Instabilities in astrophysics

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Outline

- Definition of stability; linear stability analysis & normal modes,
- Hydrodynamical instabilities in astrophysics,
- Turbulence,
- magneto-rotational instability,
- Rossby modes of neutron stars.

The importance of instabilities in (astro)physics

- Instabilities involve energy exchange ("energy release") - they feed off of free energy in the system,
- Instabilities are necessary to "makes great things happen"- they facilitate the mechanisms that initiate sometimes violent energy exchange processes.

From stability to instability

Transition from linear regime to turbulence in case of flow:

- ⋆ Laminar flow: symmetric, steady, simple etc.,
- ⋆ The development of instability: the flow loses symmetry and becomes unsteady,
- ⋆ Turbulent flow: multi-scale, non-periodic, unpredictable

Usually, laminar flow has the same symmetry as the problem, but is not often observed. Instead, real solutions are less symmetric.

Definition of stability: evolution and perturbations

- Evolution equation of a state vector ϕ

$$\frac{d\phi}{dt} = f(\phi, \mathbf{r}),$$

where $\mathbf{r}$ denotes the parameter(s) of the problem.

- Assume that state ϕ is composed of a basic state Φ and a perturbation ϕ':

$$\phi = \Phi + \phi', \quad \text{and} \quad \frac{d\Phi}{dt} = f(\Phi, \mathbf{r}),$$

- Perturbation evolution equation:

$$\frac{d\phi'}{dt} = f(\Phi + \phi', \mathbf{r}) - f(\Phi, \mathbf{r})$$

- To quantify the perturbations, one defines the norm - a scalar "perturbation amplitude", for example

$$\left\|\phi'\right\|(t) = \sqrt{\int_V (\phi' \cdot \phi')\, dV} \qquad \text{or} \qquad \left\|\phi'\right\|(t) = \sup_V |\phi'|$$

Definitions of stability

- ⋆ Lyapunov stability definition: base solution Φ is stable if

$$\forall \epsilon > 0 \text{ there exists } \delta(\epsilon) > 0 \text{ such that}$$
$$\text{if } \left\|\phi'\right\|(t=0) < \delta, \text{ then } \left\|\phi'\right\|(t \geqslant 0) < \epsilon,$$

- ⋆ Asymptotic stability: base solution Φ is asymptotically stable if it is stable in the sense of Lyapunov, and

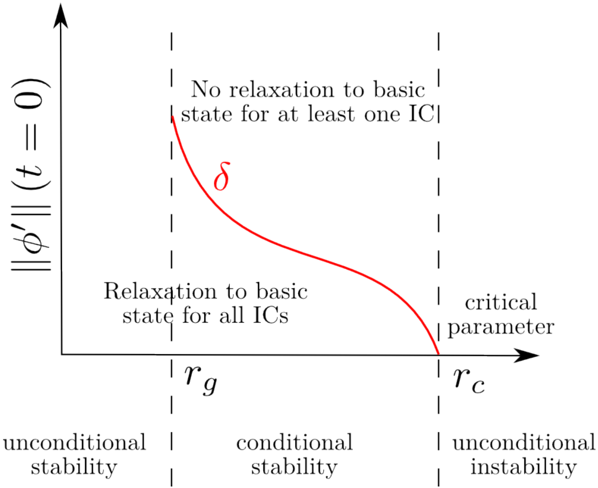
$$\lim_{t\to\infty} \left\|\phi'\right\| = 0.$$

- ⋆ Unconditional stability: base solution Φ is unconditionally stable if it is stable, and

$$\forall \left\|\phi'\right\|(t=0) \implies \lim_{t\to\infty} \left\|\phi'\right\| = 0.$$

- ⋆ Single initial condition is sufficient to prove the instability,
- ⋆ To prove stability, one has to prove it for all possible initial conditions.

Dependence on the control parameter **r**



$r_g < r_c$ here, but it could be also that $r_c = \infty$ or $r_g = r_c$.

Linearization of equations

For a given perturbation equation,

$$\frac{d\phi'}{dt} = f(\Phi + \phi', \mathbf{r}) - f(\Phi, \mathbf{r})$$

Taylor expansion of the r.h.s in the vicinity of the base state, neglecting the higher order terms:

$$f(\Phi + \phi') = f(\Phi) + \frac{df}{d\phi}(\Phi)\phi' + \mathcal{O}(\|\phi'\|^2)$$

One can define the linearized evolution equations

$$\frac{d\phi'}{dt} = L\phi' \quad \text{where} \quad L(\Phi, \mathbf{r}) = \frac{df}{d\phi}(\Phi, \mathbf{r})$$

is the linearized Jacobian operator.

Linear stability: base solution Φ is linearly stable if the solutions of the evolution equations linear near Φ are stable.

Linear stability analysis

In order to check the linear stability, one usually decomposes the solution on a basis of *fundamental* solutions:

$$\phi'(t) = \sum c_j(IC)\phi'^j(t)$$

where ϕ'^j are members of the complete set of linearly independent solutions, and coefficients c_j depend on initial conditions (IC). Obviously, system is

- *linearly unstable* if at least one fundamental solution ϕ'^j is unstable,
- *linearly stable* if all fundamental solutions ϕ'^j are stable.

General method of solution for steady flows (time-independent L) is called the method of normal modes.

Method of normal modes

Consider a system of N 1st-order ordinary differential equations with constant coefficients:

$$\frac{d\phi'}{dt} = L\phi',$$

where state ϕ is an N-dim vector, and L is linear and time-independent.

For such system one has eigenvectors ψ and eigenvalues s,

$$L\psi = s\psi \quad \text{(direction of } \psi \text{ unchanged by } L\text{)}$$

For homogeneus system, there are non-trivial solutions if

$$\det(L - s\mathbb{I}) = 0 \quad \rightarrow \quad L\psi^j = s^j\psi^j$$

(characteristic equations for N roots for $s \rightarrow N$ eigenvalue-eigenvector pairs)

Decomposition into modes

For N distinct eigenvalues, one has N linearly independent eigenvectors that form an eigenvector basis. The perturbation ϕ' expressed with such basis

$$\phi'(t) = \sum_{j=1}^{N} q_j \psi^j \quad (q_j \text{ - modal amplitudes})$$

For $d\phi'/dt = L\phi'$ one has

$$\sum_{j=1}^{N} \psi^j \frac{dq_j}{dt} = \sum_{j=1}^{N} L\psi^j q_j \rightarrow \sum_{j=1}^{N} \psi^j \frac{dq_j}{dt} = \sum_{j=1}^{N} s^j \psi^j q_j \rightarrow$$

$$\sum_{j=1}^{N} \left(\frac{dq_j}{dt} - s^j q_j \right) = 0 \rightarrow \frac{dq_j}{dt} = s^j q_j$$

N independent equations for modal amplitudes q_j.

