Relativistic stars 2

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Outline

- ⋆ Newtonian rotation,
- ⋆ Slow-rotation approximation in general relativity,
- ⋆ Numerical general relativity: 3+1 formalism,
- ⋆ Stationary axisymmetric rotating stars,
- ⋆ Astrophysical results.

Newtonian star. Equation of motion in the rotating frame

Consider a rigidly rotating body. The relation between the inertial (nonrotating, laboratory) frame and the rotating frame is

$$\frac{D\mathbf{r}}{Dt} = \frac{d\mathbf{r}}{dt} + \Omega \times \mathbf{r}$$

$$\frac{D^2\mathbf{r}}{Dt^2} = \left(\frac{d\mathbf{r}}{dt} + \Omega\times\right)^2 \mathbf{r} = \frac{d^2\mathbf{r}}{dt^2} + 2\Omega \times \frac{d\mathbf{r}}{dt} + \Omega \times (\Omega \times \mathbf{r}),$$

assuming that $d\Omega/dt = 0$. The equation of motion in the inertial frame is

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P - \rho\nabla\psi,$$

where $\mathbf{u}$ is the fluid velocity (fluid element's rate of change of position $\mathbf{r}$ in time). In the rotating frame:

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla P - \nabla\psi - \underbrace{2\Omega \times \mathbf{u}}_{Coriolis} - \underbrace{\Omega \times (\Omega \times \mathbf{u})}_{Centrifugal\ \ acc.}.$$

Equilibrium equations of a slowly rotating body

Assumptions:

- ⋆ rotation induces a weak distortion of the shape from the spherically symmetry,
- ⋆ the body is axisymmetric about the rotation axis.

In the frame rotating with the body, in equilibrium $\mathbf{u} = 0$ and $\partial/\partial t = 0$. If $\Omega = \Omega\mathbf{e_z}$ (z direction is along the rotational axis), then

$$\Omega \times \mathbf{r} = \Omega r \sin\theta e_\phi, \quad \text{and}$$

$$-\Omega \times (\Omega \times \mathbf{r}) = \Omega^2 r \sin^2\theta \mathbf{e_r} + \Omega^2 r \sin\theta \cos\theta \mathbf{e}_\theta = \nabla \left(\frac{1}{2}\Omega^2 r^2 \sin^2\theta \right),$$

i.e., centrifugal force can be expressed as the gradient of potential. The equation of motion is then

$$\nabla P = -\rho \nabla \Phi, \quad \text{with} \quad \Phi = \psi - \frac{1}{2}\Omega^2 r^2 \sin^2\theta.$$

→ the surfaces of constant P and ρ coincide with surfaces of constant Φ.

The Roche model

On the surface, $\Phi = \psi - \frac{1}{2}\Omega^2 r^2 \sin^2\theta$ is constant ($P = 0$). Assume that

$$\psi = -GM/r \quad \text{at the surface and outside the star.}$$

This corresponds to a centrally-condensed star (all mass at $r = 0$). For a stellar surface described by $r = R(1 + \epsilon(\theta))$, the potential

$$\Phi_{surface} = -\frac{GM}{R(1+\epsilon(\theta))} - \frac{1}{2}\Omega^2 R^2 (1+\epsilon(\theta))^2 \sin^2\theta$$

is constant (does not depend of θ). Neglecting high-order terms of Ω^2 and ϵ^2,

$$\epsilon(\theta) = \frac{1}{2}\frac{\Omega^2 R^3}{GM}\sin^2\theta + const.$$

describes the ratio of centrifugal ($\Omega^2 R$) to gravitational (GM/R^2) acceleration.

The Roche model

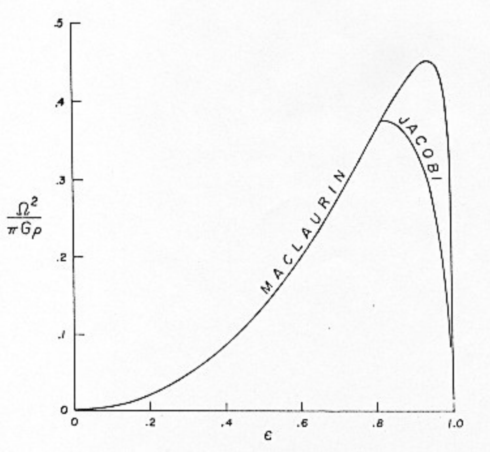
The shape of such star can be evaluated as follows. The relative difference between equatorial and polar radii (shape distortion) is

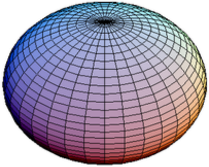
$$\frac{R(1+\epsilon(\pi/2)) - R(1+\epsilon(0))}{R} = \frac{1}{2}\frac{\Omega^2 R^3}{GM}.$$

The above is a good toy-model for estimating the shape (but note the assumption of $\psi = -GM/r$).
Proper treatment leading to a figure of equilibrium of a slowly rotating star is the *Chandrasekhar-Milne expansion*.

Newtonian figures of equilibrium: Maclaurin spheroid

Great deal of classical research - equilibria of rotating bodies with uniform density: Laplace, Jacobi, Liouville, Riemann, Poincaré, Kelvin, Jeans...





$A_1 = A_2 = A$

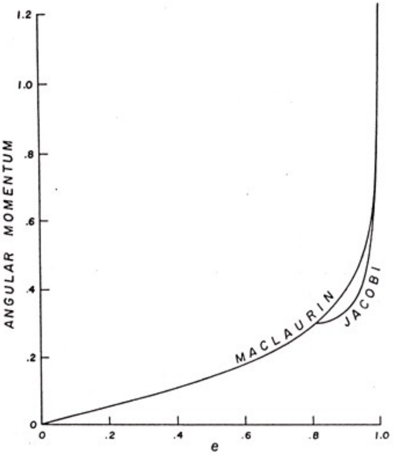
Moment of inertia along z axis
$I = \frac{2}{5} M A^2$, $M = \frac{4}{3}\pi\rho A^2 A_3$

Equation of state: $\rho = const.$

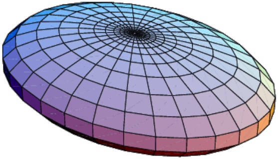
Poisson equation: $\nabla^2\Phi = 4\pi G\rho$

Kinetic energy $T = \frac{1}{2} I \Omega^2$, potential energy $W = \frac{1}{2}\rho\int \Phi d^3x$

Newtonian figures of equilibrium: Jacobi ellipsoid



eccentricity $e = \sqrt{1 - \frac{A_3^2}{A_1^2}}$



Transition from a Maclaurin spheroid to a lower $E = T + W$ state:

- ⋆ Secular instability (occurs at the dissipation timescale, Chandrasekhar-Friedman-Schutz instability, CFS): $e = 0.812760$, $T/|W| = 0.1375$,
- ⋆ Dynamical instability (dynamical timescale, Maclaurin spheroids unstable): $e = 0.952887$, $T/|W| = 0.2738$

Rotating stars in general relativity

Geometry of spacetime

Consider a stationary and axisymmetric metric $g_{\alpha\beta}$,

$$g_{\alpha\beta}dx^\alpha dx^\beta = -e^{2\nu}dt^2 + e^{-2\nu}B^2r^2\sin^2\theta(d\phi - \omega dt)^2 + e^{2\mu}\left(dr^2 + r^2d\theta^2\right),$$

with ν, B, ω and μ being metric functions of r and θ. This is a general form of a metric assuming:

- ⋆ the spacetime has a timelike Killing vector field ξ^α,
- ⋆ a second Killing vector field χ^α corresponding to axial symmetry (symmetries - Noether theorem),
- ⋆ the spacetime is asymptotically flat - at spatial infinity:

$$\xi_\alpha\xi^\alpha = -1, \quad \chi_\alpha\chi^\alpha = \infty, \quad \xi_\alpha\chi^\alpha = 0.$$

- ⋆ Quasi-isotropic coordinates, where

$$\xi_\alpha\xi^\alpha = g_{tt}, \quad \chi_\alpha\chi^\alpha = g_{\phi\phi}, \quad \xi_\alpha\chi^\alpha = g_{t\phi}.$$

and $e^{2\mu}$ called the *conformal factor* that characterizes the 2-surface (r,θ) geometry.

