Introduction: general relativity

Michał Bejger
N. Copernicus Center, Warsaw

Outline

- ⋆ Equivalence principle,
- ⋆ Free-fall and geodesic equations,
- ⋆ Measurements in curved space,
- ⋆ Einstein equations.

Why general relativity?

Maxwell's equations (1863) describe electromagnetism and optical phenomena within the theory of waves:

- ⋆ A special medium, „*luminiferous ether*", needed for the EM waves to propagate (like water for water waves); Ether almost doesn't interact with matter, but is supposedly carried along with astronomical objects,
- ⋆ Light propagates with a finite speed, but is not invariant in all frames,
- ⋆ Especially, Maxwell's equations are **not invariant** under Galilean transformations:

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

- ⋆ To make electromagnetism compatible with classical mechanics, light has speed $c = 3 \times 10^8$ m/s only in frames where source is at rest.

- ⋆ Rømer determination of the finite value of the speed of light,
- ⋆ Star light aberration: a small shift in apparent positions of distant stars due to the finite speed of light,
- ⋆ Fizeau-Foucault (1850): velocity of light in air and liquids
- ⋆ Michelson-Morley (1887): to detect the motion of the Earth through ether
- ⋆ Lorentz-Fitzgerald contraction hypothesis (1894): speeding bodies get compressed in the direction of motion by a factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Lorentz transformation, as opposed to simpler-looking Galilean transformation, mixes space and time. Example boost in x-direction

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma\left(x - vt\right) \\ y' &= y \\ z' &= z \end{aligned}$$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

with $\beta = v/c$.

Einstein: an idea of fixing the Maxwell equations by accepting that

- ⋆ the laws of physics are invariant (i.e., act identically) in all inertial systems (non-accelerating frames of reference),
  $\rightarrow$ no experiment can measure absolute velocity,
- ⋆ the speed of light in a vacuum is the same for all observers.

By accepting this, one gets

- ⋆ length contraction $\Delta l' = \Delta l / \gamma$,
- ⋆ time dilation $\Delta t' = \Delta t \gamma$,
- ⋆ relativistic mass $m\gamma$,
- ⋆ mass–energy equivalence $E = mc^2$,
- ⋆ universal speed limit,
- ⋆ relativity of simultaneity.

Gravity and acceleration

What is the difference between Newtonian and Einsteinian theory?

- ⋆ **Newton viewpoint**: mass tells gravity how to exert a force, force tells mass how to accelerate

$$F = -\frac{GM_g m_g}{r^2}, \quad F = m_i a$$

$$a = -\frac{GM_g}{r^2}\frac{m_g}{m_i}$$

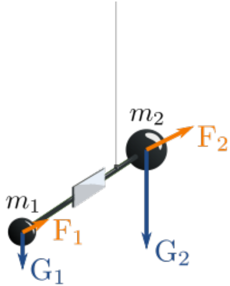
- ⋆ is gravitational mass m_g equal to inertial mass m_i?
- ⋆ Instantaneous action at a distance,
- ⋆ **Einstein viewpoint**: Mass (energy) tells spacetime how to curve, curved spacetime tells mass (energy) how to move (J. Wheeler) - geometry is related to mass distribution.

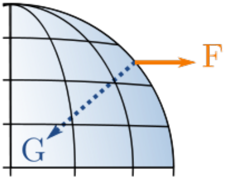
Equivalence principle

Weak equivalence principle: testing the equivalence of gravitational mass and inertial mass Eötvös parameter η for two different test bodies A and B (aluminum and gold, for example):

$$\eta(A,B) = 2\frac{\left(\frac{m_g}{m_i}\right)_A - \left(\frac{m_g}{m_i}\right)_B}{\left(\frac{m_g}{m_i}\right)_A + \left(\frac{m_g}{m_i}\right)_B}$$

From the times of Galileo (no difference „by eye”) till present (Eöt-Wash group) $\eta < 10^{-13}$



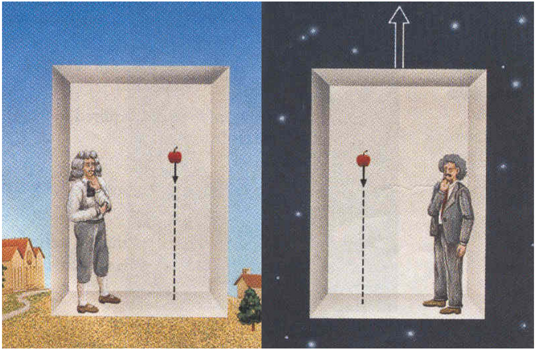


Equivalence principle

Strong equivalence principle:

- ⋆ The outcome of any local (gravitational or not) experiment in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime,
- ⋆ the laws of gravitation are independent of velocity and location,
- ⋆ Locally, the effects of gravitation (motion in a curved space) are the same as that of an accelerated observer in flat space,
- ⋆ Falsifiability: testing GR in the Solar System and near black holes - different regimes should give consistent answers (also recently discovered triple system with 2WD and NS, PSR J0337+1715)

Equivalence principle



Gravitation is a form of acceleration; locally, the effects of gravitation (motion in a curved space) are the same as that of an accelerated observer in flat space.