Modal solution to the initial value problem

Lets assume a form of modal amplitudes in order to solve the modal amplitude equations:

$$\frac{dq_j}{dt} = s^j q_j, \quad \text{with} \quad q_j(t) = e^{s^j t} q_j(0).$$

where $q_j(0)$ are the intial values of the amplitude coefficients. This gives us the solution

$$\phi'(t) = \sum_{j=1}^{N} e^{s^j t} \psi^j q_j(0)$$

The solution is generally composed of

$$\underbrace{e^{s_r^j t}}_{envelope} \underbrace{\left(\cos s_i^j t + i \sin s_i^j t\right)}_{oscillations}$$

- ⋆ *Linear instability* if at least one eigenvalue with $s_r > 0$ (unlimited growth of a fundamental solution),
- ⋆ *Linear stability* if all eigenvalues have $s_r < 0$.

Linear stability analysis: checking the real parts of eigenvalues computed from L.

Normal modes example

Consider 1D reaction-diffusion equation, describing an unconfined flow

$$\frac{\partial \phi'}{\partial t} = r\phi' + \frac{\partial^2 \phi'}{\partial x^2}$$

and try to solve it with

$$L = r\mathbb{I} + \frac{d^2}{dx^2}. \quad \text{The solution is} \quad \psi(k,x) = e^{ikx}.$$

Eigenfunctions are Fourier modes with a real wavenumber k, $k \in \mathbb{N}$. The dispersion relation $s = s(k,r)$ is found by replacing the eigenfunctions in $L\psi = s\psi$:

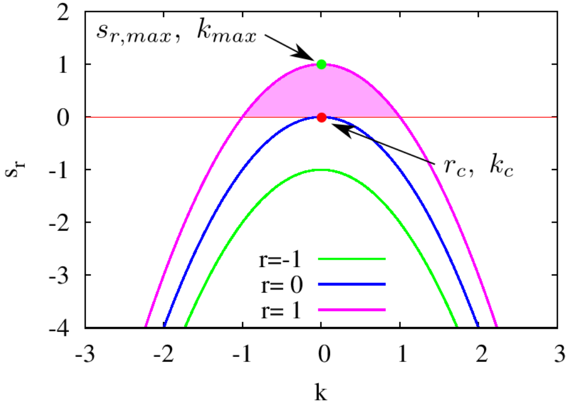
$$s = r - k^2$$

The solution is expressed with modal decomposition of $q_j(t) = e^{s^j t} q_j(0)$ (here, inverse Fourier transform) as

$$\phi'(x,t) = \int_{-\infty}^{\infty} q(k,t)\psi(k,x)dx = \int_{-\infty}^{\infty} q(k,t)e^{ikx}dk$$

Stability plot: growth rate

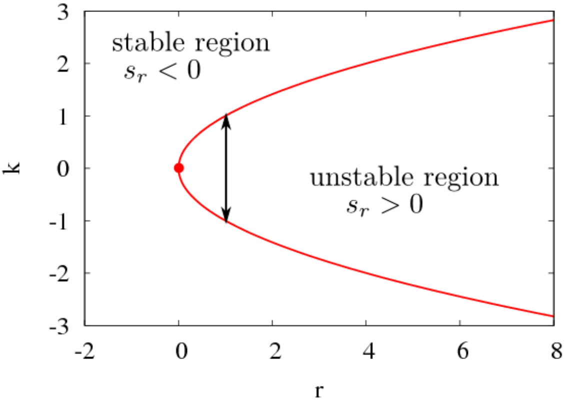
Growth rate s_r vs the wavenumber k for selected values of r:



- ⋆ curves from $s_r = r - k^2$ for selected r,
- ⋆ magenta region: unstable waveband for $r = 1$,
- ⋆ green dot: maximum growth rate s_r and most amplified k
- ⋆ red dot: instability appears for critical r_c, k_c.

Stability plot: neutral curve

Neutral curve $s_r = 0$ separates regions with positive growth rate from regions of negative growth rate in the $r - k$ plane



- ⋆ neutral curve from the dispersion relation $s_r = r - k^2$ $\rightarrow r_{neut}(k) = k^2$,
- ⋆ red dot: critical r_c, k_c, defined by $\min_k(r_{neut}(k))$,
- ⋆ black arrow: unstable waveband for $r = 1$.

Nonlinear development of instabilities

- ⋆ Linear theory is not the whole story... Linear stage lasts only for so long, but is then followed by nonlinear physics,
- ⋆ Long-term nonlinear fate of an instability may depend on many factors.

Saturation/relaxation to a new equilibrium:

- ⋆ no new energy is supplied,
- ⋆ new equilibrium has lower energy and lower symmetry,
- ⋆ examples: kink instability.

Development of turbulence:

- ⋆ free energy is supplied continuously,
- ⋆ leads to marginal stability,
- ⋆ examples: MRI, convection instability

System disruption:

- ⋆ No lower-energy equilibrium available,
- ⋆ System runs away until catastrophic disruption,
- ⋆ example: Rayleigh-Taylor instability.

Examples of instabilities in astrophysics

- Large-scale fluid instabilities (macroscopic):
  - hydrodynamic (gravitational, Rayleigh-Taylor, Kelvin-Helmholtz, convective),
  - ideal-MHD (kink/sausage/spaghetti, MRI, Parker instability),
  - resistive-MHD (tearing instability).
- Small-scale (2-fluid and microscopic):
  - electromagnetic kinetic (Weibel)
  - Pressure-anisotropy-driven (firehose/mirror).

