Relativistic stars

Michał Bejger
N. Copernicus Center, Warsaw

Outline

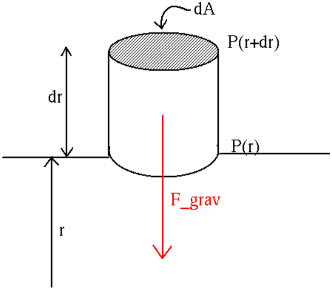
- ⋆ Newtonian hydrostatic equilibrium,
- ⋆ HR diagram, stars of different masses,
- ⋆ White dwarfs: electron degeneracy, maximum mass,
- ⋆ Neutron stars: pulsars - rotating compact stars,
- ⋆ Equation of state and structure, neutron drip, deep interior
- ⋆ NS structure from TOV, constant density star, mass limit, NS vs WD maximum mass,
- ⋆ Current affairs: 2 $M_{\odot}$ observations,
- ⋆ Spectral methods for solving PDEs.

Hydrostatic equilibrium of stars

Equilibrium conditions from simple considerations

A cylinder of

- ⋆ density $\rho(r)$,
- ⋆ volume $V = dAdr$,
- ⋆ mass $dm = \rho V$,



placed in gravitational field of a mass $M(r)$.

Forces acting on the cylinder:

- ⋆ Gravity:

$$F_{grav} = -\frac{GM(r)dm}{r^2} = -\frac{GM}{r^2}\rho drdA.$$

- ⋆ Pressure P:

$$F_{press} = (P(r+dr) - P(r))dA = dPdA.$$

In equilibrium, $F_{press} = F_{graV}$,

$$dPdA = -\frac{GM}{r^2}\rho drdA$$

that is

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

(+ equation of state $P(\rho, T\ldots)$)

Equilibrium conditions from simple considerations

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

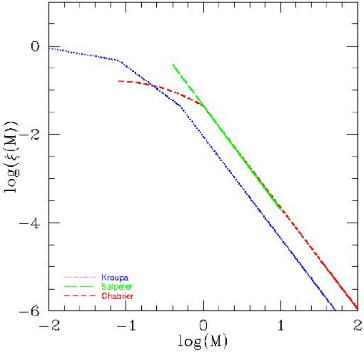
We could guess the above from the Euler/Navier-Stokes equation i.e., the *momentum conservation: the rate of change of total fluid momentum in some volume equals to the sum of forces acting on the volume.*

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla \cdot \mathbf{u} = -\nabla P + \rho \mathbf{f}$$

Which stars are relativistic?

Initial mass function of stars

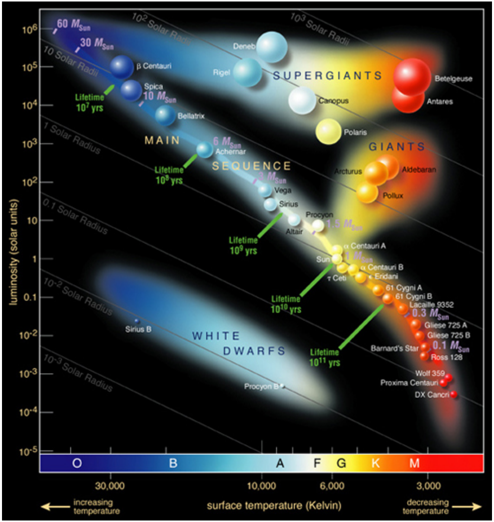
Number of stars within mass range $(m, m + dm)$ proportional to $m^{-\alpha}$:



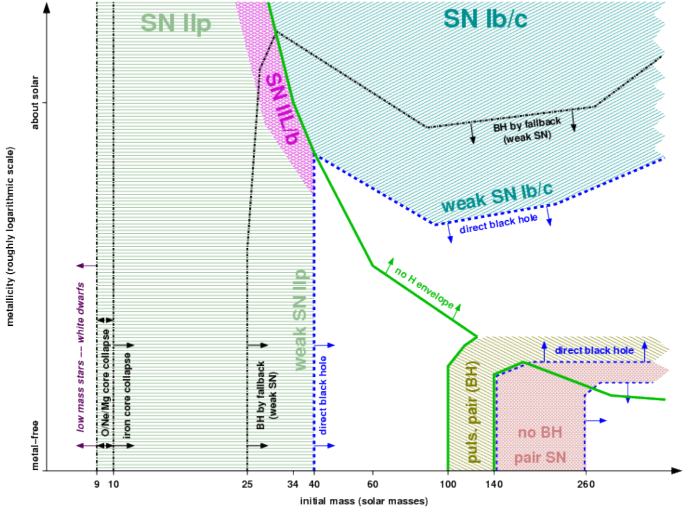
E.g., Salpeter (1955) IMF:

$$\xi(m)\Delta m = \xi_0 \left(\frac{m}{M_{\rm sun}}\right)^{-2.35} \left(\frac{\Delta m}{M_{\rm sun}}\right)$$

Hertzsprung–Russell diagram:



Life of stars with different initial masses



(massive enough stars produce neutron stars and black holes at the end of their lifes)

White dwarfs

Dim and hot stars, e.g., Sirius B,

- ⋆ mass 1 $M_{\odot}$,
- ⋆ luminosity 0.03 $L_{\odot}$,
- ⋆ temperature 25000 K.

Power emitted, from Stefan-Boltzmann law ($\propto \sigma T^4$) estimates the radius, $\simeq R_{\oplus} \rightarrow \rho \sim 3 \times 10^6 \ g/cm^3$

From the hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{4}{3}\pi G \rho^2 r$$

$$\int_{P_c}^{0} dP = -\int_0^R \frac{4}{3}\pi G \rho^2 r dr = \frac{4}{3}\frac{R^2}{2}\pi G \rho^2$$

$$\rightarrow P_c = \frac{2}{3}\pi G \rho^2 R^2 \simeq 10^{23} \ dyne/cm^2$$

Such high pressure cannot come from thermal movement of particles, it is an effect of electron degeneracy.

- ⋆ Pauli exclusion principle,
- ⋆ Heinsenberg principle $\Delta p \Delta x \geqslant \hbar/2$

For average density n_e, the separation $\Delta x \simeq n_e^{-1/3}$, and momentum $p \sim \Delta p \simeq \hbar n_e^{1/3}$.

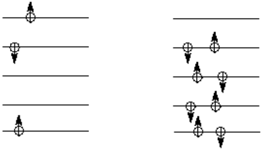
Pressure (of non-relativistic electrons):

$$P \sim n_e p v \sim n_e \frac{p^2}{m_e} \sim n_e^{5/3}$$

Relativitic electrons:

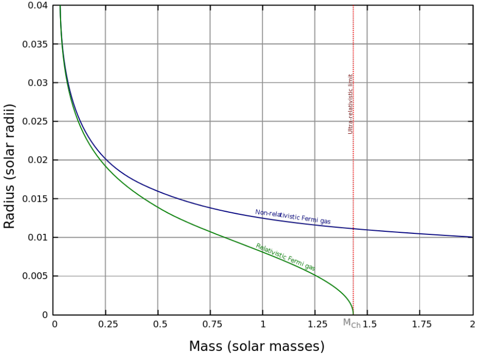
$$P \sim n_e p c \sim n_e^{4/3}$$

White dwarfs



Regular gas: many unfilled energy levels. Particles free to move about and change energy levels.

Degenerate gas: all lower energy levels filled with two particles each (opposite spins).



