Gravitational waves

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Outline

- ⋆ Wave equation in linearized GR,
- ⋆ the quadrupolar nature of gravitational waves,
- ⋆ Detection principle,
- ⋆ Astrophysical sources.

GR is nonlinear & fully dynamical $\rightarrow$ not so clear a distinction between waves and the rest of the metric. Speaking about waves is 'safe' in in certain limits:

- ⋆ linearized theory,
- ⋆ as small perturbations of a smooth background metric (gravitational lensing of waves, cosmological perturbations),
- ⋆ in post-Newtonian theory (far-zone, i.e., more than one wave- length distant from the source).

Wave equation in linearized GR

Since we anticipate the gravitational-wave component w.r.t the otherwise stationary (e.g., Minkowski) metric η to be small, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$ let's linearize the Einstein's equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad \text{where} \quad R = g^{\mu\nu} R_{\mu\nu}, \quad R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}$$

$$R^{\nu}_{\mu\rho\sigma} = \partial_\rho \Gamma^{\nu}_{\mu\sigma} - \partial_\sigma \Gamma^{\nu}_{\mu\rho} + \Gamma^{\nu}_{\lambda\rho} \Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\nu}_{\lambda\sigma} \Gamma^{\lambda}_{\mu\rho}, \quad R_{\nu\mu\rho\sigma} = g_{\nu\rho} R^{\rho}_{\mu\rho\sigma},$$

$$\Gamma^{\nu}_{\mu\rho} = \frac{1}{2} g^{\nu\lambda} (g_{\lambda\mu,\rho} + g_{\lambda\rho,\mu} - g_{\mu\rho,\lambda}).$$

At linear order in $h_{\mu\nu}$ the connection coefficients are

$$\Gamma^{\nu}_{\mu\rho} = \frac{1}{2} \eta^{\nu\lambda} (h_{\lambda\mu,\rho} + h_{\lambda\rho,\mu} - h_{\mu\rho,\lambda})$$

and

$$R^{\nu}_{\mu\rho\sigma} = \partial_\rho \Gamma^{\nu}_{\mu\sigma} - \partial_\sigma \Gamma^{\nu}_{\mu\rho} + \mathcal{O}(h^2) \rightarrow$$

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} \left(\partial_{\rho\nu} h_{\mu\sigma} + \partial_{\sigma\mu} h_{\nu\rho} - \partial_{\rho\mu} h_{\nu\sigma} - \partial_{\sigma\nu} h_{\mu\rho} \right).$$

Wave equation in linearized GR

To simplify previous expressions, the *trace-reversed* tensor is introduced:

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h, \quad \text{where} \quad h = \eta_{\alpha\beta}\, h^{\alpha\beta} \quad \text{and} \quad \bar{h} = -h.$$

With these changes, the Einstein equations are

$$\Box\bar{h}_{\nu\sigma} + \eta_{\nu\sigma}\,\partial^\rho\,\partial^\lambda \bar{h}_{\rho\lambda} - \partial^\rho\,\partial_\nu \bar{h}_{\rho\sigma} - \partial^\rho\,\partial_\sigma \bar{h}_{\rho\nu} + \mathcal{O}(h^2) = -\frac{16\pi G}{c^4}\, T_{\nu\sigma},$$

where $\Box = \eta_{\rho\sigma}\,\partial^\rho\,\partial^\sigma$ is the d'Alambert (wave) operator. In Cartesian terms

$$\Box = \eta_{\rho\sigma}\,\partial^\rho\,\partial^\sigma = -\frac{1}{c^2}\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2.$$

Further simplification is the use of *gauge* freedom; by imposing the Lorentz (de Donder, harmonic) gauge condition, $\partial_\nu \bar{h}^{\mu\nu} = 0$, one obtains

$$\Box\bar{h}_{\nu\sigma} = -\frac{16\pi G}{c^4}\, T_{\nu\sigma}.$$

Gauge transformation freedom

Consider an infinitesimal coordinate transformation,

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x^{\beta}), \quad \text{with } \xi^{\alpha} \text{ small in the sense that } |\partial_{\beta}\xi^{\alpha}| \ll 1.$$