Special relativity in Minkowski spacetime

How we evaluate the distance in space in the usual 3D geometry? Let's consider spherical coordinates,

$$\begin{aligned} x^1 &= r \sin\theta \cos\phi \\ x^2 &= r \sin\theta \sin\phi \\ x^3 &= r \cos\theta \end{aligned}$$

$$\text{and call such an object, } g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

the *metric tensor*. An infinitesimal distance between (r, θ, ϕ) and $(r + dr, \theta + d\theta, \phi + d\phi)$ is then,

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

Special relativity in Minkowski spacetime

Let's consider now a 4D space, with a following coordinate system:

$$\begin{aligned} x^0 &= ct \;(= \text{t for c=1}) \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z \end{aligned}$$

and introduce the following *metric tensor*

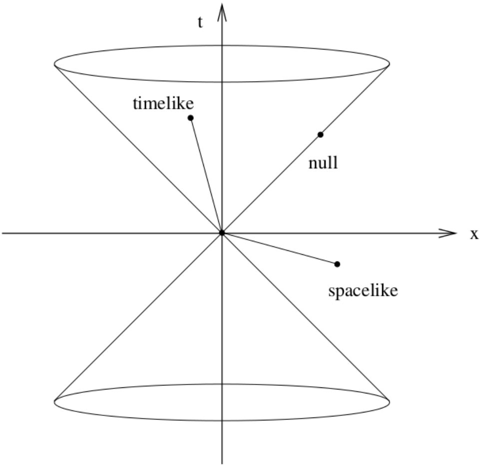
$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

that can be used to calculate the distances in an usual way

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = -dt^2 + dx^2 + dy^2 + dz^2.$$

Mind the signature (-+++)! Such a manifold - set of points in a topological space - is called *pseudo-Riemannian* manifold: the metric tensor is not positive-definite.

Special relativity in Minkowski spacetime



timelike $ds^2 < 0$;   spacelike $ds^2 > 0$;   null $ds^2 = 0$.

Special relativity in Minkowski spacetime

In this space we can measure proper time (length of the spacetime curve) by integrating over the spacetime interval:

$$\tau = \int \sqrt{-ds^2} = \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda,$$

define 4-vectors, e.g., the 4-velocity

$$u^\mu = \frac{dx^\mu}{d\tau}, \quad \text{(normalized:} \quad \eta_{\mu\nu} u^\mu u^\nu = -1),$$

and 4-momentum for a particle of mass m:

$$p^\mu = m u^\mu.$$

- ⋆ The particle energy E is the timelike component, p^0 (for a particle at rest $E = p^0 = mc^2$),
- ⋆ In a moving frame (x-direction, say) from the Lorentz transformation,

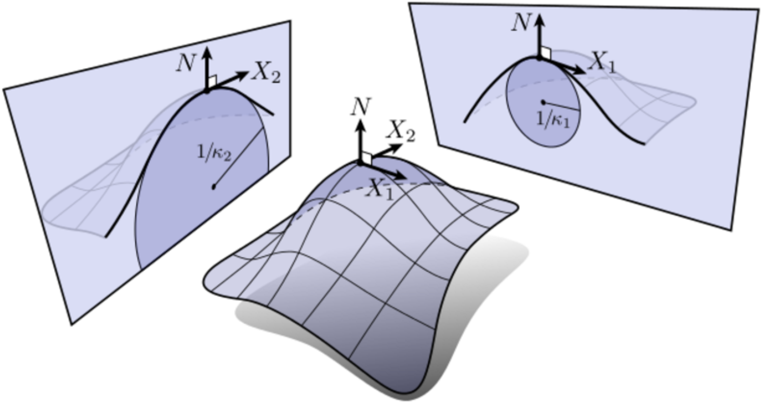
$$p^\mu = (m\gamma, vm\gamma, 0, 0), \qquad \text{where } \gamma = 1/\sqrt{1 - v^2},$$

For small v, $p^0 = mc^2 + mv^2/2$ and $p^1 = mv$.