Jeans instability

Jeans instability causes the collapse of a gas cloud due to the lack of pressure support or high-enough mass - it's the departure from the hydrostatic equilibrium described by

$$\frac{dp}{dr} = -\frac{GM\rho}{r^2}.$$

For a spherical distribution of mass (radius R, mass M):

$$t_s = \frac{R}{c_s} \quad \text{(soundwave crossing timescale, for sound speed } c_s\text{)},$$
$$t_f = \frac{1}{\sqrt{G\rho}} \quad \text{(free-fall timescale)}.$$

The condition for collapse is

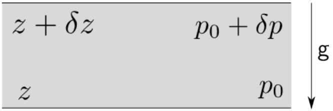
$$t_f < t_s \quad \text{(free-fall takes less time than sound to cross the region)}$$

This results in characteristic Jeans radius R_J and Jeans mass M_J:

$$R_J = \frac{c_s}{2\sqrt{G\rho}}, \quad M_J = \left(\frac{4\pi}{3}\right)\rho R_J^2.$$

Convective instability: the Schwarzschild criterion

Consider a 1D fluid at rest, with $\rho(z)$ stratification and under gravity $\mathbf{g} = -g e_z$. What happens when fluid piece is displaced from z to $z + \delta z$? We assume pressure equilibrium and no heat exchange (adiabatic process, $p \propto \rho^\gamma$)



$$\frac{p_0 + \delta p}{p_0} = \left(\frac{\rho_0 + \delta \rho}{\rho_0} \right)^\gamma$$

By linearizing the equations we have for density perturbation $\delta\rho$:

$$\delta\rho = \frac{\rho_0}{\gamma p_0} \delta p = \frac{\rho_0}{\gamma p_0} \delta z \frac{dp}{dz}$$

The fluid element sinks and returns to its original place if

$$\rho_0 + \delta\rho > \rho_0 + \delta z \frac{d\rho}{dz} \rightarrow \frac{\rho_0}{\gamma p_0} \delta z \frac{dp}{dz} > \delta z \frac{d\rho}{dz} \rightarrow \frac{1}{\gamma p} \frac{dp}{dz} > \frac{1}{\rho} \frac{d\rho}{dz}$$

$$\rightarrow \frac{1}{\gamma} < \frac{d \ln \rho}{d \ln p}$$ otherwise the system is convectively unstable.

Convective instability: the Schwarzschild criterion

The energy source for this instability is the potential energy of the initially unstable stratification. Convective instability, if

$$\frac{1}{\gamma} > \frac{d \ln \rho}{d \ln p}$$

The density must increase sufficiently fast with depth to stabilize the convection. For stable situation, equation of motion for the parcel is

$$\rho \frac{d^2 \delta z}{dt^2} = \underbrace{\left(\rho + \delta z \frac{\partial \rho}{\partial z}\right) g}_{buoyancy\ \ force} - \underbrace{(\rho + \delta \rho)\, g}_{weight} = N^2 \delta z$$

with the Brunt-Väisälä (buoyancy) frequency:

$$N^2 = g \left(\frac{1}{\gamma} \frac{d \ln \rho}{dz} - \frac{d \ln p}{dz} \right) = \frac{\rho g^2}{p} \left(\frac{d \ln \rho}{d \ln p} - \frac{1}{\gamma} \right).$$

System is unstable if

$$N^2 < 0.$$

Convective instability: the Schwarzschild criterion

For an uniform chemical composition and perfect gas $pV \propto T$,

$\ln p = \ln \rho + \ln T + const.$ the instability criterion is $\dfrac{d \ln T}{d \ln p} > 1 - \dfrac{1}{\gamma}$,

sometimes written as

$$\nabla > \nabla_{ad} = \left(\frac{d \ln T}{d \ln p} \right)_S = 1 - \frac{1}{\gamma}$$

This is the Schwarzschild criterion for convective instability.
If there is a vertical gradient of chemical composition,

$$\nabla_\mu = \frac{d \ln \mu}{d \ln p}, \quad \text{then} \quad N^2 = \frac{g\delta}{H_p} \left(\nabla_{ad} - \nabla + \nabla_\mu \right)$$

with $H_p = p|dp/dz|^{-1}$ the pressure scale height, and $\delta = (\partial \ln \rho / \partial \ln T)_p$,

$$\nabla > \nabla_{ad} + \nabla_\mu$$

is called the Ledoux criterion for convective instability.

Convective instability: effects of dissipation

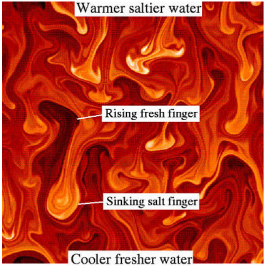
Convective instability is a *dynamical* process - it does not require dissipation to run; Dissipation changes the instability criterion to

$$N^2 < -\frac{C}{t_v t_d},$$

where $C > 0$ is a constant (depending on geometry) and t_v and t_d are viscosity and heat diffusion timescales.
Inside stars,

- ⋆ $t_d < t_v$, the diffusion damps oscillations in stable regions,
- ⋆ Prandl number $Pr = \nu/\kappa = 10^{-9} - 10^{-6}$,

Example: double-diffusive convection (fluid with two different density gradients which have different rates of diffusion e.g., heated water with salinity gradient).



Convective instability: the Boussinesq approximation

In order to model convection, a following approximation is used:

- ⋆ Variables, such as pressure fluctuation p' change about their means,
- ⋆ Velocity $\mathbf{u}$ is considered a fluctuation,
- ⋆ Density fluctuations are ignored in the continuity equation (anelastic approximation: $\partial\rho/\partial t = 0$, filtering the high-frequency sound waves),
- ⋆ differences in inertia are negligible,

in order to get the following set of equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p' - \frac{\rho'}{rho}\mathbf{g} + \nu\nabla^2\mathbf{u},$$

$$\nabla\cdot\mathbf{u} = 0,$$

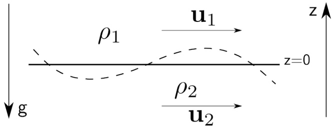
$$\frac{\partial T'}{\partial t} + \mathbf{u}\cdot\nabla T' - \beta\mathbf{e_z}\cdot\mathbf{u} = \text{radiative exchange term},$$

with $\beta = (T/H_p)(\nabla - \nabla_{ad})$ is called the superadiabatic lapse rate.

Good approximation when $H_p,\ H_\rho \gg l$ (small changes in p and ρ, sometimes not true for stellar convection).

The Kelvin-Helmholtz instability

This instability occurs at the interface between two fluids moving w.r.t each other with velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ (shear instability).