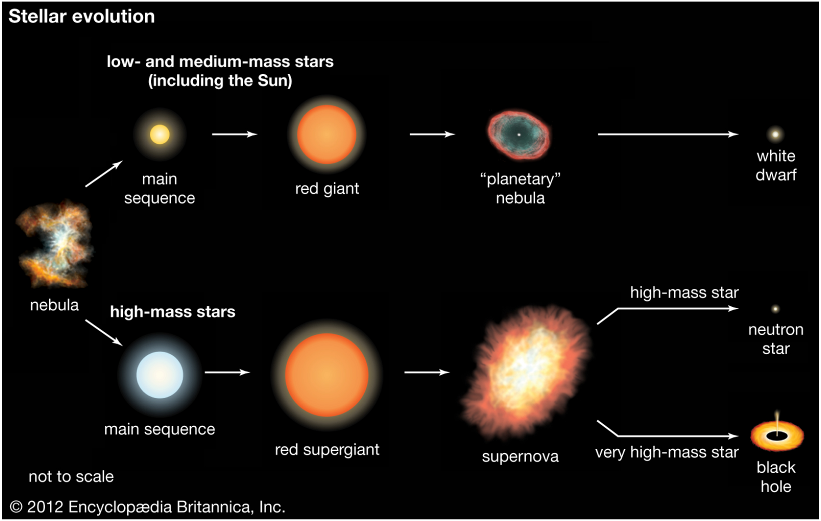
Non-relativistic electrons, $P_c \simeq P_e$:

$$P \sim \rho^2 R^2 \sim \rho^{5/3} \quad \rightarrow \quad R^2 \sim \rho^{-1/3} \sim \frac{R}{M^{1/3}} \rightarrow R \propto M^{-1/3}$$

Relativistic electrons, $P_c \simeq P_{e,rel}$

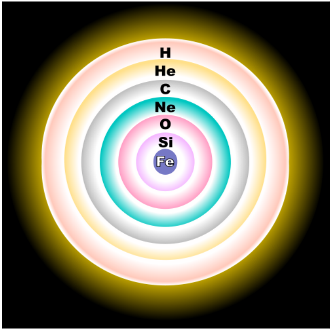
$$P \sim \rho^{4/3} \quad \rightarrow \quad R^2 \sim \rho^{-2/3} \sim \frac{R^2}{M^{2/3}} \rightarrow M \propto \text{const.} \text{ (the Chandrasekhar mass)}$$

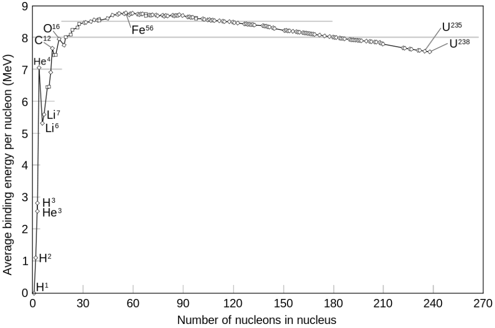
Relativistic stars



Core-collapse supernova

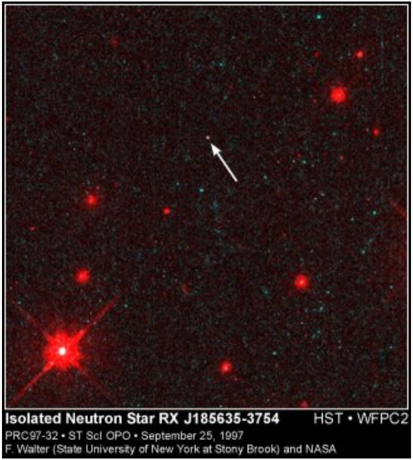
Star with $M > 8 - 10\ M_\odot$ on ZAMS produce interesting objects, NSs and BHs.





Supernova explodes because there is no gain in energy from combining two Fe nuclei (Fe are well bound; the boundary between fusion and fission regimes)

Neutron stars: orders of magnitude

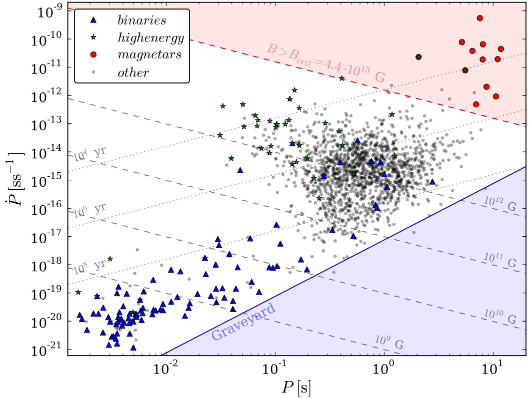


- ⋆ mass $1 - 2M_\odot$,
- ⋆ $N \simeq 10^{57}$ baryons,
- ⋆ radius $\simeq 10$ km,
- ⋆ mean density $\sim 10^{14}$ g/cm^3,
- ⋆ magnetic field 10^8 G $< B <$ 10^{15} G,
- ⋆ rotation $\sim 1000/s$,
- ⋆ compactness $r_g/R \simeq 0.25$ ($r_g = 2GM/c^2$),
- ⋆ Pressure by degenerate nucleons (mostly neutrons)!

There are stars that are dense and compact ($M/R \lesssim 1$), effects of their gravity on spacetime not-negligible.

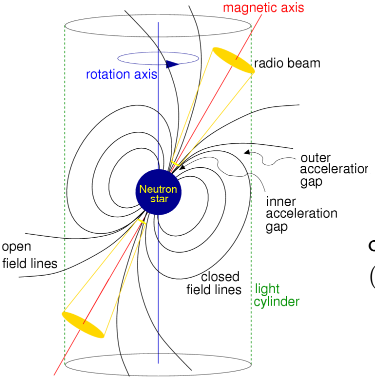
Pulsar = a magnetized, rotating neutron star. First approximation: rotating, radiating EM dipole. From observed P and $\dot{P}$, estimates of the magnetic field B and characteristic age τ:

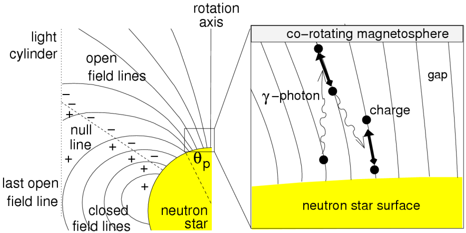
$$B > \left(\frac{3c^3 I}{8\pi^2 R^6}\right)^{1/2} \sqrt{P\dot{P}}, \qquad \tau = P/(2\dot{P})$$



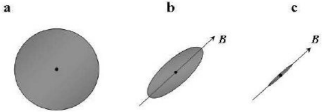
Series of consecutive pulses of PSR B1919+21 ($P = 1.3373$ s)

Pulsar „lighthouse” model



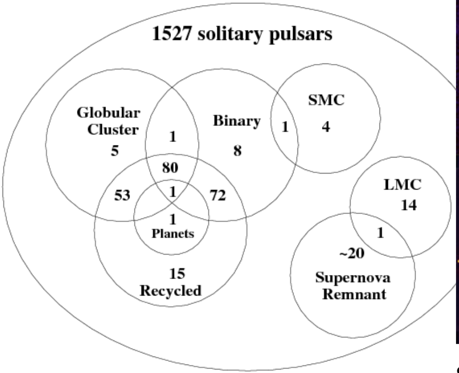


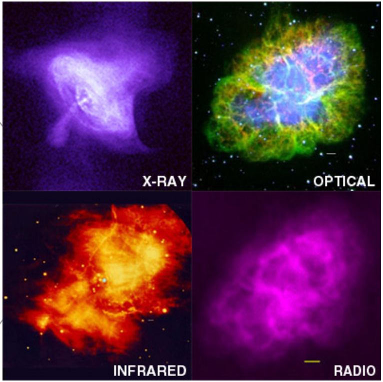
Surface effects: in strong field, electrons/ions occupy the *Landau orbitals*, characteristic scale $(\hbar c/eB)^{1/2}$



Hydrogen atom: (a) $B \ll 10^{9}$ G, (b) $B \sim 10^{10}$ G, (c) $B \sim 10^{12}$ G

