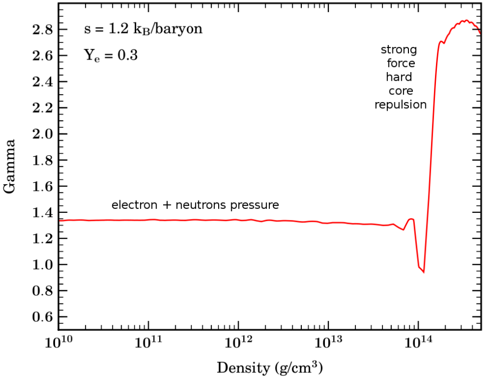
Equation of state for $T \neq 0$ is $P = P(\mathcal{E}, T, Y_e)$ - nuclear statistical equilibrium.

- ⋆ Strong forces make the EOS relation stiff at $\simeq 2 \times 10^{14}\ g/cm^3$ (nuclear saturation density):

$$\rho_{nuc} = \frac{4Am_b}{4\pi R_{nuc}^3}$$

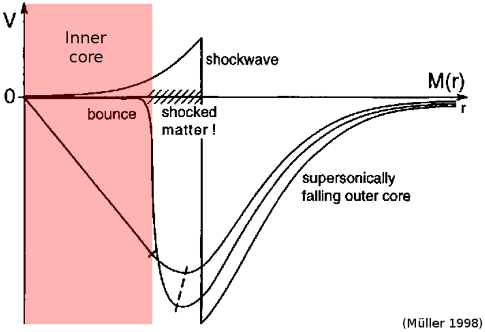
where $R_{nuc} = A^{1/3} r_0$, $r_0 = 1.25\ fm$.

Above ρ_{nuc} nuclei are so close to each other, that the repulsion "hard core" makes the EOS stiff (wiggle is the transition region).



Collapse: bounce and shock wave

- ⋆ EOS stiffens due to nuclear interactions,
- ⋆ sound wave is created → propagates through the inner core,
- ⋆ steepening of waves at the sonic point → shock!



Mass of the inner core, $M_{ic} \propto Y_l^2$, determined by nuclear physics and weak interactions (also GR, rotation and thermal corrections). Quite universally,

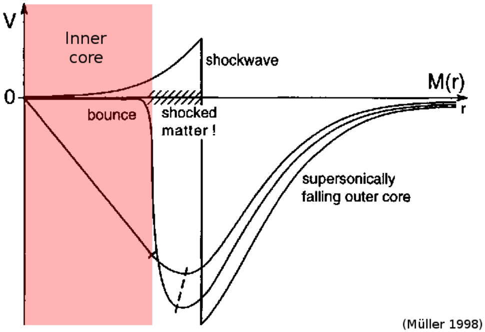
$$M_{ic} \simeq 0.5\ M_\odot,$$

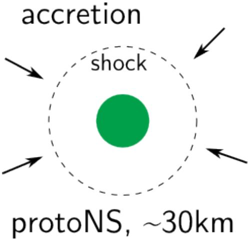
and *quite independent* of the progenitor details.

Collapse: bounce and shock wave

From the point of view the dynamics of collapse, the parameters of the inner core are of importance:

- ⋆ M_{ic} sets the kinetic energy transfered to the shock,
- ⋆ R_{ic} and M_{ic} define the angular momentum reservoir,
- ⋆ no problem from the point of view of nuclear physics - well known nuclear forces stabilize the core,
- ⋆ the collapse doesn't lead directly to black hole formation (no "prompt" collapse),
- ⋆ $M_{Ch} - M_{ic}$ is the amount of material that falls back in later stages of explosion (fallback).