Conformal geometry. Frame-dragging

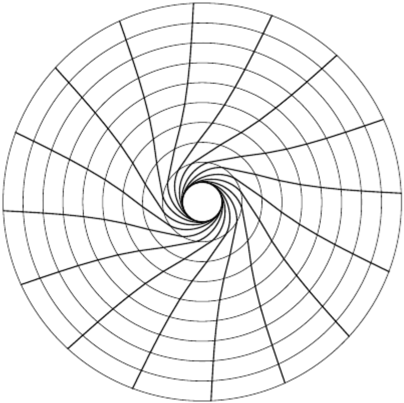
Conformal geometry - angle preserving transformations on a space (Escher's "Angels and daemons".)

Two main effects of rotation:

- ⋆ The shape of the star is flattened by centrifugal forces (second order in the rotation rate),
- ⋆ Local inertial frames are dragged by the rotation of the source of the gravitational field.

Second effect is purely relativistic (also called frame-dragging).

Conformal geometry. Frame-dragging



Radial infall in rotating spacetime metric - not radial anymore.

Two main effects of rotation:

- ⋆ The shape of the star is flattened by centrifugal forces (second order in the rotation rate),
- ⋆ Local inertial frames are dragged by the rotation of the source of the gravitational field.

Second effect is purely relativistic (also called frame-dragging).

ZAMO & frame-dragging

For the metric

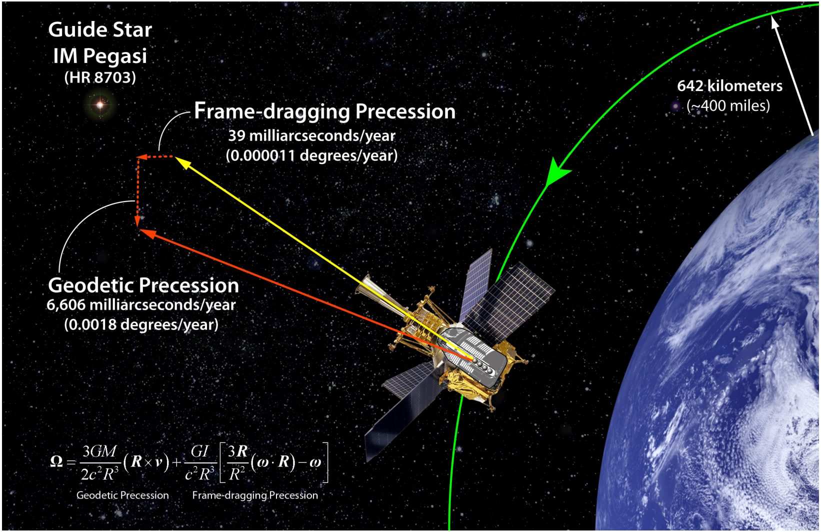
$$ds^2 = -e^{2\nu}dt^2 + e^{-2\nu}B^2r^2\sin^2\theta\,(d\phi - \omega dt)^2 + e^{2\mu}\left(dr^2 + r^2d\theta^2\right),$$

How to know the spacetime is dragged with the body? ZAMO (Zero-Angular-Momentum-Observer, see Bardeen 1972) with worldlines normal to $t = const.$ is an analogue of the Eulerian observer:

- ⋆ ω is the angular velocity of the local ZAMO w.r.t to an observer at rest at infinity,
- ⋆ $e^{-\nu}$ is the time dilation factor between the proper time of the local ZAMO and coordinate time t (proper time at infinity) along a radial coordinate line,
- ⋆ $Br\sin\theta e^{-\nu}$ is the proper circumferential radius of a circle around the axis of symmetry.

Frame-dragging is related to the existence of an ergosphere ($g_{tt} > 0$; spacetime dragging so strong, particles can have negative energies w.r.t infinity).

Frame-dragging experiment: Gravity Probe B



The rotating fluid

The asymptotic behaviour of the metric functions ν and ω is

$$\nu \sim -\frac{M}{r} + \frac{Q}{r^3} P_2(\cos\theta) \quad \text{and} \quad \omega \sim \frac{2J}{r^3},$$

with M, J and Q are the gravitational mass, angular momentum and quadrupole moment of the star. For a perfect fluid energy-momentum tensor,

$$T^{\alpha\beta} = (\rho + P) u^\alpha u^\beta + P g^{\alpha\beta},$$

one can write the fluid 4-velocity u^α as

$$u^\alpha = \frac{e^{-\nu}}{\sqrt{1 - v^2}} \left(\xi^\alpha + \Omega \chi^\alpha \right),$$

where v is the fluid 3-velocity w.r.t a local ZAMO,

$$v = (\Omega - \omega) B r \sin\theta e^{-2\nu} \quad \text{and} \quad \Omega = d\phi/dt = u^\phi / u^t,$$

with Ω, the angular velocity of the fluid in the coordinate frame (as seen from infinity).

Slow rotation approximation (Hartle 1967)

Assumptions: slow rigid rotation, treated as a perturbation of a spherically-symmetric model (up to quadrupole $l = 2$). Spin frequency $\Omega \ll \Omega_{\rm max}$ (angular rotation frequency Ω measured by the observer at spatial infinity)

A general, axisymmetric stationary spacetime, the metric is

$$ds^2 = e^{2\nu(r,\theta)}c^2dt^2 - e^{2\lambda(r,\theta)}dr^2 - e^{2\gamma(r,\theta)}r^2\left(d\theta^2 + \sin^2\theta(d\phi - L(r,\theta)dt)^2\right)$$

How to guess the form of L:

- From the form of metric, time reversal corresponds to $\Omega \to -\Omega$: L is a function of odd powers of Ω,
- Let ω be the frequency of local inertial (zero angular momentum) observers (ZAMO), $\omega \propto \Omega$,

$$L(r,\theta) = \omega(r,\theta) + \mathcal{O}(\Omega^3)$$

Slow rotation approximation (Hartle 1967)

From the dimensional analysis:

$$\omega \propto GJ/c^2 r^3,$$

where J is the total angular momentum of the star.

Frame-dragging of the inertial observers is:

$$\bar{\omega}(R) = \Omega - \frac{2GJ}{R^3 c^2}$$

Slow rotation approximation can be also used to provide the moment of inertia of the star for the TOV (nonrotating) solution. By retaining first order terms in Ω in

$$R^t_\phi - \frac{1}{2} R g^t_\phi = 8\pi T^t_\phi$$

one gets the following equation

$$\frac{1}{j^4} \frac{d}{dr} \left(r^4 j \frac{d\bar{\omega}}{dr} \right) + \bar{\omega} \frac{4}{r} \frac{dj}{dr} = 0$$

Slow rotation approximation (Hartle 1967)

$$\frac{1}{j^4}\frac{d}{dr}\left(r^4 j\frac{d\bar{\omega}}{dr}\right)+\bar{\omega}\frac{4}{r}\frac{dj}{dr}=0$$

where the function $j(r)$ may be already defined using the Schwarzschild metric,

$$j(r)=\begin{cases} e^{-(\nu+\lambda)}=e^{-\lambda}\sqrt{1-2Gm(r)/rc^2} & r<R \\ 1 & r\geqslant R \end{cases}$$

