Introduction: equations of hydrodynamics

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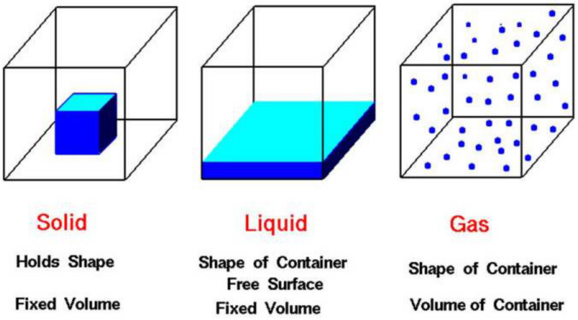
Outline

- ⋆ Derivation of the equations:
  - ⋆ from the Boltzmann equation,
  - ⋆ by other methods,
- ⋆ Regimes of applicability,
- ⋆ some remarks about the advection equation.

What is a fluid?

The difference between fluid, and other phases, gas and solid:

- ⋆ fluid flows,
- ⋆ not resists deformation (not elastic), though may be viscous,
- ⋆ forms free surface (gases don't)



Fluid description

- ⋆ system composed of many bodies, that can be described as a continuum,

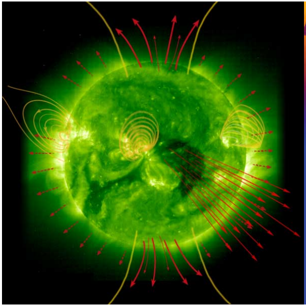
$$\text{mean free path } \lambda \ll \textit{fluid element size} \ll \text{system size}$$

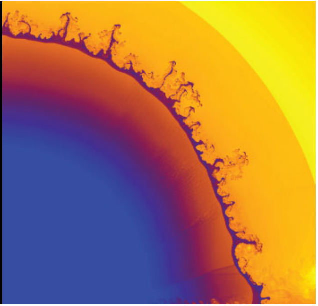
$$\text{In general } \lambda = (n\sigma)^{-1}$$

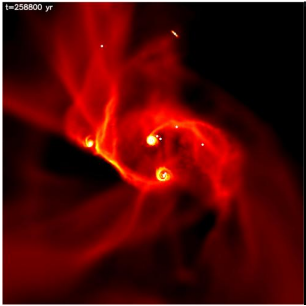
- ⋆ Forces between particles are short-range (saturation, screening).

Some examples:

	λ
Air	10^{-5} cm
water	10^{-9} cm
space	10^{15} cm



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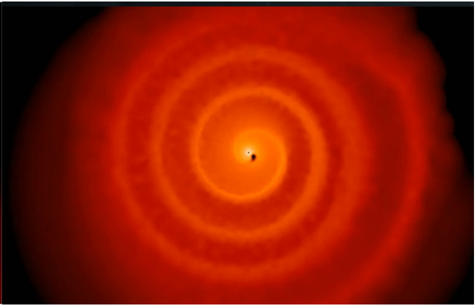
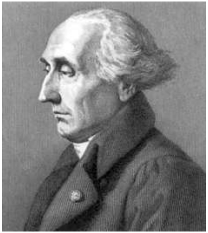


Euler and Lagrange viewpoints

Eulerian approach: An observer looks at fluid motion from a specific location in space through which the fluid flows as time passes.



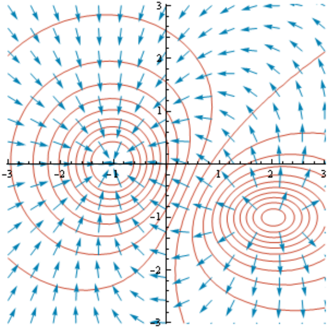
Lagrangian approach: An observer looks at fluid motion following an individual fluid parcel as it moves through space and time.

Some definitions

Vector operators: gradient

Gradient represents a rate of change of a scalar field (derivative) in space. It is a "vector" (technically a differential form):

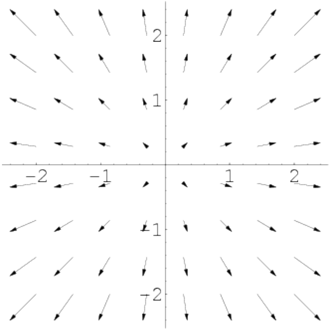
$$\nabla f = \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{e}_n$$



Vector operators: divergence

Divergence is the volume density of the outward flux from an infinitesimal volume around a given point. The result is a scalar:

$$\text{For } \mathbf{F} = F^i \mathbf{e_i}, \quad \nabla \cdot \mathbf{F} = \nabla_i F^i = \frac{\partial F^i}{\partial x^i} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