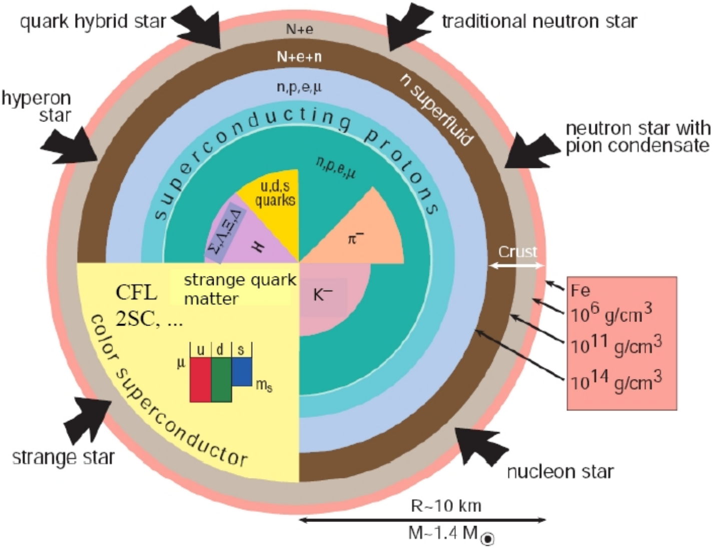
NS population





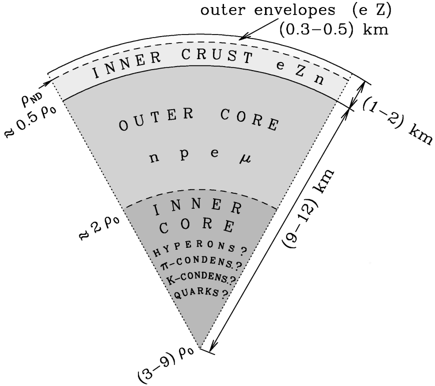
Crab nebula, M1 (supernova of 1054CE)

The interior



(by F. Weber)

Neutron star structure



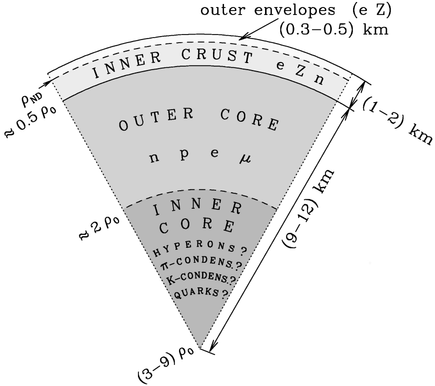
Outer layers

- ⋆ **Atmosphere**: Thickness $\simeq$ *mm* for $T \simeq 10^5~K$, $\simeq$ *cm* for $T > 10^6~K$
- ⋆ **Outer crust**: Thickness $\simeq 100~m$, pressure due to strongly degenerated electrons, non-relativistic for $\leqslant 10^8~\mathrm{g/cm^3}$ ($\gamma \simeq 5/3$), ultra-relativistic for $> 10^8~\mathrm{g/cm^3}$ ($\gamma \simeq 4/3$),

  Atomic nuclei are becoming neutron-rich with density, neutron drip point $4.3 \times 10^{11}~\mathrm{g/cm^3}$,

  Total mass $\simeq 10^{-5}~M_\odot$

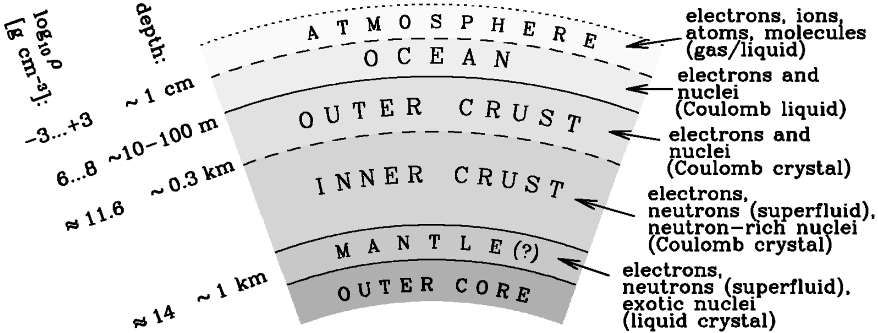
Neutron star structure



Inner layers

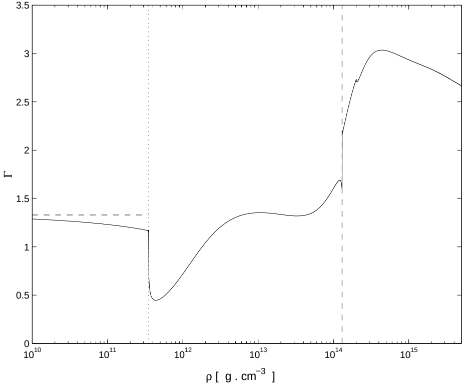
- **Inner crust**: Free neutron gas, electrons + neutron-rich nuclei with possibly non-spherical shapes. Near $\sim 10^{14}$ g/cm^3 strong interactions stiffen the matter. Neutrons superfluid. Mass $1-2\%$ $M_\odot$,
- **Outer core**: $\rho > 10^{14}$ g/cm^3 nuclei 'dissolve', all constituents are strongly degenerated, nucleons superfluid (protons in addition superconductive),
- **Inner core**: $\rho > \rho_{nuc} = 2.8 \times 10^{14}$ g/cm^3, possible new states of matter, new phases (de-confined quark matter, strange baryons, condensates, ???...)

Structure of the crust



Neutron star crust structure ($T \sim 10^8\ K$).

Stiff vs soft: adiabatic index $\Gamma = (n+1)/n$



$\Gamma = \mathrm{d}\ln P/\mathrm{d}\ln n_{\mathrm{b}} = (n+1)/n$

pressure $P = \kappa n_b^{\Gamma}$,

baryon density n_b,

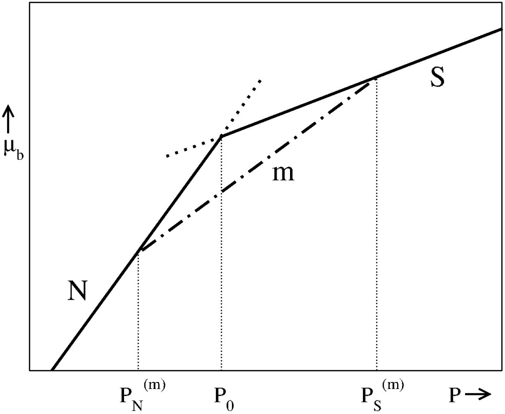
energy density $\mathcal{E} = P/(\Gamma - 1) + n_{\mathrm{b}} m_b c^2$,

...from the first law of thermodynamics,

$$d\left(\frac{\mathcal{E}}{n_b}\right) = -Pd\left(\frac{1}{n_b}\right) + Tds$$

Γ measures the "stiffness" of the equation of state

Cold "catalized" matter

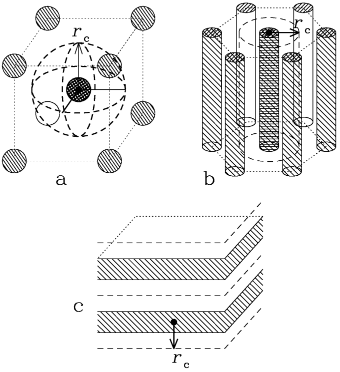


Minimising the chemical potential $\mu_b(P) = \partial\mathcal{E}/\partial n_b$, at a given pressure P, with respect to independent other variables.