By differentiating the above equation and using the TOV equations we get

$$\frac{dj}{dr}=-4\frac{G}{c^4}\pi r e^{-\lambda}\left(P+\rho c^2\right)/\sqrt{1-2Gm(r)/rc^2}$$

also:

$$\left(r^4 j\frac{d\bar{\omega}}{dr}\right)_R=-\int_0^R 4r^3\frac{dj}{dr}\bar{\omega}dr$$

and

$$\bar{\omega}\propto GJ/c^2r^3,\quad \frac{d\bar{\omega}}{dr}\propto\frac{3GJ}{c^2r^4}\quad \text{that is}\quad J\propto\frac{c^2}{3G}\int_0^R 4r^3\frac{dj}{dr}\bar{\omega}dr.$$

Slow rotation approximation (Hartle 1967)

For the slow-rotation we may accept the relation between J and Ω, $J = I\Omega$, with I the moment of inertia. The Newtonian moment of inertia is

$$I_{Newt.} = \frac{8\pi}{3} \int_0^R \rho r^4 dr.$$

By collecting the previous equations we finally get the slow-rotation moment of inertia:

$$I = \frac{J}{\Omega} = \frac{8\pi}{3} \int_0^R \frac{P + \rho c^2}{\sqrt{1 - 2Gm(r)/rc^2}} \frac{\bar{\omega}}{\Omega} \mathrm{e}^{-\lambda} r^4 dr$$

(integrating outwards the relation for J; the prefactor comes from the Newtonian limit for $2Gm/rc^2 \ll 1$ and $P \ll \rho c^2$).

Thermodynamics of simple fluids

First law of thermodynamics

In the comoving (Lorentz) frame, convenient when using baryon number density n_b, the first law of thermodynamics is:

$$d\left(\frac{\rho}{n_b}\right) = Tds - Pd\left(\frac{1}{n_b}\right) + \sum_i \mu_i d\left(\frac{n_i}{n_b}\right),$$

$$\left(\text{from the total energy per baryon } \frac{\rho}{n_b} = Ts - \frac{P}{n_b} + \sum_i \mu_i \left(\frac{n_i}{n_b}\right)\right),$$

where ρ is the mass-energy density, P is the fluid pressure, T temperature, s the entropy per baryon, and μ_i the chemical potentials of species i:

$$P = -\left(\frac{\partial \rho/n_b}{\partial 1/n_b}\right)_{s,n_i} = n_b^2 \left(\frac{\partial \rho/n_b}{\partial n_b}\right)_{s,n_i}$$

$$T = \left(\frac{\partial \rho/n_b}{\partial s}\right)_{n_i,n_b} \quad \text{and} \quad \mu_i = \left(\frac{\partial \rho/n_b}{\partial n_i/n_b}\right)_{s,n_b} = \left(\frac{\partial \rho}{\partial n_i}\right)_{s,n_b}$$

Chemical potential

Chemical potential μ_i reflects the change in the energy density ρ due to the change in the number density n_i (with s and n_b constant). Chemical equilibrium means that

$$\sum_i \mu_i d\left(\frac{n_i}{n_b}\right) = 0,$$

with $ds = 0$ in equilibrium (2nd law) and assuming no work is done on the system.

For one component system and $T = 0$ ($\sim$ barotropic neutron star), chemical potential can be defined as

$$\mu = \frac{\rho + P}{n_b}.$$

Enthalpy, Gibbs energy and Gibbs-Duhem relation

Lets define an auxiliary function, *pseudo-enthalpy, log-enthalpy*:

$$H(P) = \int_0^P \frac{dp}{\rho(p) + p} = \ln\left(h(P)\right) = \ln\left(\frac{\mu}{\mu_0}\right),$$

where

$$h(P) = \frac{\rho + P}{m_0 c^2 n_b} \quad \text{and} \quad \mu_0 = \mu(P=0) = m_0 c^2.$$

The Gibbs free energy per baryon is

$$g = \frac{\rho}{n_b} - Ts + \frac{P}{n_b} = \sum_i \mu_i d\left(\frac{n_i}{n_b}\right),$$

derivative of it compared with the 1st law, for $T = 0$ gives the *Gibbs-Duhem* relation

$$dP = \sum_i n_i d\mu_i \quad \text{or, for one component fluid} \quad \frac{dP}{dH} = \rho + P.$$

Gibbs-Duhem relation in EOS treatment

The Gibbs-Duhem relation is used to construct *strictly* thermodynamically consistent equation of state.

- **The problem:** EOSs usually provided as tables - simple interpolation between table points introduces numerical errors and thermodynamical inconsistency.
- **The solution (Swesty 1996):**
  - Calculate log-enthalpy $H(P) = \ln\left(\frac{\rho+P}{m_0 c^2 n_b}\right)$ and $dP/dH = \rho + P$,
  - Interpolate values of $P(H)$ and $dP/dH(H)$ (e.g., with the Hermite polynomials)

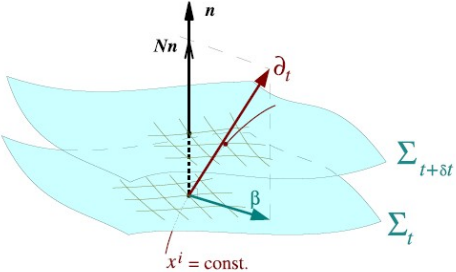
The last step is to recover ρ and n_b:

$$\rho = \frac{dP}{dH} - P, \quad \text{and} \quad n_b = (\rho + P)\, e^{-H}.$$

Resulting EOS ($\rho - P - n_b$ relation) fulfills the laws of thermodynamics.

Numerical general relativity

Numerical general relativity: 3+1 formalism



Hypersurfaces of constant time Σ_t, each with its own coordinate system. 3-metric induced on Σ_t:
$\mathbf{h} = \mathbf{g} + \mathbf{n} \otimes \mathbf{n}$, where $\mathbf{n}$ in normal to Σ_t.

Evolution is described by auxiliary parameters:

Time "lapse" N, $\mathbf{n} = N\nabla t$ and space "shift" $\beta = -\mathbf{h} \cdot \xi$.

The general metric:

$$g_{\mu\nu} dx^\mu dx^\nu = -(N^2 - \beta_i \beta^i) dt^2 - 2\beta_i dt\, dx^i + h_{ij} dx^i\, dx^j$$

3+1 formalism: equations to solve

The normal vector $\mathbf{n}$ is a unit timelike vector; let us consider the family of observers whose 4-velocity is $\mathbf{n}$ - these are the Eulerian (ZAMO) observers.

The stress-energy-momentum tensor $\mathbf{T}$, $T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}$ can be decomposed in terms of physical quantities measured by ZAMO.

$$T = \mathbf{S} + \mathbf{n} \otimes \mathbf{p} + \mathbf{p} \otimes \mathbf{n} + E\mathbf{n} \otimes \mathbf{n},$$

where

$$E = T_{\mu\nu} n^\mu n^\nu, \quad p_i = -h_i^\mu T_{\mu\nu} n^\nu, \quad S_{ij} = h_i^\mu h_j^\nu T_{\mu\nu}$$

are the energy density E, the momentum density vector $\mathbf{p}$ and the stress tensor $\mathbf{S}$.

3+1 formalism: equations to solve

By projecting the Einstein equation twice onto Σ_t, twice along $\mathbf{n}$, and once on Σ_t and $\mathbf{n}$ one gets, respectively the **evolution equation**:

$$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\mathbf{N}} K_{ij} = -D_i D_j N + N \left({}^3R_{ij} \right.$$
$$\left. -2K_{ik}K^k_j + KK_{ij} + 4\pi\{(S-E)h_{ij} - 2S_{ij}\} \right),$$

and the **conservation (constraints) equations**:

$${}^3R + K^2 - K_{ij}K^{ij} = 16\pi E, \quad D_j K^{ij} - D^i K = 8\pi p^i,$$

also called the Hamiltonian constraint and the momentum constraint.