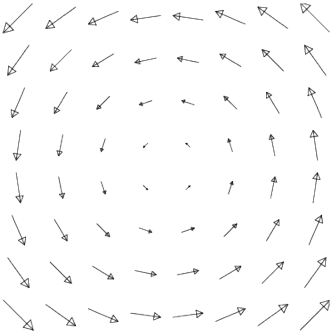


Vector operators: curl

Curl describes the infinitesimal rotation of a 3-dimensional vector field. The result is a vector:

$$\nabla \times \mathbf{F} = \boldsymbol{e}_i \epsilon^{ijk} \partial_j F_k$$

$$\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$



Material/substantial derivative

Fluid properties are generally functions of position and time. We will denote

- $\star$ $\partial/\partial t$ the rate of change w.r.t. time at some *fixed position*.
- $\star$ $\mathrm{D}/\mathrm{D}t$ the rate of change w.r.t. time while traveling with a fluid element.

$\frac{\mathrm{D}}{\mathrm{D}t}$ is called a material/substantial derivative. For some quantity f (vector field, scalar field etc.), it equals

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f,$$

Why like that? Imagine we follow a change in f over a short time δt. Fluid element moved from $\mathbf{r}$ at t to $\mathbf{r} + \mathbf{u}\delta t$ at $t + \delta t$

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \lim_{\delta t \to 0} \frac{f(\mathbf{r} + \mathbf{u}\delta t, t + \delta t) - f(\mathbf{r}, t)}{\delta t}$$

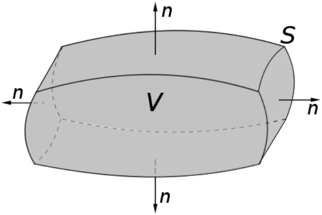
Continuity equation

Mass conservation: the rate of change of fluid mass inside a volume equals to the net rate of fluid flow into the volume. Often expressed as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Continuity equation

Consider volume V enclosed by a surface S, with $\mathbf{n}$ normal vector. Total mass of fluid in V is



$$\int_V \rho dV \quad \text{and} \quad \frac{d}{dt}\int_V \rho dV = -\int_S (\rho\mathbf{u})\cdot\mathbf{n} dS$$

(mass flux across S).

$$\frac{d}{dt}\int_V \rho dV \underset{V\ fixed}{=} \int_V \frac{\partial \rho}{\partial t} dV \underset{Stokes}{=} -\int_V \nabla\cdot(\rho\mathbf{u})dV$$

True for all $V \rightarrow$ the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \quad \text{also written as} \quad \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{u} = 0$$

Continuity equation

Example: advection equation from continuity equation.

$$\text{Advection operator } \mathbf{u} \cdot \nabla = u_i \frac{\partial}{\partial x^i}$$

We obtain the advection equation

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla(\rho) = 0$$

from the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

by assuming $\nabla \cdot \mathbf{u} = 0$ ($\mathbf{u}$ solenoidal, incompressible, divergence-free vector; $\mathbf{u} \equiv \nabla \times \mathbf{A}$)

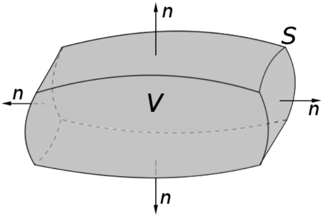
Momentum equation (equation of motion)

Momentum conservation: the rate of change of total fluid momentum in some volume equals to the sum of forces acting on the volume. For an inviscid fluid, it's called the Euler equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla \cdot \mathbf{u} = -\nabla P + \rho \mathbf{f}$$

Momentum equation (equation of motion)

Consider volume V *moving with the fluid* (no flow through the boundary S). Fluid momentum is



$$\int_V \rho \mathbf{u} dV. \quad \text{Its rate of change,} \quad \frac{d}{dt}\int_V \rho \mathbf{u} dV$$

equals to the force acting on V.

Two types of forces - *body* and *surface*:

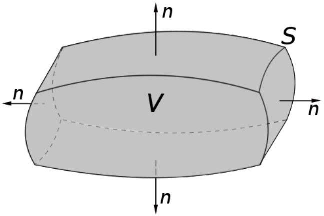
$$\underbrace{\int_V \rho \mathbf{f} dV}_{body\ \ force,\ e.g.,\ gravity} \quad \text{and} \quad \underbrace{-\int_S P\mathbf{n} dS}_{surface\ \ force} \quad \text{(inviscid fluid, force normal to S)}$$

$$\frac{d}{dt}\int_V \rho \mathbf{u} dV \underset{\rho dV\ cons.}{=} \int_V \rho \frac{D\mathbf{u}}{Dt} dV = \int_V \rho \mathbf{f} dV - \int_S P\mathbf{n} dS$$

$$\underset{Stokes}{=} \int_V \rho \mathbf{f} dV - \int_V \nabla P dV$$

Momentum equation (equation of motion)