This is the ground state of matter at P: **cold & catalized**

"Funny phases"

While looking for the minimum of energy, the shape of nuclei has to be treated as a thermodynamical variable. It corresponds to $\mathcal{E}$ at given n_{b} (possible occurrence: near the crust-core interface, $1-2\times10^{14}\ \mathrm{g/cm^3}$)

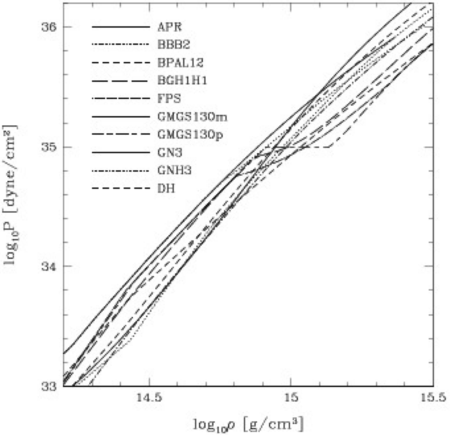


Shaded areas: nuclear matter, white: free neutron gas

In jargon, "pasta phases": cylinders - spaghetti, plates - lasagna, bubbles- Swiss cheese...

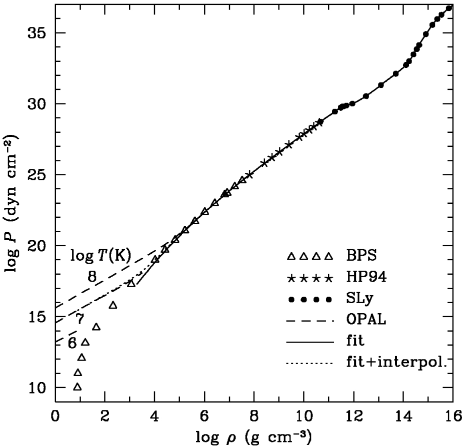
Another possiblity of pasta phases: deep core.

How to obtain the dense matter equation of state



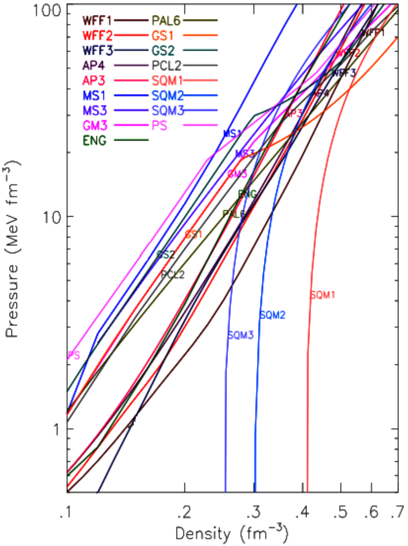
- ⋆ **Brueckner-Bethe-Goldstone theory, Green functions theory:**
  Perturbative approach:
  $\hat{H} = \hat{H}_{kin} + \hat{H}_{int} = \hat{H}_0 + \hat{H}_1$
- ⋆ **Variational methods:**
  Minimisation of the expectation value of the system Hamiltonian in the trial functions space,
- ⋆ **Relativistic mean field theory:**
  Interaction between nucleons described by fields, coupling of scalar and vector fields (representing bosons carrying interactions).
- ⋆ **Efective energy density functionals:**
  Minimisation of the energy density w.r.t. one-particle number density.

Exemplary equation of state: crust + core

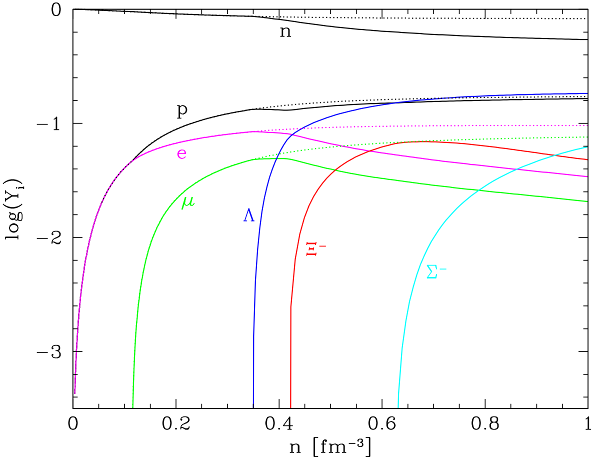


- ⋆ Ground state of matter for the outer part and the atmosphere, Fe body-centered-cubic crystal, at $P = 0$, density $\rho = 7.86\ \mathrm{g/cm^3}$.
- ⋆ Surface temperature for adult neutron stars $\sim 10^6\ K$, for young ones ($<$ one year) $> 10^7\ K$, $\sim 10^{12}\ K$ at birth.

The dense matter equation of state



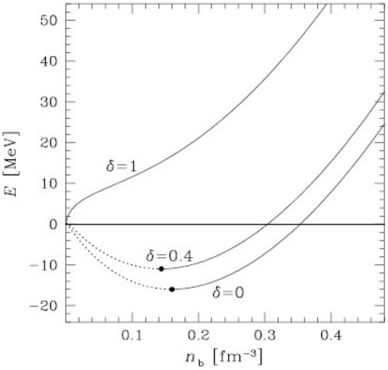
The dense matter equation of state: sample composition



Relativistic mean field model with hyperons (BM165)

Properties of nuclear matter: liquid drop model

Assymetry: $\delta = (n_n - n_p)/n_\mathrm{b}$



Energy per nucleon:

$$E(n_\mathrm{b}, \delta) \simeq E_0 + S_0\delta^2 + \frac{K_0}{9}\left(\frac{n_\mathrm{b}-n_0}{n_0}\right)^2$$

Binding energy at the saturation density:

$B_0 = -E_0$

Symmetry energy:

$S_0 = \frac{1}{2}\left(\frac{\partial^2 E}{\partial \delta^2}\right)_{n_\mathrm{b}=n_0, \delta=0}$

Compresibility:

$K_0 = 9\left(n_\mathrm{b}^2 \frac{\partial^2 E}{\partial n_\mathrm{b}^2}\right)_{n_\mathrm{b}=n_0, \delta=0}$

Experimentally:

$n_0 = 0.16 \pm 0.01\ \mathrm{fm}^{-3}$
$B_0 = 16.0 \pm 1.0\ \mathrm{MeV}$
$S_0 = 32 \pm 6\ \mathrm{MeV}$
$K_0 \simeq 230\ \mathrm{MeV}$

Approximate evaluation of the neutron drip point

Let us neglect Coulomb and surface effects etc., the energy per nucleon in a nucleus, without the rest mass part ($\delta = (N - Z)/A$):
$E_{\mathcal{N}}(A, Z)/A \simeq E_0 + S_0\,\delta^2$.
Nucleon chemical potentials:

$$\mu'_n = \partial E_{\mathcal{N}}/\partial N = E_0 + (2\delta + \delta^2)\ S_0\ , \qquad \mu'_p = \partial E_{\mathcal{N}}/\partial Z = E_0 + (-2\delta + \delta^2)\ S_0\ .$$

δ value corresponding to the neutron drip density $\rho_{\rm ND}$ can be obtained from $\mu'_n = 0$:

$$\delta_{\rm ND} = \sqrt{1 - (E_0/S_0)} - 1.$$

For $E_0 = -16\ {\rm MeV}$ i $S_0 = 32\ {\rm MeV} \to \delta_{\rm ND} = 0.225$. On the other hand, from β equilibrium: $\mu_n = \mu_p + \mu_e$:

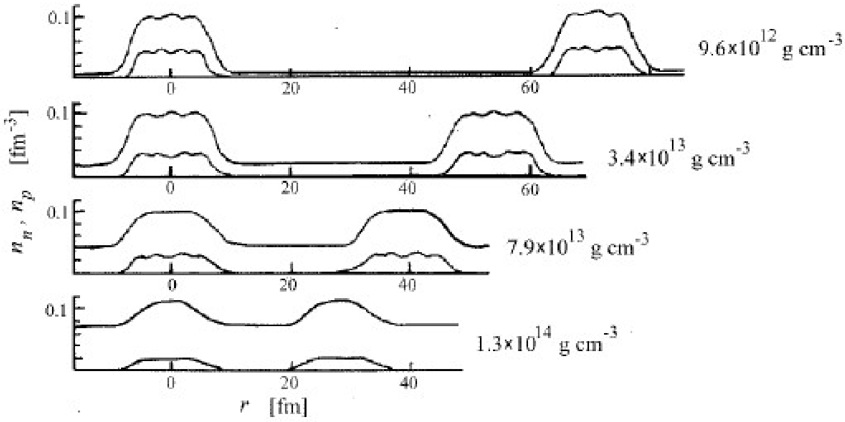
$$\mu_e = \mu_n - \mu_p \simeq 4S_0\,\delta.$$

Electron chemical potential equals $\mu_e = 0.516\ (\rho_6 Z/A)^{1/3}\ {\rm MeV}$, and we get

$$\rho_{\rm ND} \simeq 2.2 \times 10^{11}\ {\rm g/cm^3},$$

which is actually quite close to the true value ($\rho_{\rm ND} = 4.3 \times 10^{11}\ {\rm g/cm^3}$)...