Collapse: supernova energetics

- ⋆ Binding energy of the neutron star of mass M and radius R:

$$E_{grav} \sim \frac{GM^2}{R} \simeq 3 \times 10^{53}\ erg.$$

- ⋆ initial shock energy ($v \sim 0.05\ c$):

$$E_{sh} = \frac{1}{2} M_{ic} v^2 \simeq 1.2 \times 10^{51}\ erg.$$

- ⋆ Shock "stalling": energy lost on dissociation (after accretion of $\sim 0.1\ M_\odot$):

$$E_{diss} \sim 10(M/M_\odot) \times 10^{51}\ erg.$$

and neutrino loses:

$$E_\nu \sim 10^{53}\ erg/s$$

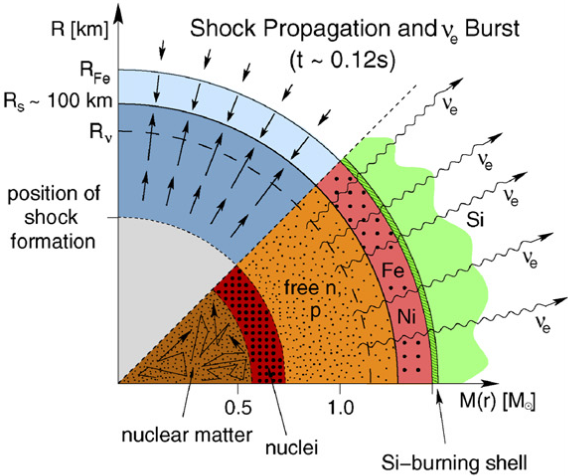
- ⋆ Binding energy of the progenitor, for 10 $M_\odot$ star it is $E_{prog,bind} \simeq 3 \times 10^{51}\ erg$

Collapse: shock stall

The energy of the bounce is not sufficient and the shock is stalled. The energy loses are:

- ⋆ Dissociation of in-falling nuclei, $\sim 9\ MeV/\text{nucleon}$,
- ⋆ neutrino flux from behind the shock and as the protoneutron star (PNS) cools (99% of energy)

Supernova energy - kinetic and internal energy of the ejecta is $\sim 10^{51}\ erg$.



→ there must be a mechanism to convert a part of the NS gravitational energy into explosion

Shock revival: neutrino heating

In other words ram pressure larger than the pressure after the shock, $p_{ram} > p_{shock}$. Additional energy deposit behind the shock is needed. Failed ideas:

- ⋆ Radiation pressure,
- ⋆ Nuclear burning (e.g., $O \rightarrow Fe$),
- ⋆ Changing the nuclear equation of state → bigger bounce,
- ⋆ Progenitor's models with steeper density profiles.

A possible delayed mechanism may be provided by neutrinos:

- ⋆ cooling rate: $Q_\nu^- \simeq 10^{20}\ (T/2MeV)^6\ erg/s/g$, dominated by the URCA processes,

$$p + e^- \rightarrow \nu_e + n, \quad n + e^+ \rightarrow \bar{\nu}_e + p,$$

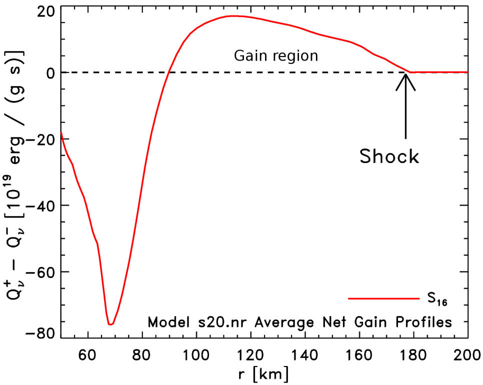
- ⋆ heating rate: $Q_\nu^- \simeq 10^{20} L_\nu (100\ km/r)^2 (T_\nu/4MeV)^2\ erg/s/g$, ($T_\nu$ does not depend on r) dominated by the inverse URCA processes,

$$\nu_e + n \rightarrow p + e^-, \quad \bar{\nu}_e + p \rightarrow n + e^+.$$

Shock revival: neutrino heating

Since cooling falls off as T^6, faster than heating at r^{-2} there is a gain region of positive $Q_\nu^+ - Q_\nu^-$ (gain region)

- ⋆ net heating adds $\sim 10^{20}\ erg/s/g$ → matter gains sufficient energy to revive the shock in 100 ms,
- ⋆ revival by pdV work of expanding matter.