This imply

$$\frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta} + \partial_{\beta}\xi^{\alpha}, \quad \text{and} \quad \frac{\partial x^{\alpha}}{\partial x'^{\beta}} = \delta^{\alpha}_{\beta} - \partial_{\beta}\xi^{\alpha} + \mathcal{O}((\partial\xi)^2).$$

Recalling that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $g'_{\mu\nu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}(x)$,

one finally gets

$$g'_{\alpha\beta} = \eta_{\alpha\beta} + \underbrace{h_{\alpha\beta} - \partial_{\alpha}\xi_{\beta} - \partial_{\beta}\xi_{\alpha}}_{h'_{\alpha\beta}} + \mathcal{O}(h\partial\xi, (\partial\xi)^2) \qquad (\xi_{\alpha} = \eta_{\alpha\beta}\xi^{\beta}).$$

Because $|\partial_{\beta}\xi^{\alpha}| \ll 1$ the metric perturbation $h'_{\alpha\beta}$ is small, the approximation is still valid. Applied to metric perturbation $\bar{h}'_{\alpha\beta}$:

$$\bar{h}'_{\alpha\beta} = \bar{h}_{\alpha\beta} - \partial_{\alpha}\xi_{\beta} - \partial_{\beta}\xi_{\alpha} + \eta_{\alpha\beta}\partial_{\mu}\xi^{\mu}.$$

Transverse-traceless gauge

We have limited 10 degrees of freedom of a symmetric 4×4 tensor $h_{\mu\nu}$ to 6 independent components by imposing the Lorentz gauge. In vacuum, where the waves propagate, $T_{\mu\nu} = 0$,

$$\Box \bar{h}_{\nu\sigma} = 0.$$

$\rightarrow$ speed of the wave equals speed of light c. Concerning the remaining degrees of freedom, in Lorentz gauge one can always consider coordinate transformations

$$\overline{h'}_{\nu\sigma} = \bar{h}_{\nu\sigma} + \xi_{\mu\nu}, \quad \text{where} \quad \xi_{\mu\nu} = \eta_{\mu\nu}\,\partial_\rho \xi^\rho - \xi_{\mu,\nu} - \xi_{\nu,\mu} \;\rightarrow\; \Box \xi_{\mu\nu} = 0$$

$\Box \xi_{\mu\nu} = 0$ means fixing 4 from 6 remaining degrees of freedom, for example choosing ξ^0 such that $\bar{h} = 0$ and ξ^i such that $h^{it} = 0$, $\partial_t h^{tt} = 0$:

$$h^{tt} = 0, \quad h^{ti} = 0, \quad \partial_i h^{ij} = 0, \quad h^{ii} = 0.$$

This is the definition of the *transverse-traceless* tensor h_{ij}^{TT}.

Transverse-traceless tensor h_{ij}^{TT}

For a propagation direction n^i, the transversality condition means $n^i\, h_{ij}^{\mathrm{TT}} = 0$ (in the TT gauge the gravitational wave is described by 2×2 matrix in the plane orthogonal to the direction of propagation $\mathbf{n}$).
Assuming the plane wave that propagates along the z-axis

$$h_{ij}^{\mathrm{TT}}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos\left(\omega t - kz\right),$$

where h_+ and $h_\times$ are two independent polarization states (two remaining degrees of freedom).
h_+ and $h_\times$ are the *helicity states* - in the form described by the TT gauge they change under rotation of ϕ around $\mathbf{n}$ as

$$h \rightarrow e^{i\mathbf{S}\cdot\mathbf{n}\phi} h \quad \text{where} \quad \mathbf{S} = \text{particle spin}$$

$$h_\times \pm i\, h_+ \rightarrow e^{\mp 2i\,\phi}\left(h_\times \pm i\, h_+\right).$$