What if the space is not flat?

How to quantify the curvature

Imagine a curved surface: at a given point, **principal curvatures** denoted κ_1 and κ_2, are the maximum and minimum values of the curvature.

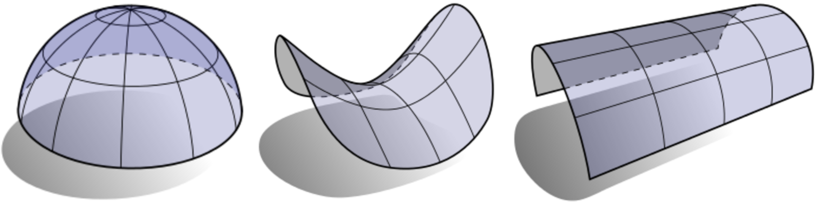


http://brickisland.net/cs177

How to quantify the curvature

- Gauss curvature $K = \kappa_1 \kappa_2$,
- intrinsic to the surface.

- Mean curvature (Sophie Germain), $H = \frac{\kappa_1 + \kappa_2}{2}$,
- requires an idea of embedding space exterior to the surface.



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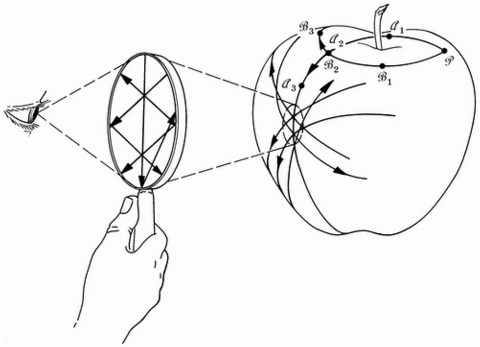
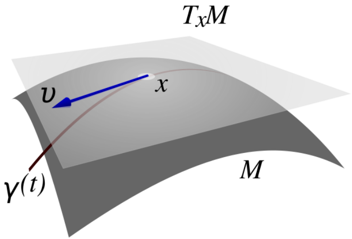
The space may not be flat in general...

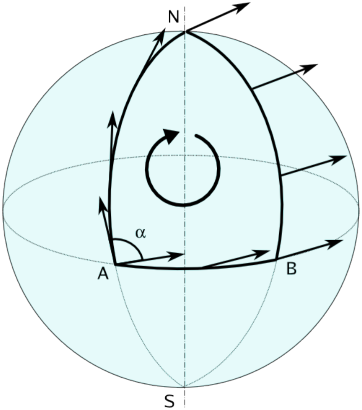


but we can assume that in a vicinity of a point, the manifold in question is locally $\mathbf{R}^n$ (choosing the coordinates to be locally emulating the Minkowski space).

The space may not be flat in general...



In all physical cases the manifold is equipped with the *tangent space* at every point. But how to compare vectors at different points?



Parallel transport of a vector along a closed curve in a curved space (a vector is moved along a curve staying parallel to itself and maintaining its magnitude).

Covariant and contravariant, forms and vectors

Vector v, expressed in two coordinate bases $\mathbf{e}_i$ and $\mathbf{e}'_i$, where

$$\mathbf{e}_i = \frac{\partial}{\partial x^i}, \quad \mathbf{e'}_i = \frac{\partial}{\partial x'^i}$$

$$\text{is } \mathbf{v} = v^i \mathbf{e}_i = v'^i \mathbf{e}'_i$$

Expressing the new components of **v** with old ones. **Contravariant transformation**:

$$v'^i = \frac{\partial x'^i}{\partial x^j} v^j$$

Basis vectors transform **coviariantly**:

$$\mathbf{e}'_i = \frac{\partial x^j}{\partial x'^i} \mathbf{e}_j$$

1-form, co-vector ω (member of a space dual to the vector space),

$$\omega = \omega_i dx^i = \omega'_i dx'^i.$$

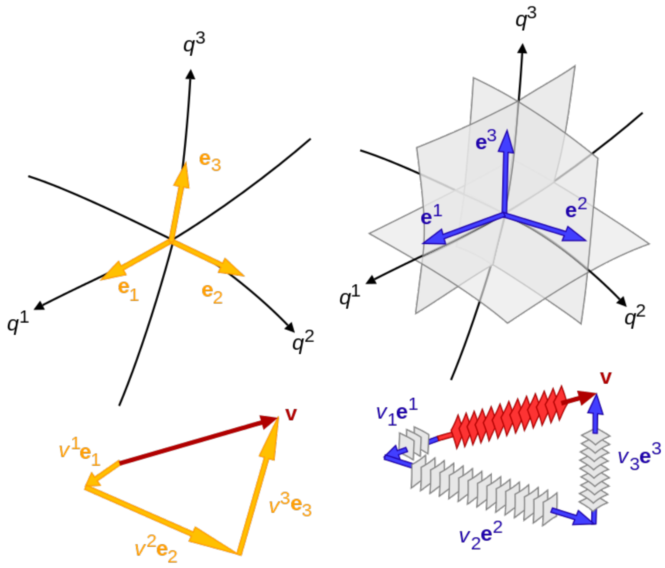
transforms its components **covariantly**:

$$\omega'_i = \frac{\partial x^j}{\partial x'^i} \omega_j$$

whereas the dual basis differentials dx^i transform **contravariantly**:

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j$$

Covariant and contravariant, forms and vectors



Covariant and contravariant, forms and vectors

Vectors and 1-forms are related to each other; 1-form is a linear transformation from the vector space of **v** to real numbers

$$\omega_\mu v^\mu \to \mathbf{R}$$

(example: $u_\mu u^\mu = -1$ in SR).
The transformation rules generalize for higher rank tensors:

$$A^{\mu'}_{\nu'\rho'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\rho}}{\partial x^{\rho'}} A^{\mu}_{\nu\rho}$$

Metric can be used to transform a vector to a form (and vice versa), by lowering/raising an index:

$$A_\mu = g_{\mu\nu} A^\nu, \quad A^\mu = g^{\mu\nu} A_\nu, \quad g_{\mu\nu} g^{\gamma\nu} = \delta^\gamma_\mu$$

(Metric tensor $g_{\mu\nu}$ is itself a 2-form: $g_{\mu\nu} dx^\mu dx^\nu \to$ spacetime distance.)

Comparing vectors in curved spaces

Why all this? We want derivatives that transform like tensors.
Unfortunately, simple $\partial/\partial x^\mu$

$$\frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial \phi}{\partial x^\mu}$$

works only for scalar fields, e.g., gradient of ϕ is a proper (0,1) tensor (1-form).

For general vectors we obtain, recalling that

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu,$$

$$\frac{\partial V^{\nu'}}{\partial x^{\mu'}} = \left(\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu}\right) \left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu\right) = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \left(\frac{\partial V^\nu}{\partial x^\mu}\right) + \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} V^\mu}_{\text{not transforming correctly}}.$$

$\rightarrow$ one cannot use $\partial/\partial x^\mu$ to compare vector field at neighboring points.

Comparing vectors in curved spaces

Consider an infinitesimal change of a vector $\mathbf{v}$ along a line parametrized by λ in a space with a coordinate basis $\mathbf{e}$:

$$\frac{d\mathbf{v}}{d\lambda} = \frac{d(v^\alpha \mathbf{e}_\alpha)}{d\lambda} = \frac{dv^\alpha}{d\lambda}\mathbf{e}_\alpha + v^\alpha \frac{d\mathbf{e}_\alpha}{d\lambda}.$$

How the vectors from the coordinate basis change with λ?

$$\frac{d\mathbf{e}_\alpha}{d\lambda} = \frac{d\mathbf{e}_\alpha}{dx^\beta}\frac{dx^\beta}{d\lambda} \quad \text{with} \quad \frac{d\mathbf{e}_\alpha}{dx^\beta} = \underbrace{\Gamma^\gamma_{\alpha\beta}}_{\textit{Christoffel}} \mathbf{e}_\gamma$$

so we can write a *total* derivative

$$\frac{d\mathbf{v}}{d\lambda} = \left(\frac{dv^\alpha}{d\lambda} + \Gamma^\alpha_{\gamma\beta} v^\gamma \frac{dx^\beta}{d\lambda}\right)\mathbf{e}_\alpha \quad \text{or} \quad \frac{Dv^\alpha}{d\lambda} = \frac{dv^\alpha}{d\lambda} + \Gamma^\alpha_{\gamma\beta} v^\gamma \frac{dx^\beta}{d\lambda}.$$

In a curved space, the changes are because of

- ⋆ physical changes of a vector field between points,
- ⋆ curvilinear coordinates.