(Ott et al., 2008)

Additional problems: increase the time for which matter is heated up, increase L_ν (convection, asymmetry?), effects of GR and multi-dimensional (new instabilities).

Neutrino transport

Proper transport solution - Boltzmann equations - is computationally very expensive (6+1D: position, momenta and energy).

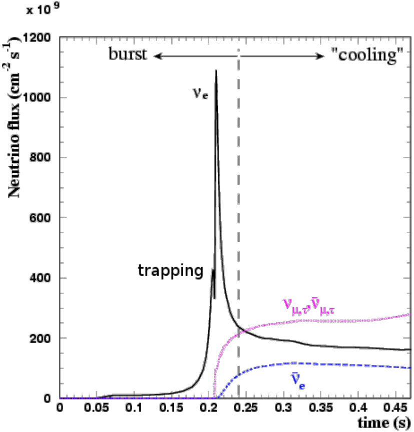
The notion of neutrino energy-depended ($\propto T_\nu^2$) neutrinosphere,

$$R_\nu = R(\tau_\nu = 2/3), \quad \tau_\nu = \int_r^\infty \frac{dr}{\lambda}.$$

Limiting cases:

- ⋆ Diffusion (isotropy, $\lambda \ll R$),
- ⋆ Free-streaming (radial outflow, $\lambda \gg R$).

Something in-between: leakage scheme (neutrino trapped above some density).



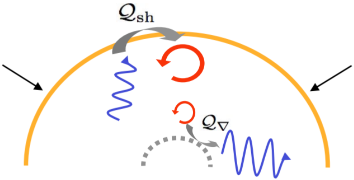
Neutrino luminosity in time (from Gil-Botella and Rubbia 2003)

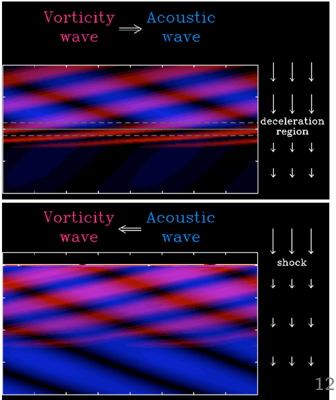
Standing Accretion Shock Instability

SASI is a low mode ($l = 1$) hydrodynamical instability Blondin et al. 2003, Scheck et al. 2004, Ohnishi et al. 2006, Burrows et al. 2006) that leads to

- ⋆ development of a spiral mode ($m = 1$),
- ⋆ accretion and outflows at the same time,
- ⋆ large scale convection (in nature, possibly together with neutrino-driven convection),
- ⋆ big asymmetry → pulsar kicks,
- ⋆ *slows down* the initial PNS spin.

Reason for the instability - advective-acoustic cycle: coupling between acoustic and advected perturbations through the flow gradients.





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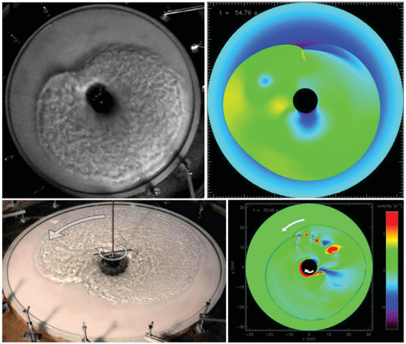


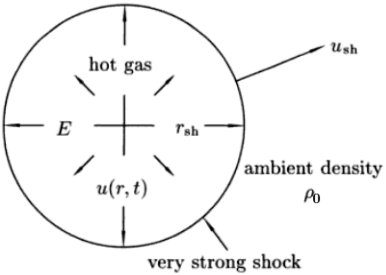
Shallow water simulations of SASI sloshing mode (Foglizzo et al.)

Blast wave from supernova explosion

Supernova releases energy E from a point source into exterior homogeneous gas of density ρ_0

- ⋆ spherical shock wave of speed u_{sh},
- ⋆ Assume energy conservation (radiative loses $\ll E$),
- ⋆ Ram pressure $\rho u_{sh}^2 \gg p_0 \simeq 0$.