GW interaction with a point particle in the TT gauge

Let's consider a test particle, at rest at $\tau = 0$. From the geodesic equation one has

$$\frac{d^2x^i}{d\tau^2}_{|\tau=0} = -\left(\Gamma^i_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right)_{|\tau=0} = -\left(\Gamma^i_{tt}\frac{dx^t}{d\tau}\frac{dx^t}{d\tau}\right)_{|\tau=0},$$

with $(dx^\mu/d\tau)_{\tau=0} = (c, 0)$ and

$$\Gamma^i_{tt} = \frac{1}{2}\eta^{ij}\left(\partial_t h_{tj} + \partial_t h_{jt} - \partial_j h_{tt}\right), \quad \text{but} \quad h_{tt} = 0,\ h_{tj} = 0 \text{ so } (\Gamma^i_{tt})_{\tau=0} = 0.$$

If at $\tau = 0$ $dx^i/d\tau = 0$, also $d^2x^i/d\tau^2 = 0$ and a particle at rest before the GW arrives remains at rest. What varies is the *proper distance* between the particles. For a plane wave in the $z-$direction,

$$ds^2 = -c^2dt^2 + dx^2(1 + h_+\cos(\omega t - kz)) + dy^2(1 - h_+\cos(\omega t - kz)) + 2dxdy\, h_\times\cos(\omega t - kz) + dz^2.$$

If particles A and B set down initially along the x-axis, we have

$$s \simeq L\left(1 + \frac{h_+}{2}\cos\omega t\right),$$

where L is the initial, unperturbed distance between particles A and B.

Newtonian tidal force & geodetic deviation

For two particles A and B falling in Euclidean space under gravitational potential Φ. At $t = 0$ separation is $\xi = \mathbf{x}_A - \mathbf{x}_B$ and $\mathbf{v}_A(t=0) = \mathbf{v}_B(t=0)$. The evolution of ξ because of $\mathbf{g} = -\nabla\Phi$

$$\frac{d^2\xi^i}{dt^2} = -\left(\frac{\partial\Phi}{dx^i}\right)_{\mathrm{B}} + \left(\frac{\partial\Phi}{dx^i}\right)_{\mathrm{A}} \simeq - \underbrace{\left(\frac{\partial^2\Phi}{\partial x^i\,\partial x^j}\right)}_{Tidal\ gravity\ tensor\ \mathcal{E}^i_j} \xi^j$$

In GR for two neigboring geodesics $x^\mu(\tau)$ and $x^\mu(\tau) + \xi^\mu(\tau)$, the geodetic equation,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}(x)\,\frac{dx^\rho}{d\tau}\,\frac{dx^\sigma}{d\tau} = 0$$

By expanding the geodesic equation of particle B around the position of particle A and subtracting it from the geodesic equation of particle A, one gets

$$\boldsymbol{\nabla}_u\,\boldsymbol{\nabla}_u\xi^\mu = -R^\mu{}_{\nu\rho\sigma}\,\xi^\rho\,\frac{dx^\nu}{d\tau}\,\frac{dx^\sigma}{d\tau}\,,$$

with $u^\beta = dx^\beta/d\tau$. Nearby time-like geodesics are tidally *deviated* proportional to the Riemann tensor.

GWs in the free-falling frame

By changing the coordinates to a system in which $\Gamma^{\mu}_{\rho\sigma}(x) = 0$,

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^{\mu}_{\rho\sigma}(x)\,\frac{dx^\rho}{d\tau}\,\frac{dx^\sigma}{d\tau} = 0 \quad \rightarrow \quad \left(\frac{d^2 x^\mu}{d\tau^2}\right)_x = 0.$$

A particle is not experiencing acceleration (free-falling, FF). Let's chose a coordinate system in which $x^j = 0$ and $x^0 = \tau$ (coordinate time is proper time), the metric at the origin is Minkowski

$$ds^2 = -c^2\,dt^2 + d\mathbf{x}^2 + \mathcal{O}\left(\frac{|\mathbf{x}|^2}{\mathcal{R}^2}\right) = -c^2\,dt^2\,\left(1 + R_{itjt}\,x^i\,x^j\right)$$
$$-\,2c\,dt\,dx^i\left(\frac{2}{3}R_{tjik}\,x^j\,x^k\right) + dx^i dx^j\left(\delta_{ij} - \frac{1}{3}R_{ijkl}\,x^k\,x^l\right)$$

where $\mathcal{R}$ is the curvature radius $\mathcal{R}^{-2} = |R_{\mu\nu\rho\sigma}|$. On Earth, the detector is not in full free fall (acceleration $\mathbf{a} = -\mathbf{g}$ w.r.t a local inertial frame), but some directions (horizontal movement) may be used.