Comparing vectors in curved spaces

$\Gamma^{\alpha}_{\gamma\beta}$ (Christoffel symbols, Levi-Civita, affine connection coefficients) describe the effects of parallel transport in curved spaces; they are functions of the metric

$$\Gamma^{\alpha}{}_{\gamma\delta} = \frac{1}{2}g^{\alpha\beta}\left(\frac{\partial g_{\beta\gamma}}{\partial x^{\delta}} + \frac{\partial g_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial g_{\gamma\delta}}{\partial x^{\beta}}\right)$$

$$= \frac{1}{2}g^{\alpha\beta}(g_{\beta\gamma,\delta} + g_{\beta\delta,\gamma} - g_{\gamma\delta,\beta})$$

Symmetric in lower indices, $\Gamma^{\alpha}{}_{\gamma\delta} = \Gamma^{\alpha}{}_{\delta\gamma}$.

Comparing vectors in curved spaces

The total derivative, similar like in hydrodynamics, is

$$\frac{Dv^\alpha}{D\lambda} = \frac{dv^\alpha}{d\lambda} + \Gamma^\alpha_{\gamma\beta} v^\gamma \frac{dx^\beta}{d\lambda} \quad \text{or in vector notation} \quad \frac{D\mathbf{v}}{D\lambda} = \nabla_{\mathbf{u}}\mathbf{v}$$

with $u^\alpha = dx^\alpha / d\lambda$, the 4-velocity/tangent vector to the curve.

Often called the *covariant* derivative:

$$v^\alpha_{;\beta} = v^\alpha_{,\beta} + \Gamma^\alpha_{\gamma\beta} v^\gamma \quad \text{or} \quad \frac{Dv^\alpha}{D\lambda} = v^\alpha_{;\beta} u^\beta$$

Covariant derivative acting on the metric return 0 (metric compatibility):

$$g_{\alpha\beta;\gamma} = 0, \quad g^{\alpha\beta}_{;\gamma} = 0.$$

Free fall and the geodesic equation

Geodesic, the straightest line in a curved space. A line is "straight", if it parallel transports its own tangent vector, which means

$$\nabla_{\mathbf{u}} \mathbf{u} = 0$$

From this and previous considerations we obtain the *geodesic equation*:

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$$

which is the force-free equation of motion of a particle in a curved space.

Free fall and the geodesic equation

Let's assume that we are in free falling coordinate system χ^α:

$$\frac{d^2\chi^\alpha}{d\lambda^2} = 0.$$

and make a coordinate transformation to arbitrary coordinates x^α,

$$\chi^\alpha = \chi^\alpha(x^\gamma) \quad \rightarrow \quad \frac{d\chi^\alpha}{d\lambda} = \frac{\partial\chi^\alpha}{\partial x^\gamma}\frac{dx^\gamma}{d\lambda},$$

so we have

$$\frac{d}{d\lambda}\left(\frac{d\chi^\alpha}{d\lambda}\right) = \frac{d}{d\lambda}\left(\frac{\partial\chi^\alpha}{\partial x^\gamma}\frac{dx^\gamma}{d\lambda}\right) = \frac{d^2x^\gamma}{d\lambda^2}\frac{\partial\chi^\alpha}{\partial x^\gamma} + \frac{\partial^2\chi^\alpha}{\partial x^\beta\partial x^\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0.$$

Multiply the above by $\partial x^\sigma/\partial\chi^\alpha$, recalling that

$$\frac{\partial\chi^\alpha}{\partial x^\gamma}\frac{\partial x^\sigma}{\partial\chi^\alpha} = \frac{\partial x^\sigma}{\partial x^\gamma} = \delta^\sigma_\gamma \qquad \text{(Kronecker delta).}$$

Free fall and the geodesic equation

From

$$\frac{d}{d\lambda}\left(\frac{\partial\chi^\alpha}{\partial x^\gamma}\frac{dx^\gamma}{d\lambda}\right)=\frac{d^2x^\gamma}{d\lambda^2}\frac{\partial\chi^\alpha}{\partial x^\gamma}+\frac{\partial^2\chi^\alpha}{\partial x^\beta\partial x^\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}=0. \qquad \Big/\cdot\frac{\partial x^\sigma}{\partial\chi^\alpha}$$

we have

$$\frac{d^2x^\gamma}{d\lambda^2}\delta^\sigma_\gamma+\frac{\partial x^\sigma}{\partial\chi^\alpha}\frac{\partial^2\chi^\alpha}{\partial x^\beta\partial x^\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}=0,$$

that is

$$\frac{d^2x^\sigma}{d\lambda^2}+\underbrace{\left(\frac{\partial x^\sigma}{\partial\chi^\alpha}\frac{\partial^2\chi^\alpha}{\partial x^\beta\partial x^\gamma}\right)}_{\text{Christoffel}}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}=0.$$