Consider volume V *moving with the fluid* (no flow through the boundary S). Fluid momentum is



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$$\frac{d}{dt}\int_V \rho \mathbf{u} dV \underset{\rho dV \ cons.}{=} \int_V \rho \frac{D\mathbf{u}}{Dt} dV = \int_V \rho \mathbf{f} dV - \int_S P\mathbf{n} dS$$

$$\underset{Stokes}{=} \int_V \rho \mathbf{f} dV - \int_V \nabla P dV$$

We recover the Euler equation:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla \cdot \mathbf{u} \right) = -\nabla P + \rho \mathbf{f}$$

Momentum equation (equation of motion)

In case of viscosity, momentum equation is called the Navier-Stokes equation. Consider the $i-$th component of the surface force;

$$-\int_S Pn^i dS \quad \text{becomes} \quad \int_S \sigma_{ij} n^j dS$$

with the *stress tensor* σ_{ij}. For simple liquids and gases, with μ called the *dynamical viscosity*, and the *second viscosity* $-2/3\mu$:

$$\sigma_{ij} = -P\delta_{ij} + 2\mu \left(\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{Strain\ \ rate\ \ tensor} - \frac{1}{3}\left(\nabla \cdot \mathbf{u}\right)\delta_{ij} \right),$$

$$\int_S \sigma_{ij} n^j dS = \int_V \frac{\partial}{\partial x_j}\sigma_{ij} dV \quad \text{(Stokes again)}$$

Assuming $\mu = const.$, we finally arrive at

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P + \rho\mathbf{f} + \mu\left(\nabla^2\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\right).$$

Conserved quantities: energy

Energy conservation: the rate of change of the total energy equals the rate at which work and other sources of energy act at the fluid element:

$$\rho\left(\frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}^2\right)+\frac{DU}{Dt}\right)=-\nabla\cdot(P\mathbf{u})+\rho\mathbf{u}\cdot\mathbf{f}+\rho\epsilon-\nabla\cdot\mathbf{F}.$$

Conserved quantities: energy

Let's multiply the momentum equation by $\mathbf{u}$:

$$\rho \mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} = \rho \frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}^2\right) = -\mathbf{u} \cdot \nabla P + \rho \mathbf{u} \cdot \mathbf{f}$$

Rate of change of the kinetic energy equals the rate at which work is done by forces acting at the fluid element.
Similarly, equation for the *total energy*, *kinetic* $\mathbf{u}^2/2$ + *internal* U is

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\mathbf{u}^2 + U\right) dV = \int_V \rho \mathbf{u} \cdot \mathbf{f} dV - \int_S \mathbf{u} \cdot (P\mathbf{n}) dS$$
$$+ \underbrace{\int_V \rho \epsilon dV}_{generated\ at\ rate\ \epsilon} - \underbrace{\int_S \mathbf{F} \cdot \mathbf{n} dS.}_{heat\ flux\ across\ S}$$

Conserved quantities: energy

Let's multiply the momentum equation by **u**:

$$\rho \mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} = \rho \frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}^2\right) = -\mathbf{u} \cdot \nabla P + \rho \mathbf{u} \cdot \mathbf{f}$$

Rate of change of the kinetic energy equals the rate at which work is done by forces acting at the fluid element.
Similarly, equation for the *total energy*, *kinetic* $\mathbf{u}^2/2$ + *internal* U is

$$\rho\left(\frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}^2\right) + \frac{DU}{Dt}\right) = -\nabla \cdot (P\mathbf{u}) + \rho \mathbf{u} \cdot \mathbf{f} + \rho\epsilon - \nabla \cdot \mathbf{F}.$$

Equation for internal energy U only (subtracting the kinetic part):

$$\frac{DU}{Dt} = \frac{p}{\rho^2}\frac{D\rho}{Dt} + \epsilon - \frac{1}{\rho}\nabla \cdot \mathbf{F}.$$

Since $V = \rho^{-1}$, we recover the *first law of thermodynamics*:

$$dU = -PdV + \underbrace{\delta Q.}_{added\ \ heat}$$

Virial theorem

Velocity is the rate of change of position following the fluid: $\mathbf{u} = \frac{D\mathbf{r}}{Dt}$.