GWs in the free-falling frame

Let's investigate the geodesic deviation in this frame:

$$\nabla_u \nabla_u \xi^\alpha = u^\beta \nabla_\beta (u^\lambda \nabla_\lambda \xi^\alpha) = u^\beta u^\lambda (\partial_{\beta\lambda}\xi^\alpha + \Gamma^\alpha_{\lambda\sigma,\beta}\, \xi^\sigma),$$

which simplifies because $\Gamma^\alpha_{\lambda\sigma} = 0$, particles are initially at rest ($u^\beta = \delta^\beta_0$), with $\xi^0 = 0$ and $\Gamma^j_{tk,t}$

$$\nabla_u \nabla_u \xi^j = \frac{d^2\xi^j}{d\tau^2} \quad \rightarrow \quad \frac{d^2\xi^j}{d\tau^2} = -R^j_{tkt}\, \xi^k$$

from the geodesic deviation equation. In linearized theory the Riemann tensor is invariant under change of coordinates, so in TT gauge

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\, (\partial_{\rho\nu} h_{\mu\sigma} + \partial_{\sigma\mu} h_{\nu\rho} - \partial_{\rho\mu} h_{\nu\sigma} - \partial_{\sigma\nu} h_{\mu\rho}) \quad \rightarrow \quad R^{\mathrm{TT}}_{jtkt} = -\frac{1}{2c^2}\, \ddot{h}^{\mathrm{TT}}_{jk}$$

$$\rightarrow \frac{d^2\xi^j}{dt^2} = \frac{1}{2} \ddot{h}^{\mathrm{TT}}_{jk}\, \xi^k$$

(also, in FF the coordinate distances and proper distances coincide)

GWs in the free-falling frame

The GW effect on a point particle can be described as an effect of a Newtonian force

$$\frac{d^2\xi^j}{dt^2} = \frac{1}{2}\ddot{h}^{\mathrm{TT}}_{jk}\xi^k \quad \leftrightarrow \quad F_i = \frac{m}{2}\ddot{h}^{\mathrm{TT}}_{ij}\xi^j .$$

Free-falling approximation to geodesic deviation applies as long as

- ⋆ $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(x^2/\mathcal{R}^2)$,
- ⋆ since $\mathcal{R}^{-2} = |R_{itjt}| \sim \ddot{h} \sim h/\lambda^2_{\mathrm{GW}}$, one has $x^2/\mathcal{R}^2 \simeq L^2\, h/\lambda^2_{\mathrm{GW}}$, and comparing with $\delta L/L \sim h \to L^2/\lambda^2_{\mathrm{GW}} \ll 1$,
- ⋆ Works for a ground based detector: for $L = 4$ km and $\lambda_{\mathrm{GW}} \sim 3000$ km,
- ⋆ Space-based detectors: $L \sim 10^6$ km set to observe GWs with wavelength comparable or shorter L - different strategy needed (time of flight, phase shifts measurements).

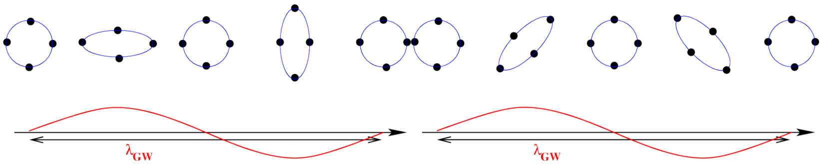
Gravitational-wave detection

GW acting on a ring of free-falling test particles, with x_0 and y_0 the unperturbed position at time $t = 0$. For $+$ polarization

$$h_{ij}^{\mathrm{TT}} = h_+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sin \omega t \,, \quad \xi_i = (x_0 + \delta x(t), y_0 + \delta y(t))$$

$$\delta x(t) = \frac{h_+}{2} x_0 \, \sin \omega t, \quad \delta y(t) = -\frac{h_+}{2} y_0 \, \sin \omega t, \quad \text{and likewise}$$