Consider a small perturbation $\zeta(x)$ of the interface (the dashed line):

$$\zeta = A \exp(ikx - i\omega t)$$

For incompressible, irrotational perturbations, the small perturbation $\mathbf{u}'$ is expressed by a scalar potential ϕ:

$$\mathbf{u}' = \nabla\phi, \quad \nabla^2\phi = 0.$$

Since the velocity potentials obey the Laplace equation,

$$\phi_1 = C_1 \exp(ikx - i\omega t - kz), \quad \phi_2 = C_2 \exp(ikx - i\omega t + kz).$$

The Kelvin-Helmholtz instability

The vertical components of the velocity on either side must match the substantial derivative of the interface x-displacement $\zeta(x,t)$:

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \zeta}{\partial t} + u_1 \frac{\partial \zeta}{\partial x}, \quad \frac{\partial \phi_2}{\partial z} = \frac{\partial \zeta}{\partial t} + u_2 \frac{\partial \zeta}{\partial x}.$$

at the interface $z = 0$. This means

$$-kC_1 = -i\omega A + ikAu_1, \quad kC_2 = -i\omega A + ikAu_2. \qquad (\#1)$$

Another condition is that the normal stress across the interface must be continuous (continuity of pressure). The momentum equation is

$$\nabla\left(\frac{\partial \phi}{\partial t}\right) + \nabla\left(\frac{1}{2}\mathbf{u}^2\right) = -\frac{1}{\rho}\nabla p - g\mathbf{e_z}.$$

To linear order

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = -\frac{p}{\rho} - gz + const.$$

The Kelvin-Helmholtz instability

The continuity of pressure is then

$$-\rho_1\left(\frac{\partial\phi_1}{\partial t}+u_1\frac{\partial\phi_1}{\partial x}+g\zeta\right)=-\rho_2\left(\frac{\partial\phi_2}{\partial t}+u_2\frac{\partial\phi_2}{\partial x}+g\zeta\right).$$

At the interface ($z=0$),

$$\rho_1\left(ikC_1u_1-i\omega C_1+gA\right)=\rho_2\left(ikC_2u_2-i\omega C_2+gA\right). \qquad (\#2)$$

Combining (#1) and (#2) we have, for $A\neq 0$:

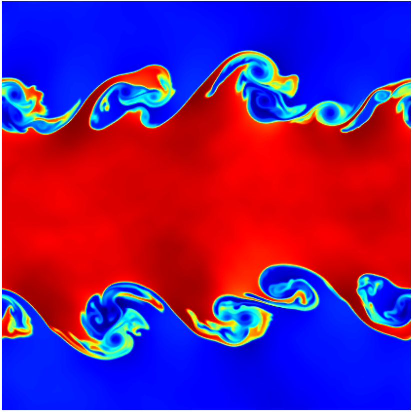
$$\rho_1\left(\omega-ku_1\right)^2+\rho_1 gk=-\rho_2\left(\omega-ku_2\right)^2+\rho_2 gk, \quad \text{that is}$$

$$\left(\omega-k\bar{u}\right)^2=\frac{(\rho_2-\rho_1)gk}{\rho_1+\rho_2}-\frac{\rho_1\rho_2(u_1-u_2)^2k^2}{(\rho_1+\rho_2)^2},$$

with $\bar{u}=\left(\rho_1u_1+\rho_2u_2\right)/(\rho_1+\rho_2)$ a density-weighted average speed. The configuration is unstable if

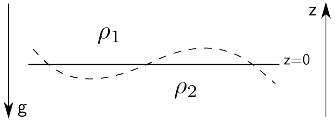
$$\left(\omega-k\bar{u}\right)^2<0.$$

The Kelvin-Helmholtz instability



The Rayleigh-Taylor instability

Rayleigh-Taylor instability can be regarded as a special case of Kelvin-Helmholtz instability for $u_1 = u_2$. If $\rho_1 > \rho_2$, the instability develops ($\mathbf{g}$ an effective gravity - accelerated shocks etc.)



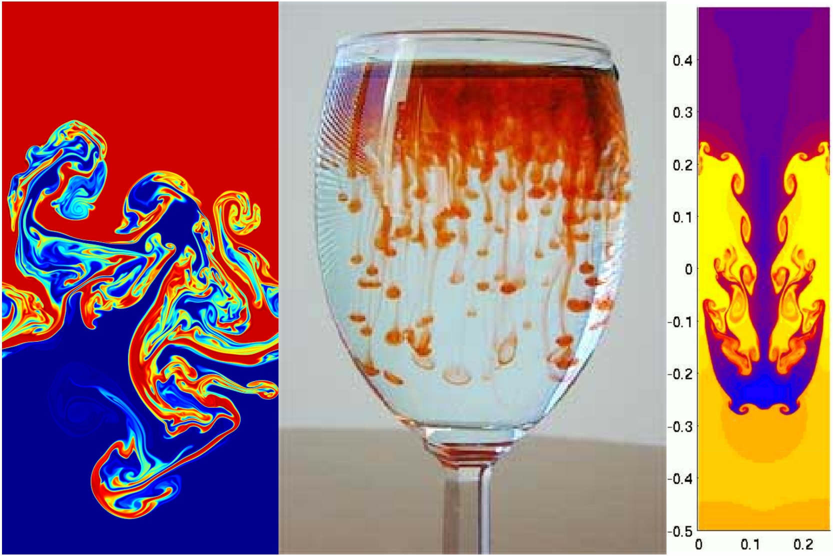
The condition is

$$\omega^2 = \frac{(\rho_2 - \rho_1)gk}{\rho_1 + \rho_2}.$$

For $\rho_1 = 0$, the dispersion relation is $\omega^2 = gk$, like for *surface gravity waves*.

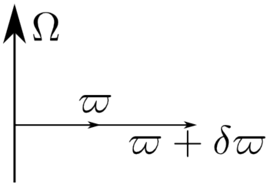
The energy source for this instability is the potential energy stored in the initial configuration (e.g., denser fluid on top).

The Rayleigh-Taylor instability



Rotational instability

Consider an instability associated with a rotation inside a star, neglecting gravity and viscosity. In cylindrical coordinates, for a distance ϖ from the axis, the equilibrium between pressure forces and centrifugal forces is



$$\frac{1}{\rho}\frac{dp}{d\varpi} + \varpi\Omega^2 = 0.$$

For a displacement of a fluid element from ϖ to $\varpi + \delta\varpi$,

- ⋆ The specific angular momentum $\varpi^2\Omega$ is conserved (no viscosity),
- ⋆ The pressure force at $\varpi + \delta\varpi$ is $(\varpi + \delta\varpi)\Omega^2(\varpi + \delta\varpi)$.