Euler equation for gravitational field: $\rho \frac{D^2\mathbf{r}}{Dt^2} = -\nabla P + \rho \mathbf{f} \underset{\mathbf{f}=-\nabla\psi}{=} -\nabla P - \rho\nabla\psi$

Let's multiply by $\mathbf{r}$ and integrate over a volume V:

$$\int_V \mathbf{r} \cdot \frac{D^2\mathbf{r}}{Dt^2} \rho dV = -\int_V \mathbf{r} \cdot \nabla P dV - \int_V \mathbf{r} \cdot \nabla\psi\rho dV.$$

$$\text{LHS: } \int_V \mathbf{r} \cdot \frac{D^2\mathbf{r}}{Dt^2} \rho dV = \frac{d}{dt}\int_V \mathbf{r} \cdot \frac{D\mathbf{r}}{Dt} \rho dV - \int_V \left(\frac{D\mathbf{r}}{Dt}\right)^2 \rho dV$$

$$= \frac{1}{2}\frac{d^2}{dt^2}\int_V |\mathbf{r}|^2 \rho dV - 2\mathcal{T}.$$

Total kinetic energy $\mathcal{T} = \frac{1}{2}\int_V \rho \mathbf{u}^2 dV$

Virial theorem

Pressure term:

$$-\int_V \mathbf{r}\cdot\nabla P dV = -\int_V \nabla\cdot(\mathbf{r}P)dV + \overbrace{\int_V P\nabla\cdot\mathbf{r}dV}^{\nabla\cdot\mathbf{r}=3} = -\underbrace{\int_S P\mathbf{r}\cdot\mathbf{n}dS}_{=0} + 3\int_V PdV$$

Also, define total gravitational energy: $-\int_V \mathbf{r}\cdot\nabla\psi\rho dV = \frac{1}{2}\int_V \psi\rho(\mathbf{r})dV = \Psi$,

and moment of inertia: $\mathcal{I} = \int_V \rho\mathbf{r}^2 dV$,

to arrive at the scalar form of the virial theorem:

$$\frac{1}{2}\frac{d^2\mathcal{I}}{dt^2} = 2\mathcal{T} + 3\int_V PdV + \Psi.$$

Distribution function

Particle's *distribution function* is a function of positions, velocities and time, $f(x, y, z, v_x, v_y, v_z, t)$, which gives the number of particles per unit volume in single-particle **phase space**.

$$\text{Number density } n(x, y, z, t) = \int f \, dv_x \, dv_y \, dv_z$$

$$\text{Total number of particles } N(t) = \int n \, dx \, dy \, dz.$$

$$\text{Normal distribution: } f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Maxwell-Boltzmann *velocity* probability distribution:

$$f = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2kT}\right)$$

describes particle speeds in idealized gases where the particles move freely inside a stationary container without interacting with one another, except for very brief collisions in which they exchange energy and momentum with each other.

Boltzmann's Equation - Derivation

For an ensemble of non-interacting point-like particles in momentum phase-space; *Liouville's theorem* states that the distribution function f is conserved along the trajectories in phase space of positions q and momenta p:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial f}{\partial p_i}\frac{\partial p_i}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i = 0$$

$\dot{p}^i = F^i$ is the "external" force, e.g., gravity, depending only on macroscopic quantities.

Collision integral. In case of collisions, phase space evolves:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i = \left(\frac{\partial f}{\partial t}\right)_{col.} \neq 0$$

Boltzmann's equation: collision integral

The form of the collision integral can be approximated by assuming:

- ⋆ Only short-range two-particle interactions should be important,
- ⋆ Collisions are elastic,
- ⋆ Collision time should be negligible.
- ⋆ No correlations between the incoming and outgoing particles.

This is *Stosszahlansatz*, (molecular chaos) assumption.

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{col.}} = \iint g \frac{d\sigma}{d\Omega}(g,\Omega)[f(\mathbf{p}'_A,t)f(\mathbf{p}'_B,t) - f(\mathbf{p}_A,t)f(\mathbf{p}_B,t)]\, d\Omega\, d^3\mathbf{p}_A,$$

where $g = |\mathbf{p}_B - \mathbf{p}_A| = |\mathbf{p}'_B - \mathbf{p}'_A|$ and $d\sigma/d\Omega$ is the differential cross-section.
In case of conserved quantities (mass, momentum, energy) the collision integral doesn't provide a contribution to averaged quantities.

Momenta of the distribution function

$$\text{Density } \rho = \int mfd^3p$$

$$\text{Momentum } \rho\mathbf{u} = \int m\mathbf{v}fd^3p$$

$$\text{Internal energy } \rho\epsilon = \int \frac{m}{2}(\mathbf{v}-\mathbf{u})^2fd^3p$$

($\mathbf{v} = \mathbf{p}/m$, $\mathbf{u}$ is mean flow)

Continuity equation

We start with $\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i = \left(\frac{\partial f}{\partial t}\right)_{col.}$

Let's multiply by a particle mass m and integrate in momentum space:

Continuity equation

We start with $\dfrac{df}{dt} = \dfrac{\partial f}{\partial t} + \dfrac{\partial f}{\partial q_i}\dot{q}_i + \dfrac{\partial f}{\partial p_i}\dot{p}_i = \left(\dfrac{\partial f}{\partial t}\right)_{col.}$

Let's multiply by a particle mass m and integrate in momentum space:

$$\int m\frac{\partial f}{\partial t}d^3p + \int mv_i\frac{\partial f}{\partial q_i}d^3p + \int m\frac{\partial f}{\partial p_i}\dot{p}_i d^3p = \underbrace{\int m\left(\frac{\partial f}{\partial t}\right)_{col.} d^3p}_{=0,\ local\ \ mass\ \ cons.}$$

$$\text{with } \dot{p}_i = F_i, \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q_i}\int mfv^i d^3p - \underbrace{m\int_S \mathbf{n}\cdot\mathbf{F} f dS}_{f\to 0\ \ for\ \ V\to\infty} = 0$$

We recover the continuity equation for a mean fluid velocity, $u^i = \langle v^i \rangle$ of a fluid element:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\langle v^i\rangle)}{\partial q_i} = 0$$

Momentum equation

Let's multiply the Boltzmann equation by $m\mathbf{v}$:

$$\int mv^j \frac{\partial f}{\partial t} d^3p + \int mv^j v^i \frac{\partial f}{\partial q_i} d^3p + \int mv^j F^i \frac{\partial f}{\partial p_i} d^3p = \underbrace{\int mv^j \left(\frac{\partial f}{\partial t}\right)_{col.} d^3p}_{=0,\ local\ \ mom.\ \ cons.}$$

$$\frac{\partial(\rho\langle v^j\rangle)}{\partial t} + \frac{\partial(\rho\langle v^i v^j\rangle)}{\partial q_i} + F^i \int_V p^j \frac{\partial f}{\partial p_i} d^3p = 0$$

$$\int_V p^j \frac{\partial f}{\partial p_i} d^3p = \int_V \frac{\partial p^j f}{\partial p_i} d^3p - \int_V \frac{\partial p^j}{\partial p_i} f d^3p = -\underbrace{\int_S f p^i n_i dS}_{=0} - \rho\delta_i^j$$

So we obtain, $$\frac{\partial(\rho\langle v^j\rangle)}{\partial t} + \frac{\partial(\rho\langle v^i v^j\rangle)}{\partial q_i} - \rho F^i \delta_i^j = 0$$

Momentum equation

$$\frac{\partial(\rho\langle v^j\rangle)}{\partial t} + \frac{\partial(\rho\langle v^i v^j\rangle)}{\partial q_i} - \rho F^i \delta_i^j = 0$$

By subtracting the continuity equation multiplied by $\langle v^j\rangle$,

$$\frac{\partial(\rho\langle v^j\rangle)}{\partial t} + \frac{\partial(\rho\langle v^i v^j\rangle)}{\partial q_i} - \rho F^j - \left(\langle v^j\rangle\frac{\partial\rho}{\partial t} + \langle v^j\rangle\frac{\partial(\rho\langle v^i\rangle)}{\partial q_i}\right) = 0$$

we obtain Navier-Stokes equation:

$$\frac{\partial(\langle v^j\rangle)}{\partial t} + \langle v^i\rangle\frac{\partial(\langle v^j\rangle)}{\partial q_i} = F^j + \frac{1}{\rho}\frac{\partial\sigma^{ij}}{\partial q_i},$$

with the stress tensor

$$\sigma^{ij} = -\rho\langle v^i v^j\rangle + \rho\langle v^i\rangle\langle v^j\rangle.$$

In case of collisions,

$$\sigma^{ij} = -P\delta_i^j + \text{viscous stress tensor}$$

Closure relations

Since hydrodynamic equations are under-determined, one needs to close the set of equations with the stress tensor, a function of ρ, P and ϵ:

- ⋆ Euler equations correspond to Maxwell distribution, $\sigma_{ij} = -P\delta_{ij}$ (neglect viscous stresses and heat conduction),
- ⋆ Chapman–Enskog theory: deviation from the Maxwell–Boltzmann distribution in the equilibrium is small, and the first order corrections are

$$\sigma_{ij} = 2\mu \left(\frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{1}{3} \left(\nabla \cdot \mathbf{u} \right) \delta_{ij} \right),$$

and the heat conduction term in the energy equation,

$$\mathbf{F} = \kappa \nabla T$$

with shear viscosity $\mu \propto \sqrt{mkT}$ and thermal conductivity $\kappa = \frac{5}{2} c_V \mu$.