These equations are complimented by the relation between the induced metric and the curvature:

$$\frac{\partial h_{ij}}{\partial t} + D^i\beta^j + D^j\beta^i = 2NK^{ij}$$

where D_i is the covariant derivative on $\mathbf{h}$, ${}^3R_{ij}$ the Ricci tensor, K_{ij} the extrinsic curvature of Σ_t, $K_{\alpha\beta} = \frac{1}{2}\mathcal{L}_n h_{\alpha\beta}$, $\mathcal{L}_n$ Lie derivative along $\mathbf{n}$.

3+1 formalism: equations to solve

This set of conservation (constraints) and evolution equations

$$
{}^3R + K^2 - K_{ij}K^{ij} = 16\pi E, \quad D_j K^{ij} - D^i K = 8\pi p^i,
$$

$$
\begin{aligned}
\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\mathbf{N}} K_{ij} &= -D_i D_j N + N \left({}^3R_{ij} \right. \\
&\left. -2K_{ik}K^k_j + KK_{ij} + 4\pi\{(S-E)h_{ij} - 2S_{ij}\} \right),
\end{aligned}
$$

$$
\frac{\partial h_{ij}}{\partial t} + D^i\beta^j + D^j\beta^i = 2NK^{ij}
$$

is quite similar to Maxwell equations:

$$
\left.\begin{array}{l} \nabla \mathbf{E} = \rho \\ \nabla \mathbf{B} = 0 \end{array}\right\} \text{constraints} \qquad \left.\begin{array}{r} \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0 \\ \nabla \times \mathbf{B} - \partial_t \mathbf{E} + \mathbf{J} = 0 \end{array}\right\} \text{evolution}
$$

Axisymmetric stationary rotating configuration

Conformally flat metric $\mathbf{h} = \Psi\eta$, where η is flat 3-metric. With a particular choice of the conformal factors:

$$g_{\alpha\beta}dx^\alpha dx^\beta = -N^2 dt^2 + A^4 B^2 r^2 \sin^2\theta(d\phi + N^\phi dt)^2 + \frac{A^4}{B^2}(dr^2 + r^2 d\phi^2)$$

where

$$\nu(r,\ \theta) := \ln(N),\quad \alpha(r,\ \theta) := \ln(A),\quad \beta(r,\ \theta) := \ln(B)$$

$$G(r,\ \theta) := NA^2B,\quad \zeta(r,\ \theta) := \nu + 2\alpha - \beta$$

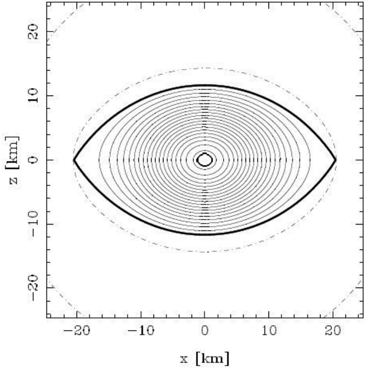
$$k_1 = \frac{B^2 r\sin\theta}{2N}\frac{\partial N^\phi}{\partial r},\quad k_2 = \frac{B^2\sin\theta}{2N}\frac{\partial N^\phi}{\partial\theta}$$

Set of equations (2D and 3D Poisson equations) to be solved is as follows:

$$\begin{aligned}
\Delta_3\nu &= \frac{A^4}{B^2}\left[4\pi(E+S) + 2(k_1^2 + k_2^2)\right] - \partial\nu\partial(\nu + 2\alpha + \beta)\ , \\
\tilde{\Delta}_3\tilde{N}^\phi &= 16\pi\frac{N}{B^4}\frac{J_\phi}{r\sin\theta} - r\sin\theta\partial N^\phi\partial(6\alpha + 3\beta - \nu), \quad \text{where } \tilde{N}^\phi = N^\phi r\sin\theta \\
\Delta_2\tilde{G} &= 8\pi\frac{NA^6}{B}r\sin\theta(S_r^r + S_\theta^\theta), \quad \text{where } \tilde{G} = Gr\sin\theta \\
\Delta_2\zeta &= \frac{A^4}{B^2}\left[8\pi S_\phi^\phi + 3(k_1^2 + k_2^2)\right] - (\partial\alpha)^2
\end{aligned}$$

where Δ_2, Δ_3 and $\tilde{\Delta}_3$ are the scalar Laplacians in 2D and 3D, and the ϕ component of 3D vector Laplacian (Bonazzola et al. 1993)

Axisymmetric stationary rotating configuration



Ideal fluid in rigid rotation:

- ⋆ 4-velocity $\mathbf{u} := u^t \xi + u^\phi \chi$,
- ⋆ angular velocity $\Omega := d\phi/dt = u^\phi/u^t$ in the coordinate frame (as seen from infinity),
- ⋆ Lorentz factor $\Gamma := -\mathbf{n}\mathbf{u} = N u^t = (1 - v^2)^{-1/2}$,
- ⋆ physical velocity of the fluid in the ϕ direction: $v = \frac{A^2 B r \sin\theta}{N}(\Omega - N^\phi) = \mathbf{u} \frac{1}{\Gamma} \frac{\chi}{|\chi|}$ (w.r.t the local ZAMO).

Axisymmetric stationary rotating configuration

Using the equations of motion - conservation of energy-momentum and baryon number density

$$\nabla^{\mu} T_{\alpha\mu} = 0, \quad \nabla_{\mu}(n_b u^{\mu}) = 0$$

for a perfect fluid one obtains the relativistic Euler equation:

$$(\rho + P) u^{\mu} \nabla_{\mu} u_{\alpha} + (\delta^{\mu}_{\alpha} + u^{\mu} u_{\alpha}) \nabla_{\mu} P = 0.$$

Using the Gibbs-Duhem relation at $T = 0$ from previous considerations:

$$dP = n_b d\mu, \quad \text{where } \mu = d\rho/dn_b = \frac{\rho + P}{n_b}$$

and dividing the Euler equation by n_b we get

$$h u^{\mu} \nabla_{\mu} u_{\alpha} + (\delta^{\mu}_{\alpha} + u^{\mu} u_{\alpha}) \nabla_{\mu} h = 0, \quad \text{or} \quad u^{\mu} \nabla_{\mu}(h u_{\alpha}) + \nabla_{\alpha} h = 0.$$

Using $u^{\mu} u_{\mu} = -1$ we recover the vorticity 2-form (measure of local spinning motion of a fluid):

$$u^{\mu} \left(\nabla_{\mu}(h u_{\alpha}) - \nabla_{\alpha}(h u_{\mu}) \right) = 0$$

Bernoulli theorem and first integral of motion

From

$$u^\mu \nabla_\mu (h u_\alpha) + \nabla_\alpha h = 0,$$

by contracting with the ξ Killing vector, one obtains

$u^\mu \nabla_\mu (h u_\alpha \xi^\alpha) = 0.$ (the scalar $h\mathbf{u}\xi$ is constant along a given fluid line)

This is the Bernoulli theorem. The stationarity of the flow is related to the first integral of motion. In case of a rigidly rotating star:

$$H(r,\ \theta) + \nu(r,\ \theta) - \ln \Gamma(r,\ \theta) = H(0,\ 0) + \nu(0,\ 0) = const.$$

where $\nu = \ln(N)$ has a meaning of gravitational potential far from the star. For comparison, in Newtonian case one has, along a fluid line

$$H + \Phi - \frac{v^2}{2} = const.$$

Global quantities

Using the property of asymptotic flatness:

⋆ **Total mass-energy** (gravitational potential $\nu(r,\ \theta)|_{r\to+\infty} \to 0$, leading term $\nu(r,\ \theta) \sim -M/r$):

$$M := \int_{\Sigma_t} (2T_{\mu\nu} - Tg_{\mu\nu}) n^\mu \xi^\nu \sqrt{h} dx^3 = \int \frac{NA^6}{B} \left(E + S_i^i + \frac{2}{N} N^\phi p_\phi \right) r^2 \sin\theta dr d\theta d\phi$$

⋆ **Number of particles inside the star:**

$$A_\mathrm{B} := -\int_{\Sigma_t} \mathbf{nu} n_\mathrm{b} \sqrt{h} dx^3 = \int \frac{A^6}{B} \Gamma n_\mathrm{b} r^2 \sin\theta dr d\theta d\phi$$

⋆ **Total angular momentum:** Leading term in frame-dragging $N^\phi(r,\ \theta)|_{r\to+\infty} \sim -2J/r^3$):

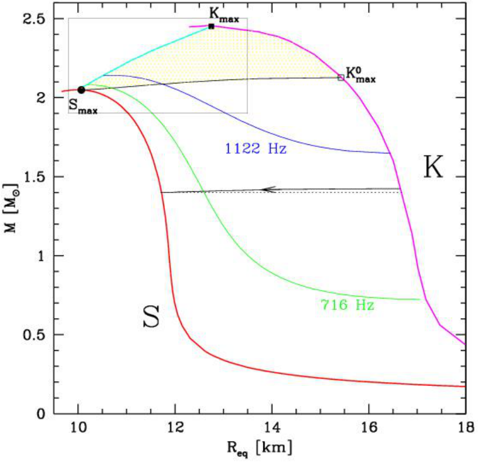
$$J := -\int_{\Sigma_t} T_{\mu\nu} n^\mu \chi^\nu \sqrt{h} dx^3 = \int \frac{A^6}{B} p_\phi r^2 \sin\theta dr d\theta d\phi$$

⋆ **Circumferential radius:**

$$R_\mathrm{eq} = A^2(r_\mathrm{eq},\ \pi/2) B(r_\mathrm{eq},\ \pi/2) r_\mathrm{eq}$$

Results related to fast rotation

Constraints from rotation on the $M(R)$ diagram

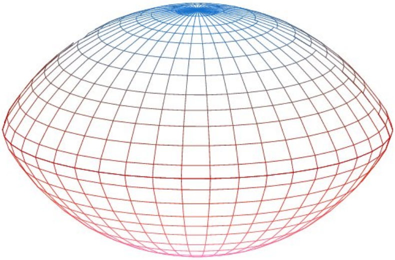


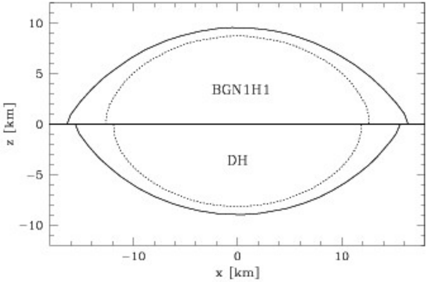
- ⋆ S: static configurations (TOV),
- ⋆ K: "Keplerian" (mass-shedding) configuration - maximally-rotating, rigid stars at a given mass,
- ⋆ in cyan: the instability line (star loses stability w.r.t. axisymmetric oscillations)

$$\left(\frac{\partial M}{\partial \xi}\right)_{J=const.} > 0,$$
$$\left(\frac{\partial M_b}{\partial \xi}\right)_{J=const.} > 0$$

where $\xi = \rho_c$ or P_c.

Constraints from rotation on the $M(R)$ diagram





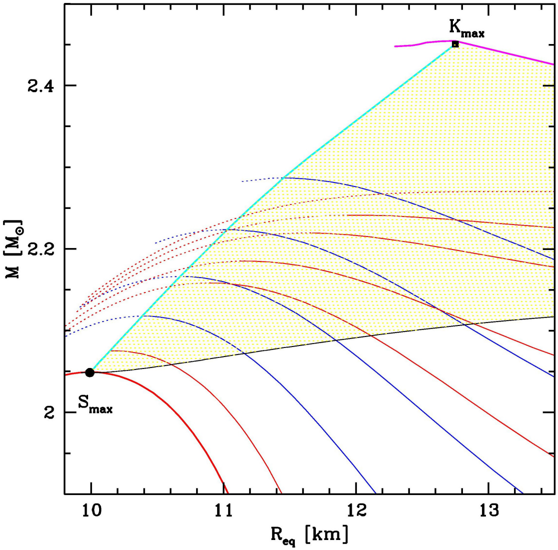
„Keplerian" rotation defined as

$$f_{rot} = f_{eq.},$$

where

- ⋆ $f_{eq.}$ is the orbital frequency of a particle on a circular orbit at the star's equator,
- ⋆ f_{rot} is star's rotational (spin) frequency.

Constraints from rotation on the $M(R)$ diagram

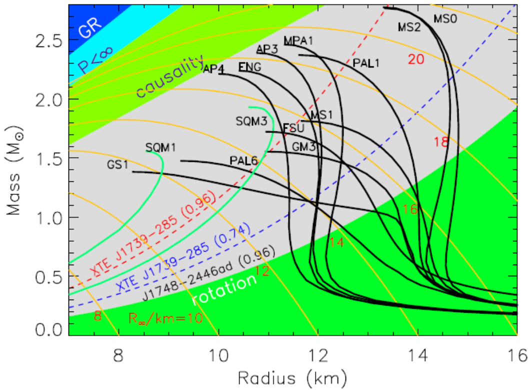


- ⋆ Constant angular momentum J lines,
- ⋆ Constant spin frequency $f = \Omega/2\pi$ lines - related phenomenon: spin-up near the instability.

Increase of the maximum mass:

- ⋆ Neutron stars $\simeq 15\%$,
- ⋆ Quark stars $\sim 30\%$.

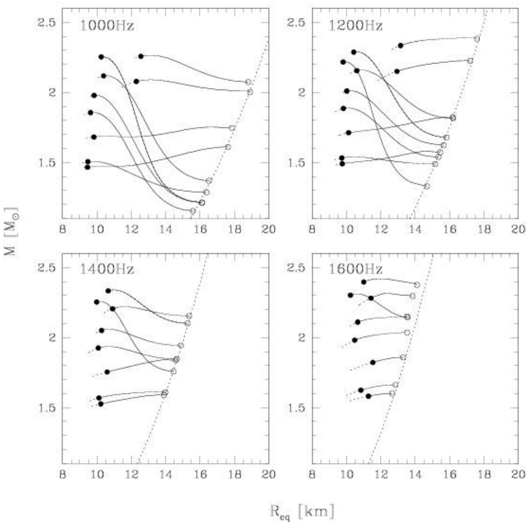
Possible (?) sub-millisecond rotation



- ⋆ $P_{min} \simeq (0.74 \pm 0.03) \left(\frac{M_\odot}{M_{sph.}}\right)^{1/2} \left(\frac{R_{sph}}{10km}\right)^{3/2}$ ms, (Koranda, Stergioulas & Friedman 1997)
- ⋆ $P_{min} \simeq 0.96 \left(\frac{M_\odot}{M_{sph.}}\right)^{1/2} \left(\frac{R_{sph.}}{10km}\right)^{3/2}$ ms, (Lattimer & Prakash 2004)
- ⋆ $f_{max}(M) \simeq 1.08 \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{R_{sph.}}{10km}\right)^{-3/2}$ kHz, (Haensel et al., 2008)

Possible (?) sub-millisecond rotation

Spin frequency of 1122 Hz inferred from bursts of XTE J1739-285 (not confirmed):



For $f > 1000$ Hz spin frequency:

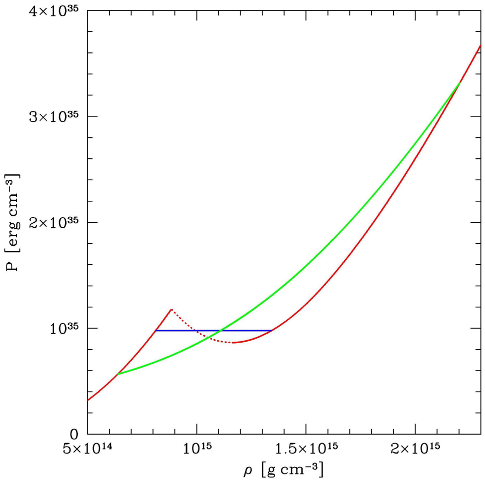
- ⋆ stellar mass M well-constrained,
- ⋆ mass-EOS-f dependence - masses separation for different models,
- ⋆ Maximal frequency strangely resembles Newtonian orbital frequency:

$$\Omega_k = 2\pi f_k = \left(\frac{GM}{R^3_{eq.}}\right)^{1/2}$$

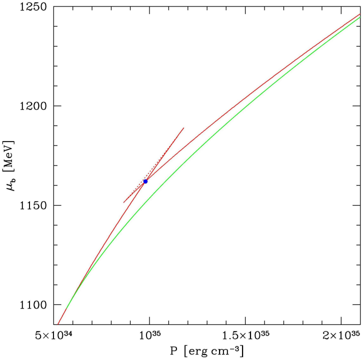
(dashed lines)

Possible phase transitions in rotating stars

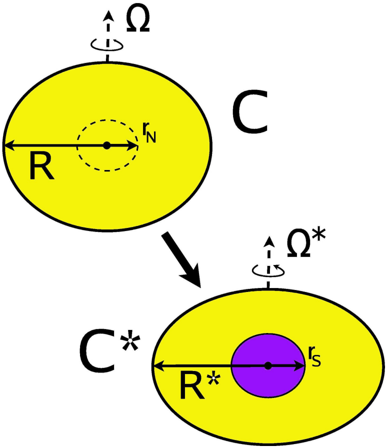
Maxwell & Gibbs phase transitions

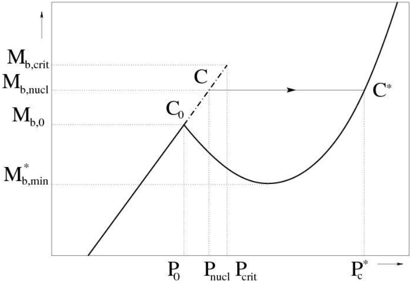


- ⋆ Maxwell construction - coexistence of two phases (density jump, e.g., when high surface tension),
- ⋆ Gibbs construction - mixed-phase.



Phase transition in a rotating star



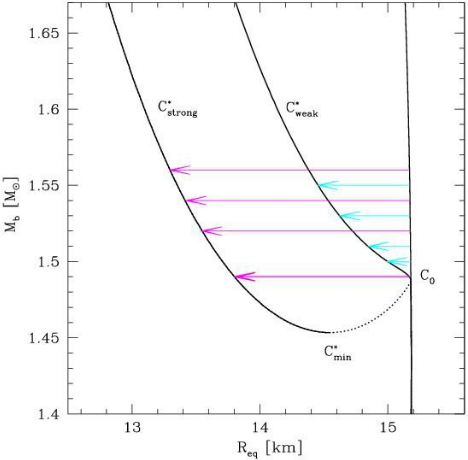


- ⋆ Increase of the central density during evolutionary processes (accretion or slowing down)
- ⋆ Formation of the metastable core in the center of the star - radius r_N, over-compression
- ⋆ Central pressure reaches P_{crit} and the configuration becomes unstable

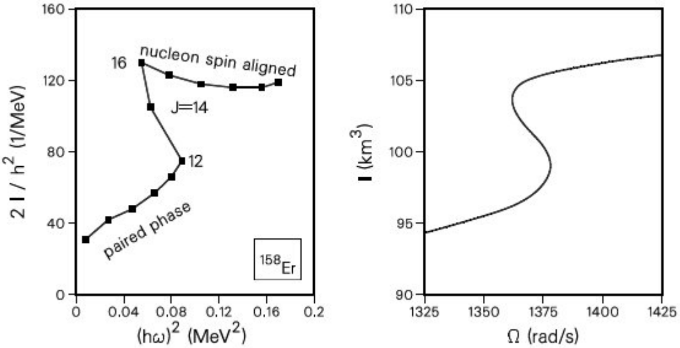
Phase transition in a rotating star

„Strong" and „weak" phase transitions, in case of the „density jump" phase transition, $\lambda = \rho_S/\rho_N$:

- ⋆ weak $\lambda < \lambda_{crit} := \frac{3}{2}\left(1 + P_0/\rho_N c^2\right)$,
- ⋆ strong $\lambda > \lambda_{crit}$.

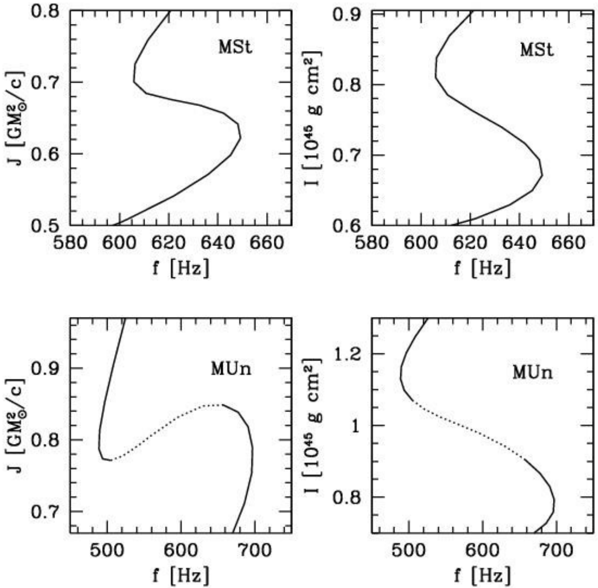


Back-bending in heavy nuclei and in neutron stars

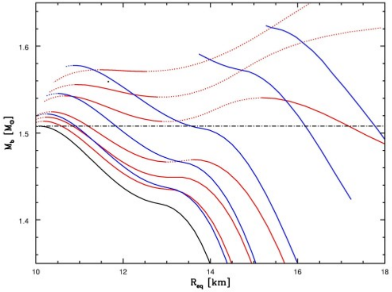


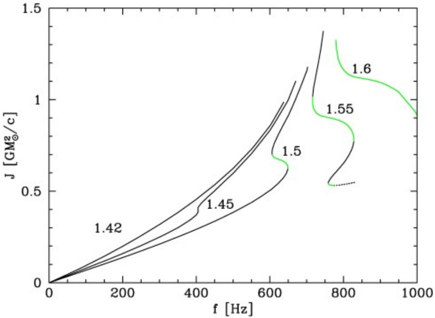
Observed for nuclei in terrestrial experiments, proposed for neutron stars (Glendening, Weber & Pei 1997)

Back-bending in heavy nuclei and in neutron stars



Softening of the EOS → back-bending of the star





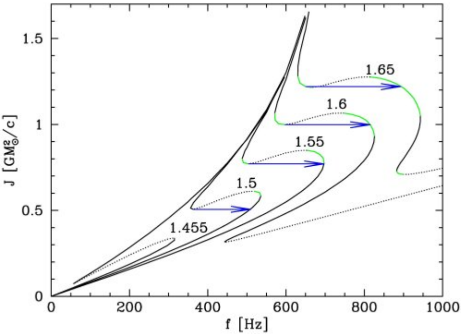
Back-bending for weak phase transitions

In this example, all configurations are stable

- ⋆ Red lines: constant Ω,
- ⋆ Blue lines: constant J (real indication of stability!),
- ⋆ Green lines: occurrence of back-bending.