The net force per unit mass felt by the fluid element displaced to $\varpi + \delta\varpi$ is, to first order in $\mathcal{O}(\delta\varpi)$

$$(\varpi + \delta\varpi)\left(\frac{\varpi^2\Omega(\varpi)}{(\varpi + \delta\varpi)^2}\right) - (\varpi + \delta\varpi)\left(\Omega(\varpi + \delta\varpi)\right)$$

$$= -\frac{1}{\varpi^3}\frac{d}{d\varpi}\left(\varpi^4\Omega^2\right)\delta\varpi = N_\Omega^2\delta\varpi$$

Rotational instability

The equation of motion of the fluid element is

$$\frac{d^2\delta\varpi}{dt^2} + N_\Omega^2\delta\varpi = 0$$

(instability for $N_\Omega^2 < 0$). From the previous slide, to be stable, the rotational profile must satisfy

$$\frac{1}{\varpi^3}\frac{d}{d\varpi}\left(\varpi^4\Omega^2\right) > 0.$$

(Sometimes called the Rayleigh discriminant). In realistic situations, strong differential rotation is needed to trigger this dynamical instability - often, other instabilities start earlier.

Critical Richardson and Reynolds numbers

The Kelvin-Helmholtz instability shows how stable stratification halts the onset of instability. For the velocity varying with z, the stabilizing effect is measured by the Richardson number:

$$Ri = \frac{\text{potential energy}}{\text{kinetic energy}} \propto \frac{gh}{u^2} \propto \frac{N^2}{(du/dz)^2}$$

In case of no dissipation, the sufficient condition for instability is $Ri < 1/4$.

The onset of turbulence (transition from a laminar flow) can be determined by the critical Reynolds number:

$$Re = Lu/\nu$$

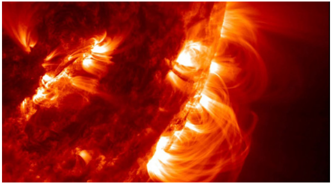
with L and u characteristic length scale and speed of the flow and ν kinematic viscosity. In absence of other forces (e.g., buoyancy) critical $Re \simeq 1000$.

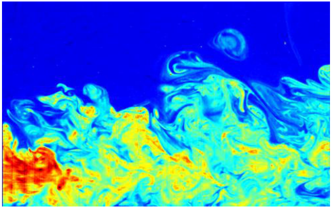
What is turbulence?

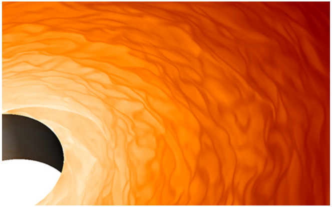
- loosely, random motion of fluid with many nonlinear interacting modes involved.
- chaotic property changes,
- low momentum diffusion, high momentum convection,
- rapid variation of p and u in space and time.

In astrophysics, plasma is usually turbulent:

- magnetic dynamo action,
- density structures in the interstellar medium,
- star formation,
- scintillation of radio sources,
- cosmic ray acceleration and scattering
- solar corona…





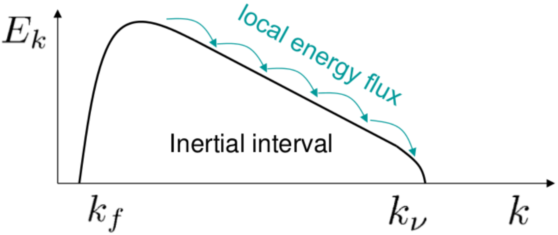


Kolomogorov phenomenology

The onset of turbulence - transition from the laminar regime in which the kinetic energy dies out due to the action of fluid viscosity.

- ⋆ the flow develops smaller and smaller scales of motion until the molecular viscosity starts operating.
- ⋆ energy is supplied at large scales, gets redistributed over fluctuations of different scales and removed (dissipated as heat) at small scales by viscosity.

In steady state rate of energy supply = rate of energy transfer = rate of energy dissipation. *Kinetic energy spectrum* is



where wavenumbers k_f is called outer (forcing) scale, and k_ν is called inner (viscous) scale.

Kolomogorov spectrum

Assume that the turbulent flow is in statistical equilibrium (averages of physical quantities independent of time; Kolomogorov theory is *mean field theory*)

- ⋆ energy supplied at rate ϵ on length scale $l_f = k_f^{-1}$ (outer scale),
- ⋆ energy per unit mass is $1/2u_f^2$.

From dimensional analysis

$$[\epsilon] = L^2 T^{-3}, \quad [\nu] = L^2 T^{-1}$$

The viscous characteristic length scale $l_\nu = k_\nu^{-1}$ should depend on ϵ and ν - one has, on dimensional grounds

$$l_\nu \sim \left(\nu^3/\epsilon\right)^{1/4}$$

l_ν is sometimes called the Kolomogorov microscale length; related are the *timescale* $\tau_\nu = (\nu/\epsilon)^{1/2}$ and *velocity scale* $u_\nu = (\nu\epsilon)^{1/4}$.

Kolomogorov spectrum

The kinetic energy per unit mass on scale l_f depends on l_f and ϵ. Again using dimensional analysis,

$$u_f^2 \sim (\epsilon l_f)^{2/3}$$

The relation between l_ν and l_f is then

$$l_\nu \sim (Re)^{-3/4} l_f. \quad \text{where} \quad Re = l_f u_f/\nu.$$

The interval between k_f and k_ν is called the *inertial range*. The kinetic energy spectrum $E_k(t)$ is defined using the average kinetic energy per unit mass,

$$\frac{1}{2}\langle u^2 \rangle = \int_0^\infty E_k \, dk.$$

Because of viscosity, the integral has a cut-off at $k = k_\nu$. With $[E_k] = L^3 T^{-2}$, in the inertial range the energy density scales as $k^{-5/3}$:

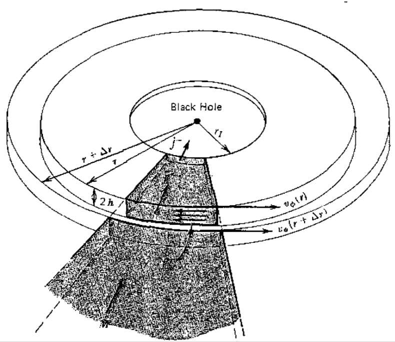
$$E_k = C\epsilon^{2/3} k^{-5/3}.$$

This is the Kolomogorov scaling for a homogeneous turbulence.