We end up with the geodesic equation:

$$\frac{d^2x^\sigma}{d\lambda^2}+\Gamma^\sigma_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}=0.$$

Geodesic equation from the action principle

One can get the geodesic equation from the Lagrangian and the action principle. The spacetime distance is

$$d\tau^2 = -\frac{1}{c^2} ds^2 = -\frac{1}{c^2} g_{\alpha\beta} dx^\alpha dx^\beta,$$

The proper time between two events along an arbitrary timelike curve is (setting $c = 1$):

$$\tau = \int_A^B d\tau = \int_A^B \frac{d\tau}{d\lambda} d\lambda = \int_A^B \left(-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda$$

Let's call this function the Lagrangian,

$$\mathcal{L}(x^\alpha, \dot{x}^\alpha, \lambda) = \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

Geodesic equation from the action principle

define the action,

$$S = \int_A^B \mathcal{L}(x^\alpha, \dot{x}^\alpha, \lambda) d\lambda \qquad (\delta S = 0)$$

and extract the Euler-Lagrange equations out of it:

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right)$$

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = -\frac{1}{2}\frac{d\lambda}{d\tau}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu \dot{x}^\nu \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = -\frac{d\lambda}{d\tau} g_{\mu\alpha}\dot{x}^\mu = -g_{\mu\alpha}\frac{dx^\mu}{d\tau}.$$

Rearanging and multiplying by $d\lambda/d\tau$ we have (just a sketch of how to go):

$$\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\mu}{d\tau}\frac{dx^\mu}{d\tau} = \frac{d}{d\tau}\left(g_{\mu\alpha}\frac{dx^\mu}{d\tau}\right) \rightarrow \quad \frac{\partial g_{\mu\nu}}{\partial d\tau} = \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\tau} \quad \rightarrow$$

$$\rightarrow \quad \text{multiply by } g^{\delta\mu} \quad \rightarrow \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

Lie derivative

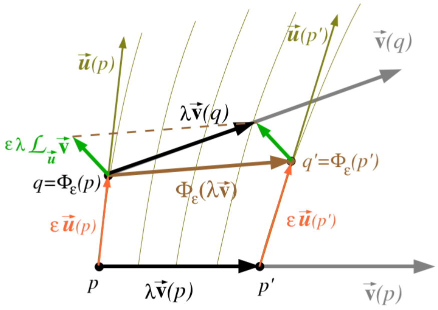
Consider two vector fields, **v** and **u**. Lie derivative measures the change of one vector field **v** along a flow of another, **u**:

$$\mathcal{L}_{\mathbf{u}}(\mathbf{v}) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left\{\mathbf{v}(q) - \Phi_\epsilon(\mathbf{v}(p))\right\}$$

- ⋆ $\mathbf{v}(q)$ - actual value of $\mathbf{v}(q)$ at q,
- ⋆ $\Phi_\epsilon(\mathbf{v}(p))$ - value transported by ϵ along the flow of **u**, $\Phi_\epsilon(\mathbf{v}(p)) = \vec{qq'}/\lambda$

$$\mathcal{L}_{\mathbf{u}} v^\alpha = u^\mu \frac{\partial v^\alpha}{\partial x^\mu} - v^\mu \frac{\partial u^\alpha}{\partial x^\mu}.$$

- ⋆ Doesn't require the *connection*,
- ⋆ generalizes to higher rank tensors.



from E. Gourgoulhon "3+1 form. and bases"

Also called the commutator (Lie bracket):

$$\mathcal{L}_{\mathbf{u}}(\mathbf{v}) = [\mathbf{u}, \mathbf{v}]$$

Symmetries and Killing fields

Noether's theorem: symmetry of the system's Lagrangian (action) corresponds to a conservation law, for example,

- ⋆ Lagrangian is symmetric w.r.t. rotations $\rightarrow$ angular momentum is conserved,
- ⋆ symmetry w.r.t. time $\rightarrow$ conservation of energy.