Dissipation in astrophysics

An estimate for the effects of viscosity, heat conduction and other effects is provided by some dimensionless numbers:

- ⋆ Knudsen number (are we in the fluid regime?):

$$Kn = \frac{\text{mean free path}}{\text{size of the system}} = \frac{\lambda}{l}$$

- ⋆ The Reynolds number ($\rightarrow$ dimensionless Navier-Stokes):

$$Re = \frac{\text{inertial forces}}{\text{viscous forces}} = \frac{\rho u l}{\mu} = \frac{ul}{\nu}$$

- ⋆ Péclet number (how the heat is transported):

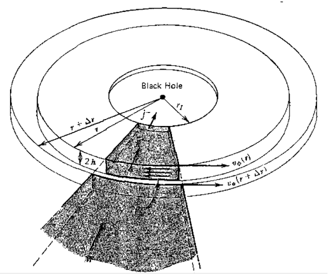
$$Pe = \frac{\text{advective transport rate}}{\text{diffusive transport rate}} = \frac{ul}{\alpha}$$

- ⋆ Prandtl number (momentum-to-thermal diffusivity):

$$Pr = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}} = \frac{\nu}{\alpha}$$

(kinematic viscosity $\nu = \mu/rho$, thermal diffusivity $\alpha = k/(\rho c_P)$

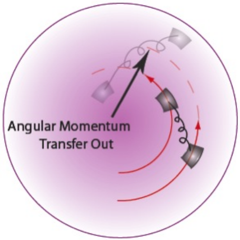
Viscous transport in disks

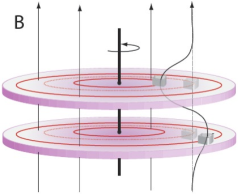


Viscosity in action - what makes the accretion so efficient?

α-viscosity prescription:

- ⋆ $\nu \propto \alpha c_s H$,
- ⋆ stress tensor in the disk, torque $T_{r\phi} \propto \rho \nu r \frac{\partial \Omega}{\partial r} \propto -\alpha P$ (at least for Shakura-Sunyaev disks).





Perhaps magnetic field plays a role? Magnetorotational instability (Balbus-Hawley)

Simplifying approximations

We have several simple flows at our disposals:

- ⋆ **Incompressible**: $\nabla \cdot \mathbf{u} = 0$, equivalent to $\rho = const.$
- ⋆ **Anelastic**: $\nabla \cdot (\rho \mathbf{u}) = 0$, equivalent to $\partial \rho / \partial t = 0$ (see continuity equation)
- ⋆ **Barotropic**: $P = P(\rho)$ instead of $P = P(\rho, T \ldots)$, which is called baroclinic. It can result from the equation of state (degeneracy, e.g. for white dwarfs and neutron stars) or may be incidental, e.g. when convection establishes $s = const.$ throughout a star.
- ⋆ **Adiabatic**: heat transfer is neglected. Simplified energy relation $d\rho s/dt = 0$

Inviscid and adiabatic flow

When neglecting viscosity and heat conduction, the momentum and energy equations are simplified. The Navier-Stokes equation reduces to the Euler equation:

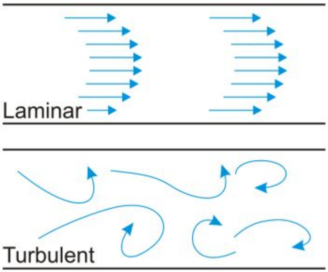
$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \rho \mathbf{f} + \mu \left(\nabla^2 \mathbf{u} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}) \right)$$

$$\rho \left(\frac{D}{Dt} \left(\frac{1}{2} \mathbf{u}^2 \right) + \frac{DU}{Dt} \right) = -\nabla \cdot (P\mathbf{u}) + \rho \mathbf{u} \cdot \mathbf{f} + \rho \epsilon - \nabla \cdot \mathbf{F}$$

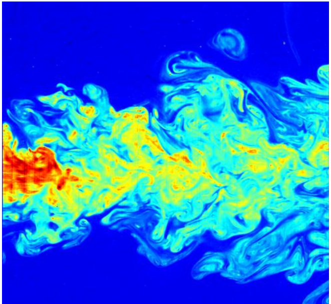
The energy equation can be rewritten as a conservation of *entropy*:

$$\frac{D\rho s}{Dt} + \nabla \cdot (\rho s \mathbf{u}) = 0.$$

Laminar and Turbulent Flows



- ⋆ Flow in which the kinetic energy decays due to the action of viscosity is called laminar flow.
- ⋆ Large Re $>$ 100-1000 $\rightarrow$ turbulent flows (often seen in astrophysical settings).
- ⋆ Problem: Resolution of turbulent flows down to the length scale where viscosity becomes important is not feasible (MRI etc.)