Density profiles above the neutron drip point



Quark matter

- Asymtotically for large densities, quarks are not bound in hadrons, but constitute weakly-interacting Fermi gas,
- "Deconfinement" of quarks: predicted, but not really well-described by current theories.

Simplest MIT "bag" model:

- massless and non-interacting quarks in a bag of QCD vacuum,
- For u, d and s quarks in equillibrium w.r.t. weak and electromagnetic interactions: $n_e = 0$, $n_u = n_d = n_s = n_b$,
- Energy density: $\mathcal{E} = \rho c^2 = b n_b^{4/3} + \mathcal{B}$,
- Pressure: $P = n_b^2 \frac{d}{dn_b}\left(\frac{\mathcal{E}}{n_b}\right) = \frac{1}{3} n_b^{4/3} - \mathcal{B}$
- Linear EOS dependence: $P \propto ac^2(\rho - \rho_s)$

TOV: Tolman-Oppenheimer-Volkoff equation

Assuming spherical & stationary metric; inside the star:

$$ds^2 = -e^{2\nu(r)}c^2dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

TOV equations are the solution of

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

$$\frac{dP}{dr} = -\frac{G(\rho+P/c^2)(m+4\pi r^3 P/c^2)}{r^2\,(1-2Gm/rc^2)}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{d\nu(r)}{dr} = -\frac{1}{P(r)+\rho(r)c^2}\frac{dP(r)}{dr},$$

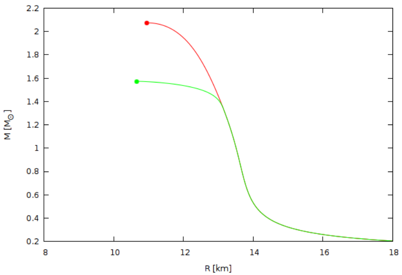
$$e^{-2\lambda(r)} = 1 - \frac{2Gm(r)}{rc^2},$$

+ equation of state $P(\rho)$

History

⋆ Tolman: analysis of spherically-symmetric metrics

⋆ Oppenheimer, Volkoff: solution for a degenerate gas of neutrons, $M_{max} \simeq 0.7\ M_\odot$



Mass-radius diagram: the effect of the softening by phase transition

Constant density solution ($\sim$ quark star)

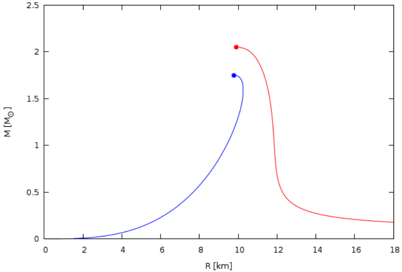
In case of incompressible matter star ($\rho = \text{const.}$), there is an analytic solution:

$$M(r) = \frac{4}{3}\pi\rho r^3$$

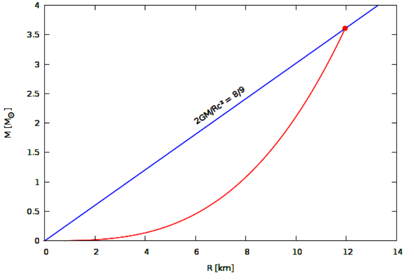
Hydrostatic equilibrium can be integrated 'by hand':

$$\frac{p(r)}{\rho} = \frac{\sqrt{1 - 2GMr^2/R^3c^2} - \sqrt{1 - 2GM/Rc^2}}{3\sqrt{1 - 2GM/Rc^2} - \sqrt{1 - 2GMr^2/R^3c^2}}$$

$p_c = p(0) \to \infty$ gives a limit on the compactness $\dfrac{2GM}{Rc^2} < \dfrac{8}{9}$



Blue: self-bound quark matter



Analytic solution: $\rho = \text{const.}$ matter

Astrophysical estimators of the EOS

Volume element in spherically symmetric spacetime is

$$dV = \frac{4\pi r^2 dr}{\sqrt{1 - 2GM(r)/rc^2}}$$

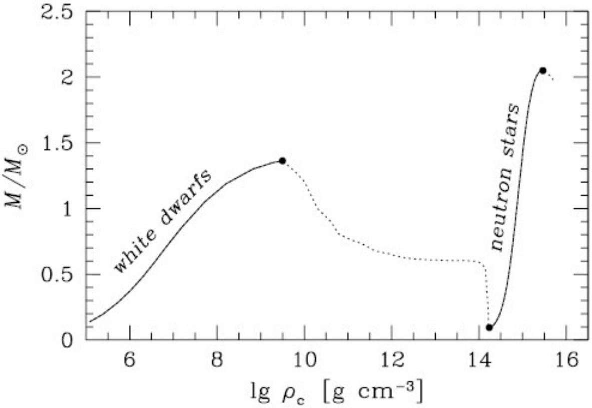
so gravitational and baryon mass are

$$M = M(R) = \int_0^R \frac{4\pi \rho(r) r^2 dr}{\sqrt{1 - 2GM(r)/rc^2}} \quad \text{and} \quad M_b = \int_0^R \frac{4\pi n_b(r) r^2 dr}{\sqrt{1 - 2GM(r)/rc^2}}$$

Some observables modified by gravity:

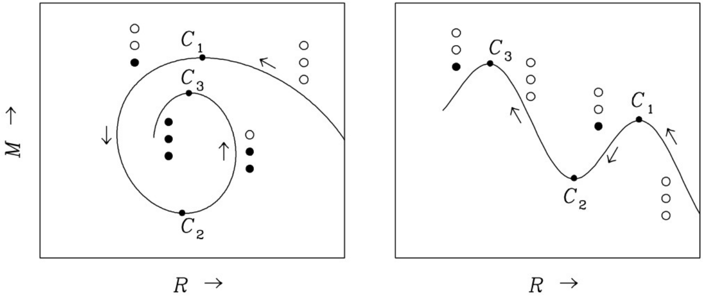
- ⋆ gravitational mass M,
- ⋆ surface redshift $z = 1/\sqrt{1 - 2GM/Rc^2} - 1$,
- ⋆ radius R (radiation radius $R_\infty = R/\sqrt{1 - 2GM/Rc^2}$),
- ⋆ surface temperature T (redshifted temperature $T_r = T\sqrt{1 - 2GM/Rc^2}$)
- ⋆ moment of inertia $I \sim MR^2$,
- ⋆ binding energy $BE = M_b - M$

Neutron stars vs white dwarfs



$\simeq 1.4\ M_\odot$ is the *Chandrasekhar mass*: the maximum mass for an equation of state (pressure-density relation) of degenerate electrons with $P = \kappa\rho^\Gamma$, $\Gamma \in (4/3, 5/3)$

Stability of configurations



- ⋆ for $M = M_{\max}$, the star becomes unstable w.r.t. radial oscillations, further extrema correspond to the lost of stability w.r.t. higher harmonics.
- ⋆ critical points on the $M(R)$ relation (extrema possible due to e.g., phase transitions).