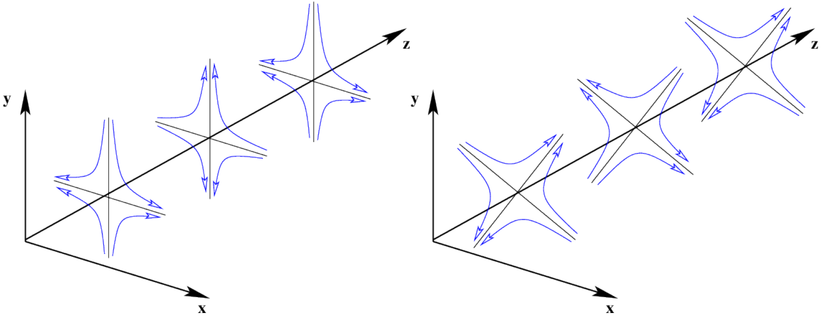
$$\text{for } \times \text{ polarization: } \delta x(t) = \frac{h_\times}{2} y_0 \, \sin \omega t, \quad \delta y(t) = \frac{h_\times}{2} x_0 \, \sin \omega t.$$



left - a wave with $+$ polarization, right - with $\times$ polarization.

Gravitational-wave detection

Lines of force for both polarizations are as follows:



The simplest detector - test mass m and an apparatus to check the change of distance L (e.g., a spring with the resonant frequency Ω and quality factor Q):

$$\ddot{\Delta L}(t) + 2\frac{\Omega}{Q}\,\dot{\Delta L}(t) + \Omega^2\,\Delta L(t) = \frac{L}{2}\left(F_+\,\ddot{h}_+(t) + F_\times\,\ddot{h}_\times(t)\right),$$

where F_+ and $F_\times$ depend on the direction of the source.

Quadrupolar nature of GWs

In electromagnetism, radiation due time changing electric dipole moment $\mathbf{d} = e\mathbf{x}$:

$$\text{Luminosity} \propto \ddot{\mathbf{d}}$$

Gravitational-wave emission in the dipole mode would mean the changing in time mass dipole moment:

$$\mathbf{d} = \sum_i m_i \mathbf{x}_i \quad \rightarrow \quad \dot{\mathbf{d}} = \underbrace{\sum_i m_i \dot{\mathbf{x}}_i}_{Momentum}$$

Conservation of momentum means no mass dipole GW radiation. Likewise, for the current dipole moment

$$\mathcal{M} = \underbrace{\sum_i m_i \mathbf{x}_i \times \dot{\mathbf{x}}_i}_{Angular\ \ momentum}$$

the conservation of angular momentum mean no current dipole GW radiation.

Estimate of wave amplitude

The wave equation for GWs,

$$\Box \bar{h}^{\alpha\beta} = \frac{16\pi G}{c^4} T^{\alpha\beta}$$

is an analogue to the Maxwell equation (Gauss law) in the Lorentz gauge, $(1/c^2)\partial_t \phi + \nabla \cdot \mathbf{A} = 0)$:

$$\nabla \cdot \mathbf{E} = \Box \phi = 4\pi\rho, \quad \text{where} \quad \mathbf{E} = -\nabla\phi + \partial_t \mathbf{A}.$$

By analogy between solutions

$$\phi(t, \mathbf{r}) = \int \frac{\rho(t - R/c, \mathbf{x})}{R} dV, \quad \bar{h}^{\alpha\beta} = \frac{4G}{c^4} \int \frac{T^{\alpha\beta}(t - R/c, \mathbf{x})}{R} dV.$$

with $R = |\mathbf{r} - \mathbf{x}|$. Far from the compact source ($r \gg \mathbf{x}$), the solution is described by the *far-field solution*:

$$\bar{h}^{\alpha\beta}(t, \mathbf{r}) = \frac{4G}{c^4 r} \int T^{\alpha\beta}(t - R/c, \mathbf{x}) dV.$$

Estimate of wave amplitude

Using the energy-momentum conservation it can be shown that

$$T^{\alpha\beta}_{,\beta} = 0 \quad \rightarrow \quad \bar{h}^{ij} \approx -\frac{2G}{c^4 r}\frac{d^2 I^{ij}}{dt^2},$$

where $I^{ij} = \int \rho x^i x^j dV$ is the moment of inertia tensor (quadrupole tensor).