Plasma hydrodynamics

- ⋆ Astrophysical fluids are quite often charged (ionized), so the neutrality condition is sometimes not satisfied,
- ⋆ There is a limit for the mean free path λ: Coulomb scattering, Debye length (screening),

$$\lambda_D = \sqrt{\frac{k_B/e^2}{n_e/T_e + \sum_i Z_i^2 n_i/T_i}}$$

  for $l \geq \lambda_D$ fluid may be considered neutral,
- ⋆ If macroscopic process slower than the plasma frequency timescale (Langmuir waves, electron density fluctuations),

$$\omega_{pe} = \sqrt{\frac{4\pi n_e e^2}{m_e}}$$

- ⋆ Effect of macroscopic magnetic fields (magnetospheres of pulsars) changes the mean free path (preferred directions, magnetic field gyration).

Magnetohydrodynamics

In case of magnetic fields, even when the fluid is electrically neutral:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \nabla \cdot \mathbf{u} = -\nabla (P + \frac{1}{8\pi} \mathbf{B}^2) + \rho \mathbf{f} + \underbrace{\frac{1}{4\pi} \mathbf{j} \times \mathbf{B}}_{Lorentz\ \ force}$$

$$\text{Maxwell equations:} \quad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{E} = \rho_e,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = 4\pi \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

+ Equation for fluid energy evolution.

Ideal MHD approximation (primes denote the fluid co-moving quantities):

- ⋆ $\mathbf{j}' = \sigma \mathbf{E}' \simeq \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B})$
- ⋆ $\frac{\partial \mathbf{E}}{\partial t} \simeq 0 \rightarrow \mathbf{j} = \frac{1}{4\pi} \nabla \times \mathbf{B}$
- ⋆ Magnetic induction equation: $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$, with magnetic diffusivity η.

Partial differential equations

Types of equations

For linear equations in two dimensions (x and t, say), one may classify the general one

$$a\frac{\partial^2 F}{\partial x^2} + 2b\frac{\partial^2 F}{\partial x \partial t} + c\frac{\partial^2 F}{\partial t^2} + d\frac{\partial F}{\partial x} + e\frac{\partial F}{\partial t} + fF + g = 0$$

For example:

$b^2 < ac$	Elliptic	Laplace equation $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial t^2} = 0$
$b^2 > ac$	Hyperbolic	Wave equation $\frac{\partial^2 F}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = 0$
$b^2 = ac$	Parabolic	Diffusion $\frac{\partial^2 F}{\partial x^2} - \frac{\partial F}{\partial t} = 0$

Naming conventions

Differences between elliptic and hyperbolic/parabolic equations:

- ⋆ elliptic equations have boundary conditions which are specified around a closed boundary,
- ⋆ derivatives are with respect to spatial variables (e.g., Laplace or Poisson),
- ⋆ for hyperbolic - boundary conditions for time variable are initial conditions.

Hyperbolic equation - wave equation

Consider a following equation, $\dfrac{\partial^2 y}{\partial t^2} - c^2 \dfrac{\partial^2 y}{\partial x^2} = 0.$

It can be rewritten as $\left(\dfrac{\partial}{\partial t} + c \dfrac{\partial}{\partial x} \right) \left(\dfrac{\partial}{\partial t} - c \dfrac{\partial}{\partial x} \right) y = 0,$

or re-casted as a system of two first-order equations,

$$\frac{\partial z}{\partial t} + c \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial y}{\partial t} - c \frac{\partial y}{\partial x} = z$$

This are examples of advection equations,

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

with boundary conditions $u(x, t) = u_0(x)$ for $t = t_0$.

Advection equation: simple attempt

Let's try a centred difference for the space derivative (subscript j) and Euler's method for the time derivative (superscript n):

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} \quad \rightarrow \quad \frac{u_j^{n+1} - u_j^n}{\delta t} = -c\frac{u_{j+1}^n - u_{j-1}^n}{2\delta x}$$

As expected, time part is 1-st order accurate:

$$u_j^{n+1} - u_j^n \approx \delta t \left.\frac{\partial u}{\partial t}\right|_j^n + \frac{1}{2}\delta t^2 \left.\frac{\partial^2 u}{\partial t^2}\right|_j^n + \ldots$$

Space part is 2-nd order:

$$u_{j+1}^n - u_{j-1}^n \approx 2\delta x \left.\frac{\partial u}{\partial t}\right|_j^n + \frac{1}{3}\delta x^3 \left.\frac{\partial^3 u}{\partial t^3}\right|_j^n + \ldots$$

Advection equation: simple attempt

Substituting the Taylor expansions in the approximation:

$$\delta t \left.\frac{\partial u}{\partial t}\right|_j^n + \frac{1}{2}\delta t^2 \left.\frac{\partial^2 u}{\partial t^2}\right|_j^n + \cdots \approx \frac{c\delta t}{2\delta x}\left(2\delta x \left.\frac{\partial u}{\partial t}\right|_j^n + \frac{1}{3}\delta x^3 \left.\frac{\partial^3 u}{\partial t^3}\right|_j^n + \ldots\right.$$

- the truncation errors are 2nd order in time and 3rd order in space.