On a pseudo-Riemannian manifold, Killing vector field preserves the metric:

$$\mathcal{L}_\xi g = 0.$$

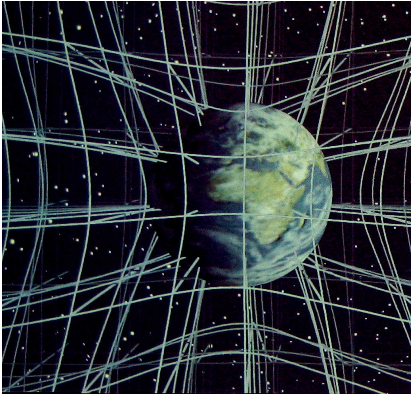
Also,

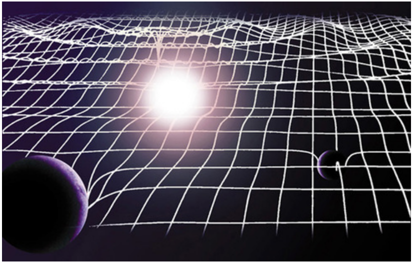
$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \qquad \text{(Killing equation)}$$

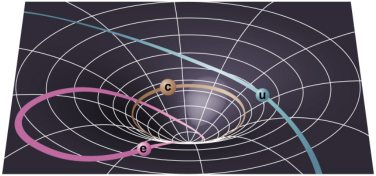
Flow generated by a Killing field generates a continuous isometry. Physical examples are

- ⋆ $\xi = \frac{\partial}{\partial t}$ in case of stationary systems,
- ⋆ $\eta = \frac{\partial}{\partial \phi}$ in case of axisymmetric systems.

Geometry of curved space







”Mass tells space-time how to curve, and space-time tells mass how to move”

Riemann curvature tensor

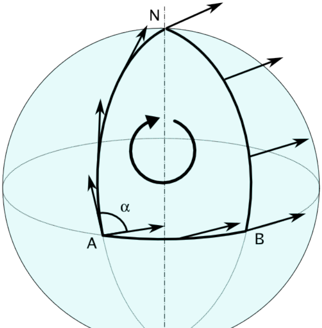
$R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$ measures a failure of derivatives to commute. In coordinates:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}$$

- ⋆ Constructed from $g_{\mu\nu}$ and its first and **second** derivatives,
- ⋆ Imagine transporting a vector **V** around a closed loop by dx^σ, dx^μ and then dx^ν; the vector will change its components w.r.t. the original ones by ΔV^i.

The Riemann tensor is roughly

$$R^{\rho}_{\sigma\mu\nu} = \Delta V^i/(dx^\sigma dx^\mu dx^\nu)$$



Riemann curvature tensor

$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$ measures a failure of derivatives to commute. In coordinates:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}$$

- ⋆ Constructed from $g_{\mu\nu}$ and its first and **second** derivatives,
- ⋆ Measures the *intrinsic* curvature → Gauss curvature, "rotation" of parallel-transported vectors ($R \equiv 0 \iff$ space is flat),
- ⋆ Measures the tidal forces acting on a body moving on the geodesic → relative acceleration between nearby bodies (geodesic deviation),
- ⋆ in 3+1 spacetime, $R^{\rho}_{\sigma\mu\nu}$ has 256 components, only 20 independent (because of the following symmetries):

$$\begin{aligned} R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} &= -R_{\sigma\rho\mu\nu}, \\ R_{\rho\sigma\mu\nu} &= R_{\mu\nu\rho\sigma}, \\ R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\mu} + R_{\rho\nu\sigma\mu} &= 0. \end{aligned}$$

Useful **Bianchi identity**: $\nabla_\gamma R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\gamma} + \nabla_\nu R_{\rho\sigma\gamma\mu} = 0.$

Ricci tensor and Ricci scalar

- ⋆ Ricci tensor is a contraction of the Riemann tensor:

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$$

If the **Riemann tensor** collects all the information about the curvature, $R_{\mu\nu}$ is kind of *average curvature*. It quantifies the amount by which a test volume differs from one in flat space,

In the vicinity of a given point, $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(x^2)$.

The difference in volume element: $dV = \left(1 - \frac{1}{6} R_{\mu\nu} x^{\mu} x^{\nu} + \mathcal{O}(x^3)\right) dV_{flat}$

- ⋆ Ricci scalar (scalar curvature) is contracted Ricci tensor:

$$R = R^{\mu}_{\mu}$$

used e.g., to compare areas of circles with those from flat space in n dimensions:

$$\frac{dS}{dS_{flat}} = 1 - \frac{R}{6n} r^2 + \mathcal{O}(r^4)$$

in 2D, $R = 2K$ (twice the Gaussian curvature).