What is the evolution of the shock front in time? From dimensional analysis,



$$r_{sh} = E^\alpha \rho_0^\beta t^\delta \quad \rightarrow \quad [r_{sh}] = L = (ML^2T^{-2})^\alpha (ML^{-3})^\beta T^\delta,$$

which gives $\alpha = 1/5$, $\beta = -1/5$, $\delta = 2/5$ and $r_{sh} = A\left(Et^2/\rho_0\right)^{1/5}$,

($A = const. \simeq 1.17$ for $\gamma = 5/3$)

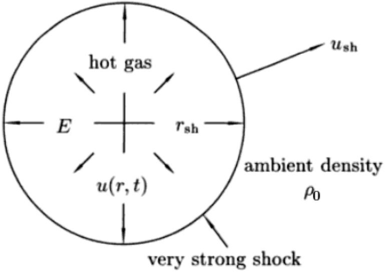
Blast wave from supernova explosion

The velocity of the shock is then

$$u_{sh} = \frac{dr_{sh}}{dt} = \frac{2}{5} A \left(E/t^3 \rho_0\right)^{1/5}.$$

In can be used to estimate the size of the supernova remnant. For

- ⋆ $\rho_0 = 2 \times 10^{-24}\ g/cm^3$ (10^6 atoms/m^3),
- ⋆ $E = 10^{51}\ erg$,
- ⋆ $t = 1\ yr$: $r_{sh} = 0.3\ pc$, $u_{sh} = 1.3 \times 10^5\ km/s$,
- ⋆ $t = 100\ yr$: $r_{sh} = 2\ pc$, $u_{sh} = 8 \times 10^3\ km/s$,
- ⋆ $t = 10000\ yr$: $r_{sh} = 13\ pc$, $u_{sh} = 5 \times 10^2\ km/s$



$T = 10^{10}(E/\rho_0)r_{sh}^{-3}\ K$ For $t \simeq 10^4\ yr$, $T \sim 10^6\ K \rightarrow$ X-ray emission.

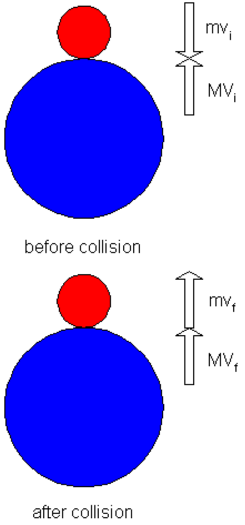
This self-similar blast wave solution is called the Taylor-Sedov solution.

Toy-model shock: two balls illustration

Consider two balls of masses M and m, dropped from an initial height h (floor and balls elastic). Just as M touches the floor:

- ⋆ downward velocities of M and m (same acceleration): $v_i = -\sqrt{2gh}$,
- ⋆ M bounces from the floor with $V_i = \sqrt{2gh}$,
- ⋆ M and m collide. Using the
- ⋆ conservation of momentum: $MV_i + mv_i = MV_f + mv_f$,
- ⋆ conservation of energy: $\frac{1}{2}MV_i^2 + \frac{1}{2}mv_i^2 = \frac{1}{2}MV_f^2 + \frac{1}{2}mv_f^2$
- ⋆ For $m < 3M$, v_f is upwards and the rebound height of m is

$$h_f = v_f^2/2g = h\left(\frac{3M - m}{M + m}\right)^2 \underset{M \gg m}{\longrightarrow} 9h$$



Further reading...

- *"An introduction to astrophysical fluid dynamics"*,
  Michael J. Thompson,
- *"Physics of Astrophysics, Vol. II: Gas Dynamics"*,
  Frank H. Shu
- *"Thermonuclear supernovae: a multi-scale astrophysical problem challenging numerical simulations and visualization"*,
  F. K. Röpke and R. Bruckschen (2008),
- *"Explosion Mechanisms of Core-Collapse Supernovae"*,
  H-T. Janka (2012).