Mini-collapse resulting from instability

On the $J(f)$ plane, in case of sufficiently-strong phase transition:

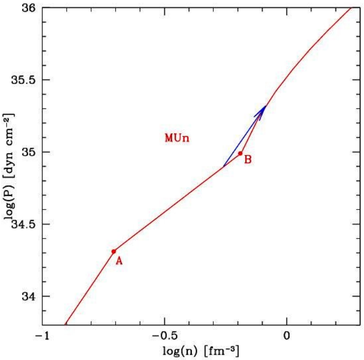


- ⋆ Star looses its angular momentum J
- ⋆ Onset of back-bending phenomenon (minimum of f)
- ⋆ The *instability point* is reached

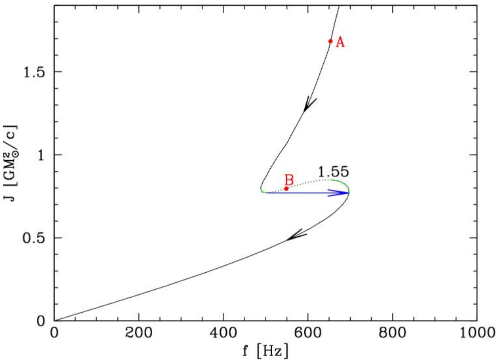
$$\left(\frac{\partial J}{\partial f}\right)_{M_{\rm B}} = 0$$

(turning point method, stability theory by Friedman, Ipser & Sorkin in the 80s)

Mini-collapse resulting from instability: EOS



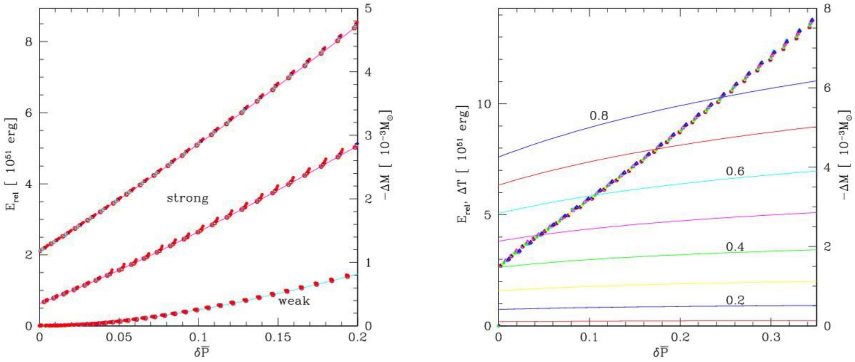
Test EOS with a softening (smaller adiabatic index γ): mixed-phase region



EOS must be sufficiently soft and then, sufficiently stiff to stabilize the star

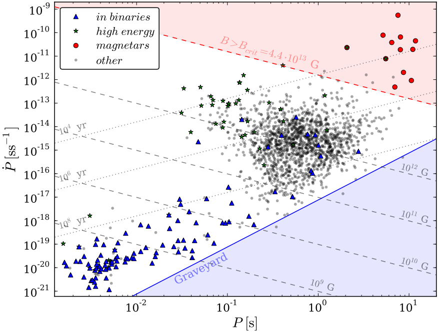
Energy release

Energy released (the difference in mass-energy), released due to the phase transition ($\Delta E = (M_C - M_{C*})c^2$):



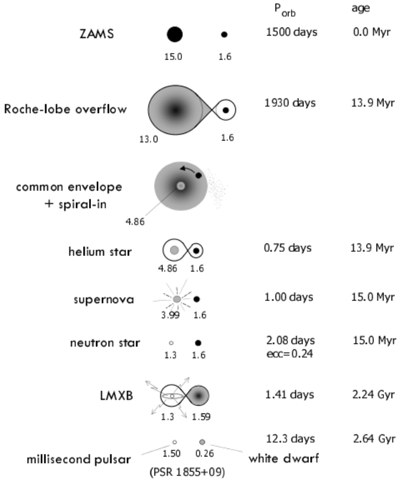
Moreover, it does not depend on the spin frequency!

Formation of millisecond pulsars



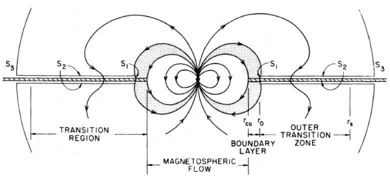
"Recycling" old pulsars to large frequencies by disk accretion. Open questions:

- ⋆ Star-disk interaction via the magnetic torque,
- ⋆ Magnetic field decay 10^{12} G $\rightarrow$ 10^{8} G,
- ⋆ Influence of $\vec{B}$ vs GW emission.

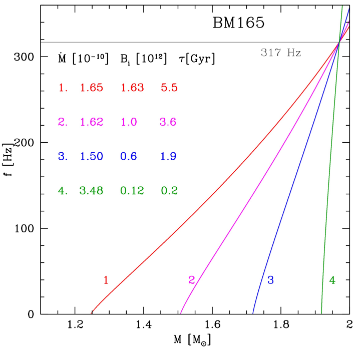


(Tauris and van der Heuvel, 2003)

Formation of millisecond pulsars



(Ghosh & Lamb 1979)



"Recycling" process ($\sim$ 0.5 *Gyr*) for pulsars with measured masses:

- ⋆ M_{ini} at the onset of accretion,
- ⋆ accretion with $\vec{B}$ results in much higher accreted masses,
- ⋆ test for SN core-collapse physics - extreme cases:
  J1614-2230 ($1.97 \pm 0.04\ M_\odot$),
  J0751+1807 ($1.26 \pm 0.14\ M_\odot$),
  (both $\simeq 3$ ms and $\simeq 10^8$ G.)

Innermost stable circular orbit

For a general 3+1 version of the metric (axisymmetric, stationary, asymptotically flat, and free of meridional currents) in a particular choice of gauge and coordinates (maximal-slicing quasi-isotropic coordinates)

$$ds^2 = -N^2 dt^2 + A^2(dr^2 + r^2 d\theta^2) + B^2 r^2 \sin^2\theta (d\phi - N^\phi dt)^2,$$

with A and B conformal factors ($A \equiv B$ in spherical symmetry). A test orbiting particle has two constants of motion of interest, energy $p_t = -E$ and angular momentum $p_\phi = l$. Using equation on p_r, one can define an effective radial potential

$$V^2 = N^2 \left(1 + \frac{l^2}{B^2 r^2}\right) + 2N^\phi El - {N^\phi}^2 l^2,$$

where the relation between the coordinate radius and the circumferential radius in the equatorial plane is

$$r_{circ} = Br,$$

and the physical velocity in the disk is (Bardeen 1970)

$$v = \frac{r_{circ}}{N}\left(\Omega - N^\phi\right).$$

Innermost stable circular orbit

With such defined effective potential

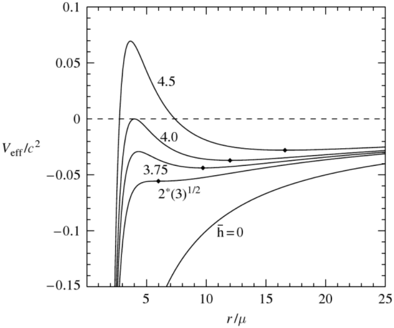
$$V^2 = N^2\left(1 + \frac{l^2}{B^2 r^2}\right) + 2N^\phi E l - {N^\phi}^2 l^2,$$

the conditions for circular orbits are

$$E^2 = V^2 \quad \text{and} \quad V_{,r} = 0.$$

Condition for the innermost stable circular orbit (ISCO) is

$$V_{,rr} = 0$$



Thin-disk accretion *without* the magnetic field

Accretion proceeds from the Innermost Stable Circular Orbit (ISCO).