Accretion disk 'problem'

Imagine matter in-falling under the influence of gravity:

- ⋆ Conservation of angular momentum,
- ⋆ In-falling matter has often too much angular momentum → accretion disk is formed:
    - ⋆ Proto-planetary disks,
    - ⋆ AGNs,
    - ⋆ LMXBs & HMXBs, Roche lobe overflow.
- ⋆ **The problem**: how to transport the angular momentum out so that the matter can fall in?



Accretion disk 'problem'

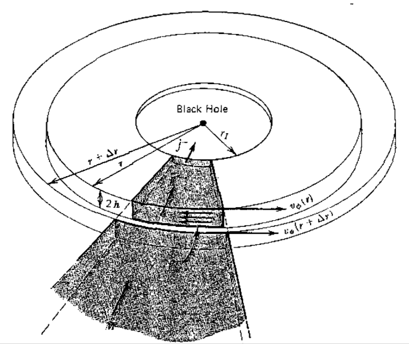
The simplest assumption - Keplerian Newtonian accretion disk,

$$\Omega(r) \propto r^{-3/2}$$

which gives

$$\frac{d\Omega}{dr} < 0.$$

Evolution equation for the specific angular momentum $l \propto r^2\Omega$,



$$\rho\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right) l = \underbrace{\frac{1}{r}\frac{d}{dr}\left(r^3\rho\nu\frac{d\Omega}{dr}\right)}_{torque}.$$

From dimensional analysis of the above, the accretion time is,

$$\frac{\rho l}{\tau} \propto \rho\nu\Omega \rightarrow \tau \propto \frac{l}{\nu\Omega} = \frac{r^2}{\nu}$$

Accretion disk 'problem'

Source of the viscosity? First guess: standard molecular viscosity, resulting from thermal collisions between individual gas particles:

$$\nu \propto a_T \lambda, \quad \text{with} \quad a_T = (kT/m)^{1/2} \quad \text{the typical thermal velocity,}$$

and λ denoting the mean free path. For 'typical' (average) values for accretion disks,

- ⋆ outer radius $R \sim 10^{10}\ cm$,
- ⋆ temperature $T \sim 10^4 K$,
- ⋆ density $n \sim 10^{16}\ cm^{-3}$

$$\lambda = \frac{k^2 T^2}{\pi n e^4} \sim 10^{-3}\ cm, \quad a_T \sim 10^6\ cm\ s^{-1}, \quad \nu \sim 10^3\ cm^2\ s^{-1}.$$

This gives the accretion rate τ

$$\tau = \frac{R^2}{\nu} \sim 10^{17}\ s = 3 \times 10^9\ yr,$$

which is much too long to explain the observed accretion rates in X-ray binaries and proto-stellar disks.

Shakura-Sunyaev α disk

- A proposition that shear-driven hydrodynamic turbulence could lead to an enhanced viscosity,
- The effective viscosity is parametrized:

$$\nu = \alpha H c_s,$$

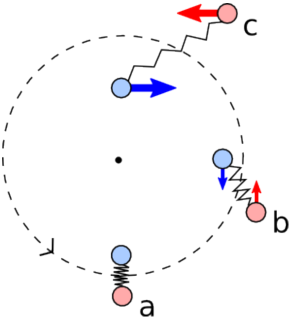
  with H the thickness of the disk and c_s the sound speed. The corresponding stress tensor component is

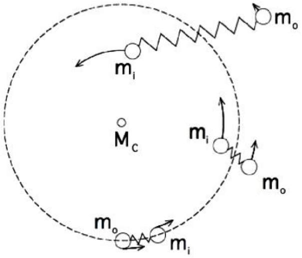
$$T_{r\phi} = \alpha p.$$

- $\alpha \in (0.01, 1)$ to match observations.
- Alas,
  - Turbulence from the shear flow, shear instabilities, barotropic/baroclinic instabilities, sound waves, shocks, finite amplitude instabilities are all not sufficient.
- An alternative is MHD turbulence: the magneto-rotational instability (MRI), discovered in the late 50s (Velikov, Chandrasekhar), used for accretion disks by Balbus & Hawley (1991).

Schematic explanation of MRI

- ⋆ Imagine two masses on nearby Keplerian orbits connected with the spring,
- ⋆ The inner mass is moving faster than the outer mass,
- ⋆ Due to the interaction through the spring, the inner mass is pulled backwards, the outer mass pulled forwards,
- ⋆ As a result, the angular momentum is transported outwards,
- ⋆ For nearby masses, MRI destabilizes the circular motion.





More detailed explanation of MRI

Imagine a fluid element on the orbit with angular velocity Ω at r_0. It is influenced by

- centrifugal force $r\Omega^2(r)$,
- centripetal force $-GM/r^2$.

For a small departure δr from r_0, in the rotating frame (one needs to take the Coriolis force, $-2\Omega \times u$ and the centrifugal force, $r\Omega^2$ into account), the net force is

$$r\left(\Omega_0^2 - \Omega^2(r_0 + \delta r)\right) \simeq -r\delta r \frac{d\Omega^2}{dr} + \mathcal{O}(\delta r^2).$$

This leads to the equations of motion in the x and y directions,

$$\frac{d^2x}{dt^2} - 2\Omega_0 \frac{dy}{dt} = -xr\frac{d\Omega^2}{dr} + f_x, \qquad \frac{d^2y}{dt^2} + 2\Omega_0 \frac{dx}{dt} = f_y,$$

with f_x and f_y the (possible) external forces per unit mass.

More detailed explanation of MRI: Rayleigh criterion

In absence of external forces, the solutions depend on time as $\exp(i\omega t)$, where ω satisfies the following dispersion relation:

$$\omega^2 = 4\Omega_0^2 + r\frac{d\Omega^2}{dr} \equiv \kappa^2.$$

(κ^2 is called *the epicyclic frequency*). It may equivalently be written as

$$\kappa^2 \propto \frac{1}{r^3}\frac{d(r^4\Omega^2)}{dr},$$

which shows that it is proportional to the radial derivative of the specific angular momentum $r^2\Omega$. For stability

$$\frac{d(r^2\Omega)}{dr} > 0 \quad \text{(specific angular momentum increases outwards).}$$

This is the Rayleigh criterion for stability.