Binding energy for polytropes

The potential energy is $U = -\int_0^s \frac{Gmdm}{r} = -\frac{1}{2}\int_0^s \frac{Gdm^2}{r} = -GM^2/2R - \frac{1}{2}\int_0^s \frac{Gm^2dr}{r^2}$

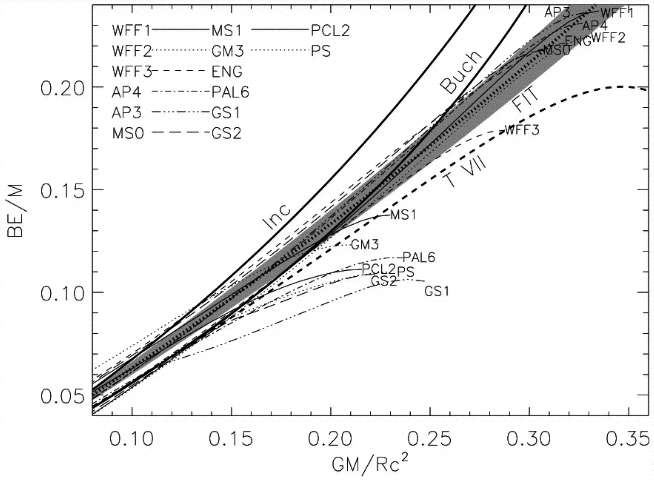
$\frac{1}{2}\int_0^s \frac{Gm^2dr}{r^2} = -\frac{1}{2}\int_0^s \frac{mdP}{\rho} = -\frac{n+1}{2}\int_0^s md\left(\frac{P}{\rho}\right)$
$= \frac{n+1}{2}\int_0^s (P/\rho)dm = \frac{n+1}{2}\int_0^s 4\pi r^2 Pdr$
$= -\frac{n+1}{6}\int_0^s 4\pi r^3 dP = \frac{n+1}{6}\int_0^s \frac{Gmdm}{r}$

- ⋆ $P = \kappa\rho^{(n+1)/n}$
- ⋆ $dP/dr = -Gm\rho/r^2$
- ⋆ $dm/dr = 4\pi\rho r^2$
- ⋆ $2T = kU$, for $U \propto r^k$

$U = -GM^2/2R + \frac{n+1}{6}U \to U = -\frac{3}{5-n}GM^2/R$

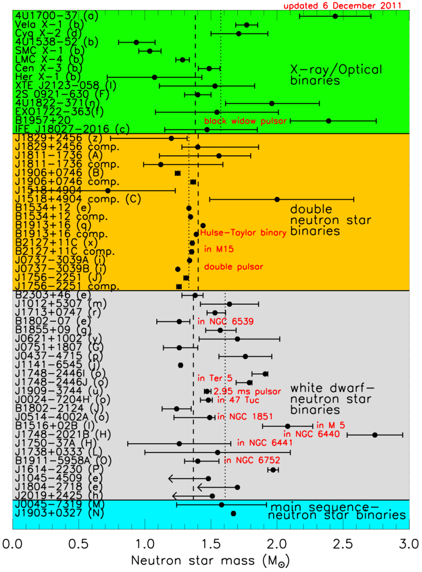
Total energy
$E = T + U = 1/2U = -\left(\frac{3}{10-2n}\right)GM^2/R.$

Binding energy for 'realistic' NSs

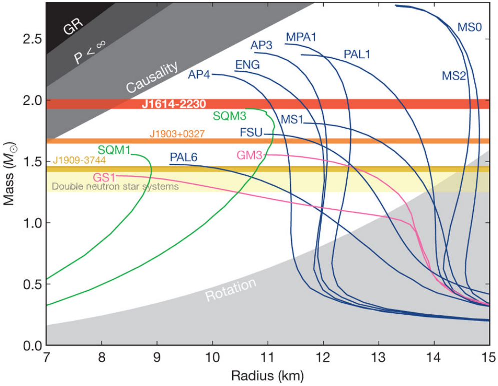


$BE = Nm_b - M$, $BE/M \simeq 0.6\beta/(1 - 0.5\beta)$, where $\beta = GM/Rc^2$.

NS mass measurements



(by J. Lattimer)



(from Demorest et al., 2010)

Masses are measured mostly in binary systems:

$$f(m_1, m_2) = \frac{4\pi^2}{G}\frac{(a\sin i)^3}{P_b^2} = \frac{(m_2 \sin i)^3}{(m_1+m_2)^2}.$$

- ⋆ J1614-2230: $1.97 \pm 0.04\ M_\odot$,
- ⋆ J1903+0327: $1.67 \pm 0.01\ M_\odot$

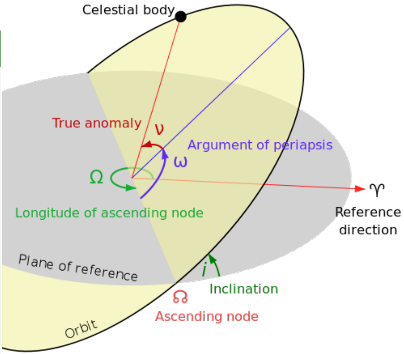
NSs in relativistic binaries

Relativistic binaries: GR effects!

Post-Keplerian parameters

- ⋆ Periastron advance:
  $$\dot{\omega} = 3\left(\frac{P_b}{2\pi}\right)^{-5/3}(T_{\odot}M)^{2/3}(1-e^2)^{-1}$$
- ⋆ Orbit decay:
  $$\dot{P}_b = -\frac{192\pi m_p m_c}{5M^{1/3}}\left(\frac{P_b}{2\pi}\right)^{-5/3} \times \left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)(1-e^2)^{-7/2}T_{\odot}^{5/3}$$
- ⋆ Shapiro delay:
  $$r = T_{\odot}m_c,$$
  $$s = \frac{a_p \sin i}{c m_c}\left(\frac{P_b}{2\pi}\right)^{-2/3}T_{\odot}^{-1/3}M^{2/3}$$
- ⋆ Gravitational redshift:
  $$\gamma = e\left(\frac{P_b}{2\pi}\right)^{1/3}T_{\odot}^{2/3}M^{-4/3}m_c(M+m_c)$$

where $T_{\odot} = GM_{\odot}/c^3$, $M = m_p + m_c$.



Keplerian parameters: eccentricity e, semimajoraxis a, inclination i, longitude of the ascending node Ω, argument of periapsis ω, mean anomaly M_o.

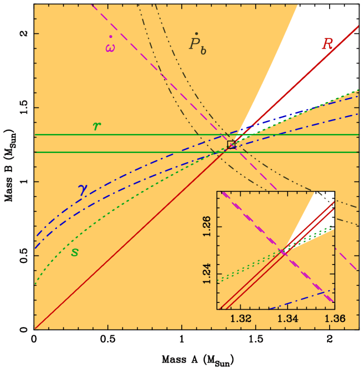
NSs in relativistic binaries

Relativistic binaries: GR effects!