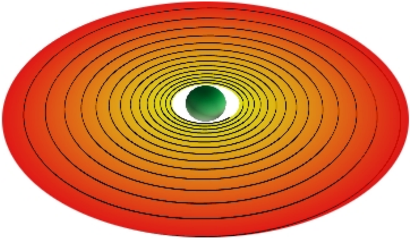
⋆ Evolution of total stellar angular momentum $J = I\Omega$ reads

$$\frac{dJ}{dM_B} = x_l l_{ISCO}, \quad x_l \in (0, 1)$$

($x_l \approx 1$ - Beckwith, Hawley & Krolik 2008, Shafee et al., 2008)

→ Ihe stellar baryon mass changes in time as

$$\Delta M_{\rm b}(t) = \int_{t_{\rm in}}^{t} \dot{M}_{\rm b}(t')\mathrm{d}t'.$$



Schwarzschild:
$r_{ISCO} = 3r_s = 6GM/c^2$.

Slow rotation:
$r_{ISCO} = 3r_s \left(1 - \frac{J}{M^2}\left(\frac{2}{3}\right)^{3/2}\right)$

Thin-disk accretion: magnetic field included

How to determine the radius r_0 from which the accretion *effectively* takes place:

- ⋆ the magnetospheric radius

$$r_m \equiv (GM)^{-1/7}\,\dot{M}^{-2/7}\mu^{4/7},$$

- ⋆ the corotation radius

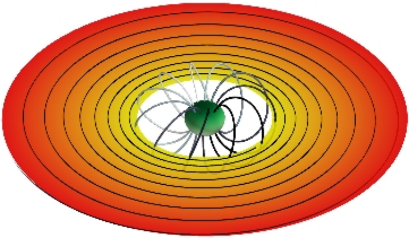
$$r_c \equiv \left(\frac{GM}{\omega_s^2}\right)^{1/3},$$

- ⋆ Torque models: Kluźniak & Rapapport (2007), Wang (1995), Aarons (1993), Ghosh & Lamb (1979). KR2007 solution for r_0:

$$\left(\frac{2}{\Omega r}\frac{dl}{dr}\right)_{r_0} = \left(\frac{r_m}{r_0}\right)^{7/2}\left(\sqrt{\frac{r_c^3}{r_0^3}} - 1\right)$$

→ Stellar angular momentum J evolution equation (μ, magnetic dipole):

$$\frac{dJ}{dM_B} = l(r_0) - \frac{\mu^2}{9r_0^3\dot{M}_B}\left(3 - 2\sqrt{\frac{r_c^3}{r_0^3}}\right)$$



- ⋆ Toroidal component of the magnetic field:
  $B_\phi = B_z\,(1 - \Omega/\omega_s)$,
- ⋆ Magnetic field decay[a]:

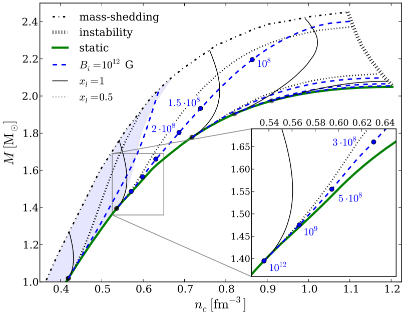
$$B = B_0/(1 + \Delta M/m_B)$$

with $m_B = 10^{-5} - 10^{-4}\ M_\odot$.

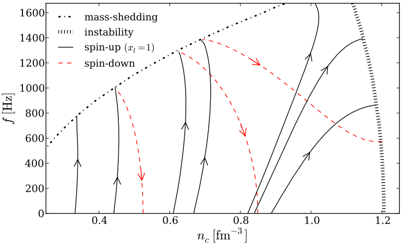
[a] van den Heuvel & Bitzaraki (1995), Taam & van den Heuvel (1986)

Accretion with $\vec{B}$ and central compression

Are dense matter phase transitions actually possible in accreting systems?



Spin-up with and without magnetic torque interacting with the disk.

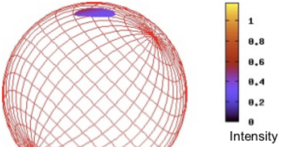


Spin-up versus spin-down and the central density change.

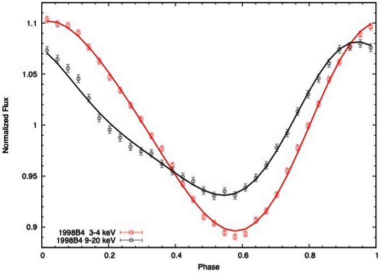
$M(R)$ from pulse profiles

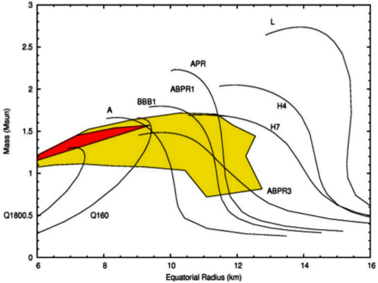
For sources that show pulsed radiation from the surface:

- ⋆ Model the emitting area (spot size, emission spectrum, beaming...),
- ⋆ Use relativistic ray-tracing in order to construct the light-curve,
  - ⋆ Light-deflection $\propto GM/Rc^2$,
  - ⋆ Rotational "Doppler" $\propto R\Omega/c$.
- ⋆ Compare with the real light-curve → "fit" the parameters of the model.



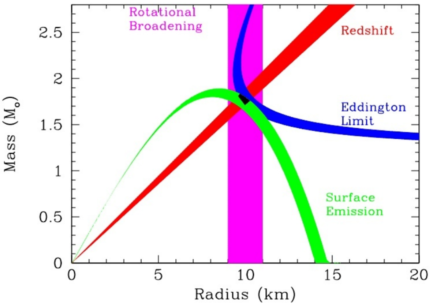
(Leahy et at., 2009)





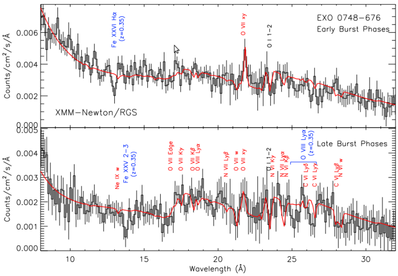
Bursts from the source SAX J1808

An example: case of EXO 0748-676



(Özel 2006)

- ★ the Eddington flux during photospheric radius expansion bursts,
- ★ the ratio F_{cool}/T_c^4 of the surface emission - asymptotes to a constant during the cooling after the thermonuclear burst,
- ★ the redshift of absorbtion lines during the burst,
- ★ line broadening due to rotation.



Cottam, Paerels, Mendez (2002)

- ★ $z = 0.35 \rightarrow M/M_\odot = 1.5R/10$ km (Cottam et al., 2002),
- ★ $M = 2.10 \pm 0.28 M_\odot$ and $R = 13.8 \pm 1.8$ km (1σ, Özel 2006),
- ★ data more consistent with $M = 1.35 M_\odot$ (Pearson et al. 2006),
- ★ $z = 0.35$ not confirmed with new data (Cottam et al., 2008).

Further reading...

- E. Gourgoulhon, "An introduction to the theory of rotating relativistic stars", arXiv:1003.5015
- N. Stergioulas, "Rotating stars in relativity", http://relativity.livingreviews.org/Articles/lrr-2003-3
- Spectral methods library LORENE (Langage Objet pour la RElativité NumériquE): http://www.lorene.obspm.fr
- LORENE school on spectral methods: http://www.lorene.obspm.fr/school