More detailed explanation of MRI: 'spring' force

In case of external restoring forces, $f_x = -Kx$, $f_y = -Ky$, the dispersion relation for solutions $x, y \propto \exp(i\omega t)$ is

$$\omega^4 - (2K + \kappa^2)\omega^2 + K\left(K + r\frac{d\Omega^2}{dr}\right) = 0.$$

Setting $\omega^2 = 0$ shows that MRI is unstable when

$$K + r\frac{d\Omega^2}{dr} < 0.$$

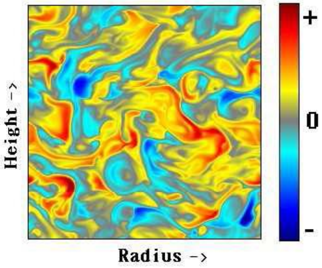
The Keplerian disk (with a weak magnetic field, small K) is unstable w.r.t the MRI instability. For $K = 0$,

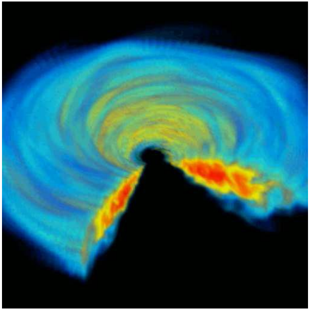
$$\Omega^2 = \frac{GM}{r^3} \quad \rightarrow \quad r\frac{d\Omega^2}{dr} = -3\Omega^2 < 0.$$

The growth rate of the fastest growing mode is

$$|\omega| = \frac{1}{2}\left|\frac{d\Omega}{d\ln r}\right|.$$

- From normal mode analysis: linearly unstable in ideal MHD,
- local behavior (insensitive to global boundary conditions),
- Triggered by weak magnetic field,
- Unstable in a regime that is Rayleigh-stable,
- Grows on a dynamical timescale.





The Plateau-Rayleigh instability

Instability of a laminar stream (jet), decrease of the total surface are under surface tension, → decomposition into droplets. For a cylinder of radius R and length $L \gg R$, what is the critical radius of spherical droplets, r?
Surface-to-volume ratios for cylinder and sphere are

$$\left(\frac{A}{V}\right)_c = \frac{2}{R}, \quad \left(\frac{A}{V}\right)_s = \frac{3}{r}.$$

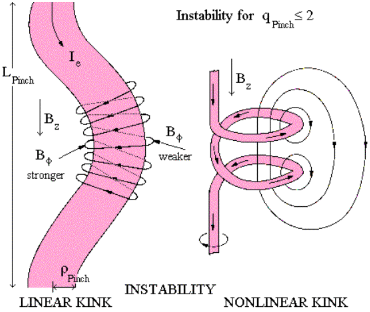
$$\text{for } V = const., \quad \frac{r}{R} \geqslant \frac{3}{2} \quad \text{or} \quad l \geqslant \frac{9}{2}R,$$

where l is the length of cylinder with volume $V = 4/3\pi r^3$.
Relation to astrophysics?

- ⋆ Systems with surface tension, like nuclear pasta in the interiors of neutron stars,
- ⋆ High-dimensional black strings/holes,
- ⋆ Astrophysical jets?

Kink instabilities

- ⋆ A class of MHD instabilities that can develop in a plasma column carrying a strong axial current,
- ⋆ Z-pinch - cylindrical plasma confinement that uses plasma electric current to compress it (Lorentz force).



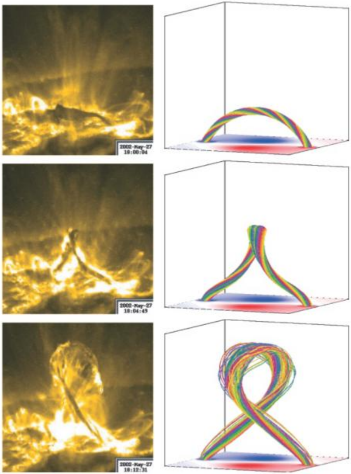


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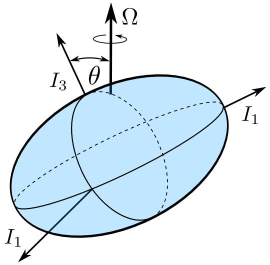
FIG. 1.—*Left*: *TRACE* 195 Å images of the confined filament eruption on 2002 May 27. *Right*: Magnetic field lines outlining the core of the kink-unstable flux rope (with start points in the bottom plane at circles of radius $b/3$) at t = 0, 24, and 37. The central part of the box (a volume of size 4^3) is shown, and the magnetogram, $B_z(x, y, 0, t)$, is included.

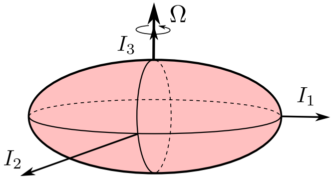
from Torok & Kliem (2004)

Instabilities of rotating neutron stars

- **Secular instability**: acts on slow (dissipative) scale. Dissipation can be viscosity or gravitational waves. For Newtonian Maclaurin spheroids ($\rho = const.$) required kinetic-to-potential energy ratio is $T/|W| = 0.1375$,
- **Dynamical instability**: acts on dynamical timescale (for NS $\sim 1\ ms$), required kinetic-to-potential energy ratio is $T/|W| > 0.26$ (may require exotic EOS and/or differential rotation),

which eventually leads to non-axisymmetric motion:





Neutron stars' oscillations

- **f-mode (fundamental mode)**: Freq. $1.5 - 3\ kHz$, damping time $\sim 1\ s$, no nodes inside the star,
- **g-modes (gravity modes)**: restoring force is buoyancy - present if temperature or stratification gradients, tangential displacement bigger than radial ones. Freq. $< 1\ kHz$, damping times long, seconds or days,
- **p-modes (pressure modes)**: restoring force is pressure, there may be many of them with node structure in the star, mostly radial movement. Freq. $4 - 7\ kHz$, damping time $\sim 1\ s$,
- **w-modes (spacetime modes)**: related to relativistic structure of the star (e.g., ergosphere). Freq. $> 5\ kHz$, fast damping,
- **r-modes (Rossby, rotational modes)**: restoring force is the Coriolis force - present in rotating stars. Freq. similar to spin frequency, damping related to composition.