Post-Keplerian parameters

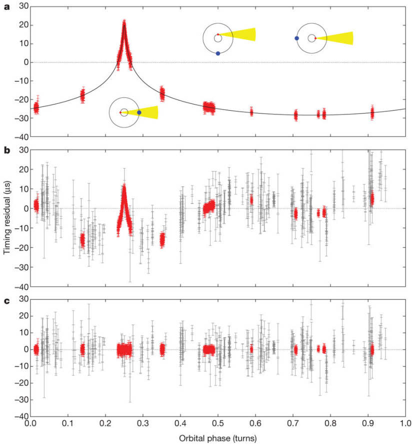
- Periastron advance:
  $$\dot{\omega} = 3\left(\frac{P_b}{2\pi}\right)^{-5/3}(T_\odot M)^{2/3}(1-e^2)^{-1}$$
- Orbit decay:
  $$\dot{P}_b = -\frac{192\pi m_p m_c}{5M^{1/3}}\left(\frac{P_b}{2\pi}\right)^{-5/3}\times$$
  $$\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)(1-e^2)^{-7/2}T_\odot^{5/3}$$
- Shapiro delay:
  $$r = T_\odot m_c,$$
  $$s = \frac{a_p \sin i}{c m_c}\left(\frac{P_b}{2\pi}\right)^{-2/3}T_\odot^{-1/3}M^{2/3}$$
- Gravitational redshift:
  $$\gamma = e\left(\frac{P_b}{2\pi}\right)^{1/3}T_\odot^{2/3}M^{-4/3}m_c(M+m_c)$$

where $T_\odot = GM_\odot/c^3$, $M = m_p + m_c$.



PSR J0737-3039 (M. Kramer)

PSR J1614-2230



Binary system with a white dwarf ($m_c = 0.5\ M_\odot$)

Almost edge-on, $\sin i \simeq 1$

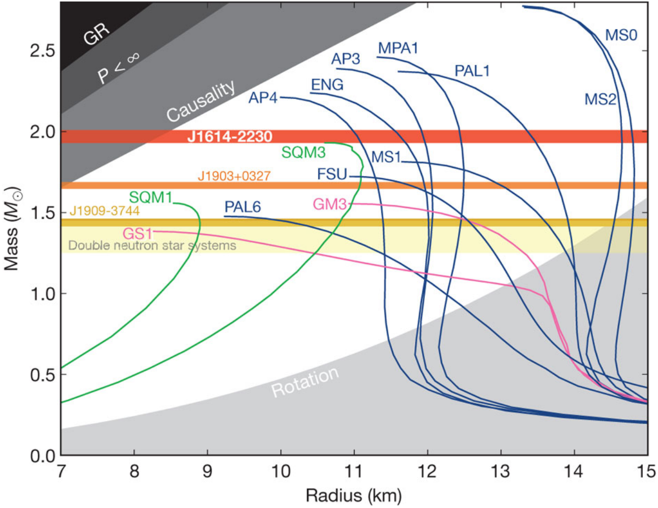
Shapiro delay parameters:

$$r = T_\odot m_c,$$

$$s = \frac{a_p \sin i}{c m_c} \left(\frac{P_b}{2\pi}\right)^{-2/3} T_\odot^{-1/3} M^{2/3}$$

$$\rightarrow M = 1.97 \pm 0.04\ M_\odot$$

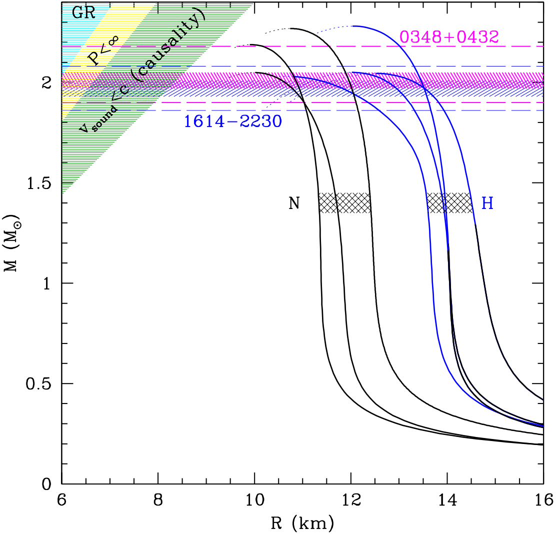
$M(R)$ diagram with $2\,M_\odot$ measurement



Demorest et al. (2010)

$M(R)$ diagram with $2\,M_\odot$ measurement

Possible solution of "the hyperon puzzle": LOFT or similar satellite (with 5% accuracy in radius measurement)



Various ways to solve a PDE

Various ways to solve a PDE

Consider the PDE with some boundary conditions

$$\begin{aligned} Lu(x) &= s(x), \qquad x \in U, \quad \text{(the equation)} \\ Bu(y) &= 0, \qquad y \in \partial U, \quad \text{(boundary conditions)}, \end{aligned}$$

with L and B linear differential operators. We search for a numerical solution $u^{(N)}(x)$, that minimizes the residual,

$$R \equiv Lu^{(N)}(x) - s(x).$$

Various ways to solve a PDE

In general, the solution $u^{(N)}$ is expressed in terms of some functions,

$$u^{(N)}(x) = \sum_{k=0}^{N} \tilde{u}_k \phi_k(x),$$

Numerical methods can be classified according to the expansion functions ϕ_k:

- **Finite differences**: overlapping local polynomials of low order,
- **Finite elements**: local smooth functions (locally non-zero polynomials of fixed degree)
- **Spectral methods**: global smooth functions (e.g., Fourier series)

$$u^{(N)}(x) = \frac{a_0}{2} + \sum_{k=1}^{N} \left(a_k \cos\left(kx\right) + b_k \sin\left(kx\right)\right)$$

Spectral methods approximate the solution to a differential equation, $u(x)$, by a truncated series

$$u(x) \simeq u^{(N)}(x) = \sum_{k=0}^{N} \tilde{u}_k \phi_k(x),$$

- where $\phi_k(x)$ are basis functions (i.e., members of a complete set of orthogonal polynomials)
- $\tilde{u}_k$ are the spectral coefficients.

What can we gain with such an approach? For example, analytical expressions for derivatives,

$$\frac{\partial u^{(N)}(x)}{\partial x} = \sum_{k=0}^{N} \tilde{u}_k \frac{\partial \phi_k(x)}{\partial x}$$

Classification of spectral methods

Many ways to evaluate the residual $R = Lu^{(N)}(x) - s(x)$, e.g., to chose functions ψ_k and calculate scalar products, such that $\forall k \in \{0, 1, \ldots, N\}$ $(\psi_k,\ R) = 0$

- **Galerkin**: $\psi_k = \phi_k$,
  ϕ_k satisfy the boundary conditions,
- **Tau/Lanczos**: ψ_k are most of ϕ_k,
  ϕ_k do not satisfy the boundary conditions, additional conditions must be added to the system,
- **Pseudospectral/collocation**: $\psi_k = \delta(x - x_k)$,
  test functions are Dirac deltas in special (collocation) points x_k, boundary conditions enforced by additional equations.

The choice of orthogonal polynomials

- ⋆ For periodic problems, Fourier expansion $(\sin, \cos)$ is the most natural and recommended → azimuthal & poloidal directions,
- ⋆ Non-periodic problems: good choice are the Chebyshev polynomials.

Chebyshev polynomials, defined on the usual numerically-evaluated interval $[-1, 1]$:

$$T_n(\cos\theta) = \cos(n\theta),$$

and satify the following Sturm-Liouville problem

$$\sqrt{1-x^2}\frac{d}{dx}\left(\sqrt{1-x^2}\,\frac{dT_n(x)}{dx}\right) = -n^2\,T_n(x).$$

First few polynomials are:

$$T_0(x) = 1,\, T_1(x) = x,\, T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x,\, T_4(x) = 8x^4 - 8x^2 + 1.$$

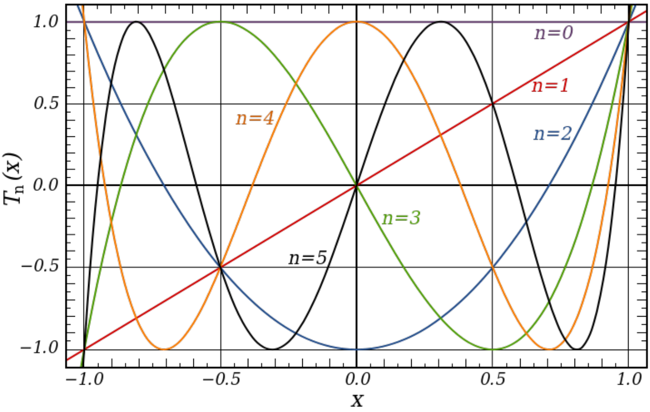
The choice of orthogonal polynomials: Chebyshev