How about the stability in time? Test solution, plane wave $v_j^n = v^n \exp(ikx_j)$ (von Neumann stability condition):

$$v^{n+1}\exp(ikx_j) = v^n \exp(ikx_j) - \frac{c\delta t}{2\delta x} v^n \left(\exp(ikx_{j+1}) - \exp(ikx_{j-1})\right),$$

For small deviation, $v^n \to v^n + \delta v^n$,

$$\delta v^{n+1} = \underbrace{\left(1 - i\frac{c\delta t}{\delta x}\sin(k\delta x)\right)}_{growth\ \ factor\ \ S}\delta v^n$$

$S = 1 - i\alpha$, so square of norm $|S|^2 = 1 + \alpha^2 \le 1$ for all wave vectors k $\to$ Problem.

Advection equation: Lax method

Minor improvement:

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} \quad \rightarrow \quad \frac{u_j^{n+1} - (u_{j+1}^n - u_{j-1}^n)/2}{\delta t} = -c\frac{u_{j+1}^n - u_{j-1}^n}{2\delta x}$$

Let's check the stability, again using the von Neumann condition:

$$\delta v^{n+1} = \underbrace{\left(\cos(k\delta x) - i\frac{c\delta t}{\delta x}\sin(k\delta x)\right)}_{growth\ \ factor\ \ S}\delta v^n,$$

so the growth factor

$$|S|^2 = \cos^2(k\delta x) + \left(\frac{c\delta t}{\delta x}\right)^2 \sin^2(k\delta x) = 1 - \sin^2(k\delta x)\left(1 - \left(\frac{c\delta t}{\delta x}\right)^2\right).$$

This means stability for all k as long as

$$\frac{\delta x}{\delta t} \geq c$$

Courant–Friedrichs–Lewy condition: the information propagation speed on the grid must be greater than all the physical speeds of the problem.

Advection equation: two more examples

Other schemes that result in the CFL condition:

- ⋆ Upwind differencing (with backwards Euler in time):

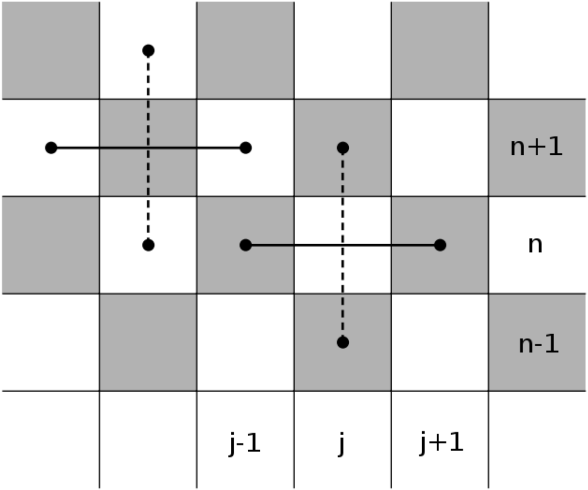
$$\frac{u_j^{n+1} - u_j^n}{\delta t} = -c\frac{u_j^n - u_{j-1}^n}{\delta x}$$

- ⋆ Staggered leapfrog (centered differences in both space and time):

$$\frac{u_j^{n+1} - u_j^{n-1}}{\delta t} = -c\frac{u_{j+1}^n - u_{j-1}^n}{\delta x}$$

second order in time and space, but prone to the mesh drift instability; grid points of odd $j + n$ and even $j + n$ decoupled

Advection equation: two more examples



Mesh drift instability

Advection equation with Lax-Wendroff

Consider again an advection equation, with a following two-step method:

First step is Lax method for $u_{j+1/2}^{n+1/2}$:

$$\frac{u_{j+1/2}^{n+1/2} - (u_{j+1}^{n} - u_{j}^{n})/2}{\delta t/2} = -c\frac{u_{j+1}^{n} - u_{j}^{n}}{\delta x}$$

Second step, the quantities at t_{n+1} are calculated using the centered expression:

$$\frac{u_{j}^{n+1} - u_{j}^{n}}{\delta t} = -c\frac{u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}}{\delta x}$$

(values $u_{j+1/2}^{n+1/2}$, $u_{j-1/2}^{n+1/2}$ auxiliary; no mesh drift instability)

Further reading...

- „*An introduction to astrophysical fluid dynamics*", Michael J. Thompson
- „*An Introduction to Astrophysical Hydrodynamics*", Steven N. Shore
- Amusing weblog: fuckyeahfluiddynamics.tumblr.com