Oscillation modes

Oscillation usually described as an Lagrangian displacement vector ξ on the (r, θ, ϕ) sphere. It's a sum of toroidal and spheroidal (axial and polar) components.

In case of a non-rotating star:

$$\xi(r, \theta, \phi, t) = A(r) Y_l^m(\theta, \phi) e^{i\omega t}.$$

- ⋆ f, g and p modes are purely spheroidal - described by a (l, m) pair,
- ⋆ mode frequency ω degenerated w.r.t m,

In case of a rotating star:

$$\xi(r, \theta, \phi, t) = A(r, \theta) e^{im\phi} e^{i\omega t}.$$

- ⋆ some non-zero toroidal components,
- ⋆ Degeneracy in m removed by *mode splitting*.

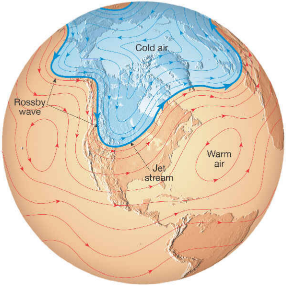
Rossby modes

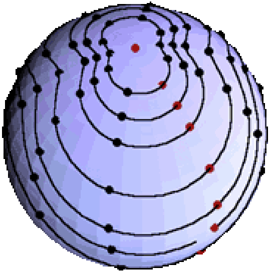
Rossby, (r-)modes of a $\rho = const.$ star:

- ⋆ non-rotating star - purely toroidal with $\omega = 0$,
- ⋆ rotating star - ξ acquires spheroidal components,

Mode frequency in a rotating frame is, to $\mathcal{O}(\Omega)$ order

$$\omega_r = \frac{2m\Omega}{l(l+1)}.$$



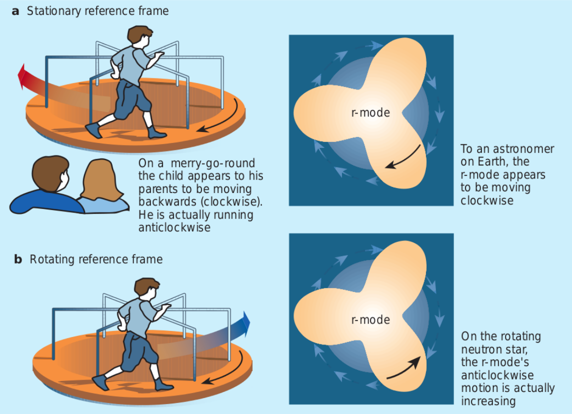


Chandrasekhar-Friedman-Schutz (CFS) instability

For non-axisymmetric modes, w.r.t distant observer:

- ⋆ if the mode rotates forward (with the star), it radiates angular momentum $J > 0$,
- ⋆ if it rotates backward, it radiates $J < 0$.

Because the rotation drags the mode, a *backward* mode with $J < 0$ may rotate *forward* w.r.t distant observer, radiate positive J $\rightarrow$ decrease J further $\rightarrow$ instability



CFS instability and r-modes

In the frame co-rotating with the star, the general perturbation is

$$\xi(r,\theta,\phi,t) = A(r,\theta)\exp(im\phi + i\omega t)$$

In the inertial (observer) frame

$$\phi' = \phi + \Omega t, \quad \text{so} \quad \xi(r,\theta,\phi',t) = A(r,\theta)\exp(im\phi' + i(\omega - m\Omega)t).$$

The mode frequency observed from far is

$$\omega_i = \omega_r - m\Omega.$$

In case of the lowest, most promising $l = m = 2$ r-mode, the observed frequency is

$$|\omega_i| = \left|\frac{2m\Omega}{l(l+1)} - m\Omega\right| = \left|\frac{4}{3}\Omega\right|.$$

Realistic star: damping timescales

In the simplest case of 'normal' *npe* matter:

- ⋆ **bulk viscosity**: matter locally off-β-equilibrium, energy loses due to modified URCA process,

$$n + n \to p + n + e^- + \bar{\nu}_e$$

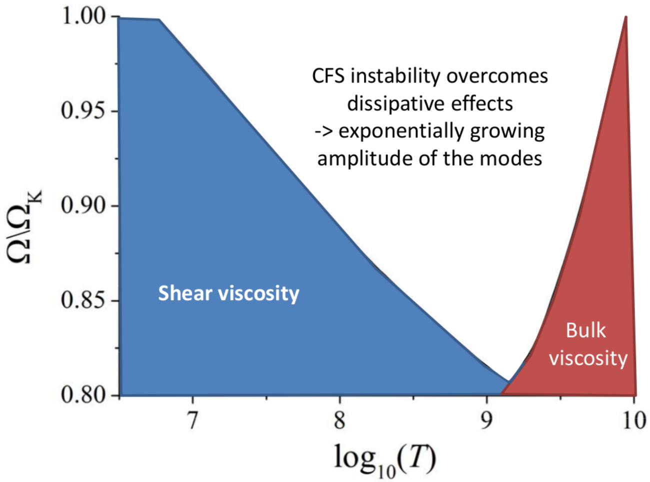
  Works at high temperatures $T > 10^9\ K$, viscosity coefficient $\propto T^6$,
- ⋆ **shear viscosity**: particles' Coulomb interactions (scattering) dissipates energy into heat. Works at low temperatures, coefficient $\propto T^{-2}$,
- ⋆ **Gravitational wave damping**: From the multipole formula (see e.g., Kokkotas & Stergioulas 1999), energy and energy loses are

$$E \propto \Omega^2, \quad \frac{dE}{dt} \propto -\omega_i^{2l+1}\omega_r \quad \text{and} \quad \tau_{GW} = \frac{1}{Im(\omega)} = -\frac{2E}{dE/dt}$$

All together, one has the damping timescale

$$\frac{1}{\tau} = \frac{1}{\tau_b} + \frac{1}{\tau_s} + \frac{1}{\tau_{GW}}$$

R-mode instability window



It could be the reason we do not observe very fastly spinning neutron stars...

Further reading...

- *"An introduction to astrophysical fluid dynamics"*, Michael J. Thompson
- *"The future of plasma astrophysics"*, `http://userpages.irap.omp.eu/~frincon/houches/program.html`
- *"Magnetorotational instability"*, Steven A. Balbus, `http://www.scholarpedia.org/article/Magnetorotational_instability`