Useful recurence relation:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \text{and} \quad T_n(-1) = (-1)^n, \quad T_n(1) = 1.$$

Recurrence relation for derivatives:

$$T_n'(x) = 2nT_{n-1}(x) + \frac{n}{n-2} T_{n-2}'(x), \quad n > 2.$$



A simple example

Consider a 1D ODE (elliptic equation):

$$\frac{d^2u}{dx^2} - 4\frac{du}{dx} + 4u = \exp(x) - 4e/(1+e^2), \quad x \in [-1, 1],$$

with the following boundary conditions:

$$u(-1) = u(1) = 0.$$

The exact solution is

$$u(x) = \exp(x) - \frac{\sinh 1}{\sinh 2}\exp(2x) - e/1(1+e^2).$$

A simple example: matrix representation of the operator

For a particular representation with the Chebyshev polynomials,

$$u^{(N)}(x) = \sum_{k=0}^{N} \tilde{u}_k T_k,$$

the operator L acting on the series $u^{(N)}$ is

$$Lu^{(N)} = \sum_{k=0}^{N} \tilde{l}_k T_k \quad \text{with} \quad \tilde{l}_k = \sum_{j=0}^{N} L_{kj} \tilde{u}_j$$

The operator $L = \frac{d^2}{dx^2} - 4\frac{d}{dx} + 4I$ can be viewed as a matrix, acting on the function coefficient vector (all methods of linear algebra apply).

A simple example: matrix representation of the operator

$L = \frac{d^2}{dx^2} - 4\frac{d}{dx} + 4I$ for $N = 4$,

$$L_{kj} = \begin{pmatrix} 4 & -4 & 4 & -12 & 32 \\ 0 & 4 & -16 & 24 & -32 \\ 0 & 0 & 4 & -24 & 48 \\ 0 & 0 & 0 & 4 & -32 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

because in the Chebyshev representation,

$$\frac{d}{dx} = \begin{pmatrix} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \frac{d^2}{dx^2} = \begin{pmatrix} 0 & 0 & 4 & 0 & 32 \\ 0 & 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & 0 & 48 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Also, higher order derivatives just by multiplying matrices:

$$\left[\frac{d}{dx}\right]^2 = \left[\frac{d^2}{dx^2}\right]$$

Solution by mean of tau method

⋆ Test functions (to evaluate the residual $R = Lu^{(N)} - s$) are T_k; they provide $N+1$ equations:

$$(T_k, R) = 0 \rightarrow \sum_{k=0}^{N} \sum_{j=0}^{N} L_{kj} \tilde{u}_j T_k = \sum_{k=0}^{N} \tilde{s}_k T_k,$$

where $\tilde{s}_k$ are the coefficients of the source (i.e., the right hand side).

⋆ The boundary conditions:

$$u(-1) = 0 \rightarrow \sum_{j=0}^{N} (-1)^j \tilde{u}_j = 0,$$

$$u(1) = 0 \rightarrow \sum_{j=0}^{N} \tilde{u}_j = 0.$$

We have $N+3$ equations; discard two of the highest order coefficients $\tilde{u}_k$ and replace them with the boundary conditions equations.

Solution for $N = 4$

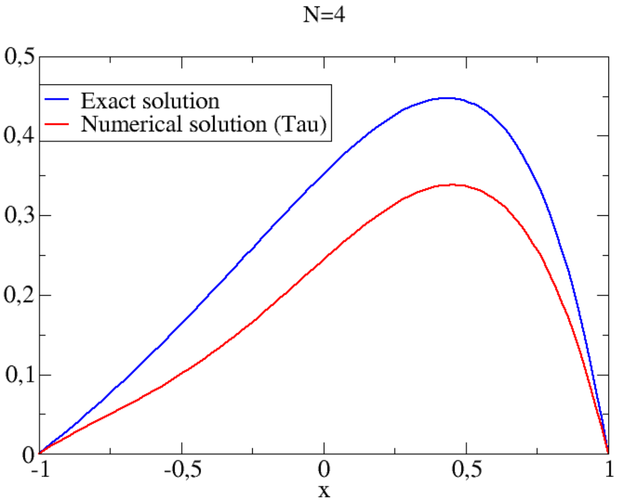
Solving for unknown $\tilde{u}_k$:

$$Lu^{(4)} = \begin{pmatrix} 4 & -4 & 4 & -12 & 32 \\ 0 & 4 & -16 & 24 & -32 \\ 0 & 0 & 4 & -24 & 48 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \end{pmatrix} = \begin{pmatrix} -0.03 \\ 1.13 \\ 0.27 \\ 0 \\ 0 \end{pmatrix}$$

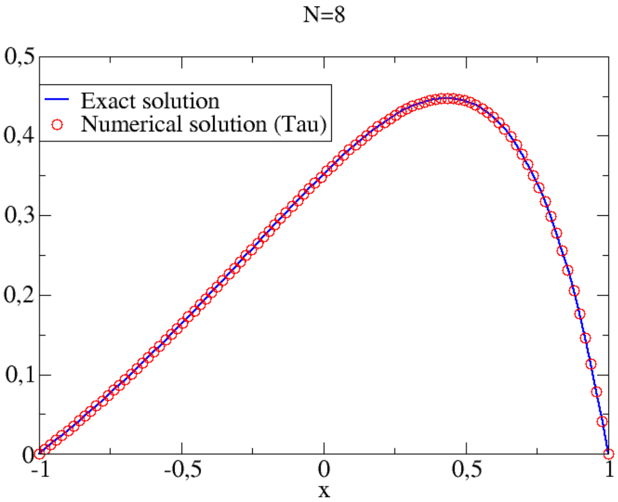
(in red, the imposed boundary conditions). The solution is

$$\begin{pmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \end{pmatrix} \simeq \begin{pmatrix} 0.146 \\ 0.079 \\ -0.122 \\ -0.079 \\ -0.024 \end{pmatrix}$$

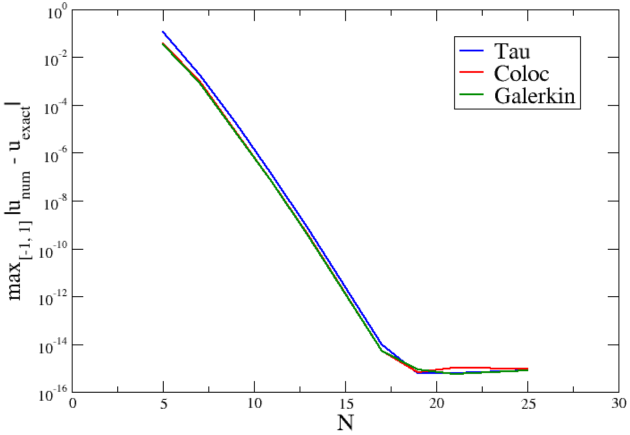
Solution for $N = 4$



Solution for $N = 8$



The evanescent error



⋆ For suffiently smooth solutions, the interpolation/truncation error falls faster than any power of $1/N$ (in practice, this means exponential decay).

⋆ For finite difference of order k, error decays as $1/N^k$.

Further reading...

- P. Haensel, A. Y. Potekhin, D. G. Yakovlev, „Neutron stars 1: equation of state and structure",
- T. W. Baumgarte, S. L. Shapiro, „Numerical Relativity: Solving Einstein's Equations on the Computer",
- S. L. Shapiro, S. A. Teukolsky, „Black Holes, White Dwarfs and Neutron Stars",
- Spectral methods library LORENE (Langage Objet pour la RElativité NumériquE): `http://www.lorene.obspm.fr`
- LORENE school on spectral methods: http://www.lorene.obspm.fr/school