Useful **Bianchi identity**: $\nabla^{\mu} R_{\alpha\mu} = \frac{1}{2} \nabla_{\alpha} R$

Energy-momentum tensor

The **energy-momentum** tensor (sometimes called the stress-energy tensor) gathers information about the matter. Most often used is the *perfect fluid* version,

$$T_{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + pg_{\mu\nu},$$

$\rightarrow$ neglect viscosity and elastic effects. Fluid which is isotropic in its rest frame ($g_{\mu\nu}u^{\mu}u^{\nu} = -1$)

$$T_{\mu}^{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

The conservation laws of $T_{\mu\nu}$ are analogs of conservation laws for energy and momenta from hydrodynamics, using the covariant derivative:

$$\nabla^{\mu} T_{\mu\nu} = 0.$$

Maxwell equations in curved spacetime

Electromagnetic field energy-momentum tensor:

$$T^{\mu\nu} = \left(F^{\mu\alpha} g_{\alpha\beta} F^{\nu\beta} - \frac{1}{4} g^{\mu\nu} F_{\delta\gamma} F^{\delta\gamma} \right)$$

that uses an antisymmetric tensor $F_{\mu\nu}$:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

and the 4-current $J^\mu = (\rho, \mathbf{J})$.

In the usual flat spacetime (Minkowski):

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho, \\ \nabla \times \mathbf{B} - \partial_t \mathbf{E} &= 4\pi \mathbf{J}, \\ \nabla \times \mathbf{E} - \partial_t \mathbf{B} &= 0, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned}$$

$$\begin{aligned} \partial_\mu F^{\nu\mu} &= 4\pi J^\nu, \\ \partial_{[\mu} F_{\nu\lambda]} &= \\ \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} &= 0. \end{aligned}$$

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and the 4-current $J^\mu = (\rho, \mathbf{J})$.

In the general curved space:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho, \\ \nabla \times \mathbf{B} - \partial_t \mathbf{E} &= 4\pi \mathbf{J}, \\ \nabla \times \mathbf{E} - \partial_t \mathbf{B} &= 0, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned}$$

Covariant derivative instead of partial derivative, $\partial_\mu \to \nabla_\mu$:

$$\begin{aligned} \nabla_\mu F^{\nu\mu} &= 4\pi J^\nu, \\ \nabla_{[\mu} F_{\nu\lambda]} &= 0. \end{aligned}$$

Einstein equations

Using the just defined tensors, we arrive at

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

(10 equations in 3+1 dimensions).

Why like that? The equations should conserve energy & momentum. We would like to have

$$\nabla^\mu T_{\mu\nu} = 0. \quad \text{It implies} \quad \nabla^\mu G_{\mu\nu} = 0.$$

Fortunately, from the contracted Bianchi identity,

$$\nabla^\mu R_{\alpha\mu} = \frac{1}{2} \nabla_\alpha R \quad \rightarrow \quad \nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = 0.$$

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Let's raise one index and contract both sides (setting $8\pi G/c^4 = 1$):

$$R^{\mu}_{\nu} - \frac{1}{2} R g^{\mu}_{\nu} = T^{\mu}_{\nu} \quad \rightarrow \quad R^{\mu}_{\mu} - \frac{1}{2} R g^{\mu}_{\mu} = T^{\mu}_{\mu} \quad \underset{g^{\mu}_{\mu}=4}{\rightarrow} \quad R - 2R = T^{\mu}_{\mu}.$$

This allows us to write the Einstein equations in an equivalent (trace-reversed) form:

$$R = -T^{\mu}_{\mu} \quad \rightarrow \quad R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} T^{\alpha}_{\alpha} g_{\mu\nu}$$

Einstein equations

What does it mean? Recall that the rate of change of a comoving test volume V in spacetime is proportional to the Ricci tensor:

$$\frac{D^2 V}{D\lambda^2} \propto -R_{\mu\nu} u^\mu u^\nu$$

In the comoving (locally Minkowski) frame $g_{\mu\nu} = \eta_{\mu\nu}$, the only non-zero component is R_{tt}, because $u^\mu = (1, 0, 0, 0)$ there. Thus

$$\frac{D^2 V}{D\lambda^2} \propto -R_{tt}.$$

On the other hand,

$$R_{tt} = T_{tt} - \frac{1}{2} T^\alpha_\alpha = \frac{1}{2} (T_{tt} + T_{rr} + T_{\theta\theta} + T_{\phi\phi})$$

($T_{\alpha\alpha}$ are momentum flows in α direction, e.g., pressure for perfect fluid).
The rate at which a free-falling test volume evolves in time is proportional to its energy density T_{tt} and sum of momenta flows in all other directions, $T_{rr} + T_{\theta\theta} + T_{\phi\phi}$.

Further reading...

- ⋆ Lecture notes of Sean Carroll (http://preposterousuniverse.com/grnotes)
- ⋆ Textbooks: Misner-Thorne-Wheeler, Wald,
- ⋆ SageManifolds examples: http://sagemanifolds.obspm.fr/examples.html