Consider two equal masses M separated by a on circular orbit in the $x - y$ with angular frequency Ω around their center of mass. Then

$$I^{xx} = \int \rho x^2 dV = 2M\left(\frac{a}{2}\cos\Omega t\right)^2 = \frac{1}{4}Ma^2\left(1 + \cos 2\Omega t\right),$$

which leads to, and other terms like that,

$$\bar{h}^{xx} = \frac{2GMa^2\Omega^2}{c^4 r}\cos 2\Omega t.$$

Two important conclusions:

- ⋆ Quadrupole radiation: GW at twice the orbital frequency,
- ⋆ Amplitude $\propto GMa^2\Omega^2/c^4 r$.

Estimate of wave amplitude

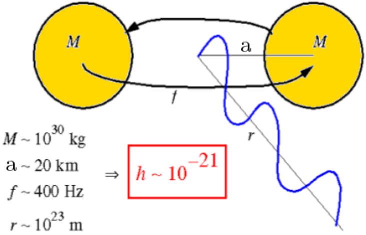
Example - binary system of two stellar-mass black holes:

- ⋆ $M = 10\ M_\odot$,
- ⋆ $a = 1\ R_\odot$,
- ⋆ $r = 8\ kpc$ (Galactic center)

From the Kepler's third law,

$$\Omega^2 = \frac{G(M+M)}{a^3} \approx 8\times10^{-4}\ \mathrm{rad}^2/\mathrm{s}^2$$

Orbital period 38 min → GW period 19 min. The GW strain amplitude is $h \sim 10^{-21}$.



Another example: binary neutron star pair, 10M light years distance (Virgo cluster), moving at $\sim 10\%$ of the speed of light → $h \simeq 10^{-21}$.

The quadrupole formula

The first term of the perturbation in the multipolar expansion far from the source is

$$\bar{h}_{\alpha\beta}(t,\mathbf{r}) \simeq \frac{4G}{c^4 R}\int T_{\alpha\beta}(t-R/c,\mathbf{x})dV.$$

By imposing the conservation equation for the energy-momentum tensor

$$\partial_\nu T^{\mu\nu} = 0 \quad \rightarrow \quad \frac{\partial T_{ij}}{\partial x^i} - \frac{\partial T_{tj}}{\partial x^t} = 0 \quad \text{and} \quad \frac{\partial T_{ti}}{\partial x^i} - \frac{\partial T_{tt}}{\partial x^t} = 0$$

it can be shown that

$$\int T_{\alpha\beta} dV \quad \text{can be expressed in terms of } T_{tt}.$$

The quadrupole formula

$$\#1: \frac{\partial T_{ij}}{\partial x^i} - \frac{\partial T_{tj}}{\partial x^t} = 0 \quad / \cdot x^k \quad \text{and integrating on all space} \;\rightarrow$$

$$\int x^k \frac{\partial T_{ij}}{\partial x^i} dV = \int x^k \frac{\partial T_{tj}}{\partial x^t} dV = \frac{\partial}{\partial x^t} \int x^k T_{jt} dV.$$

Integrating the left side by parts: $-\int T_{ij}\delta_i^k dV = \frac{\partial}{\partial x^t} \int x^k T_{jt} dV$

Symetrization: $\int T_{kj} dV = -\frac{1}{2}\frac{\partial}{\partial x^t} \int (x_k T_{tj} + x_j T_{tk}) dV$

$$\#2: \frac{\partial T_{ti}}{\partial x^i} - \frac{\partial T_{tt}}{\partial x^t} = 0 \quad / \cdot x_j x_k \quad \text{and integrating on all space} \;\rightarrow$$

$$\frac{\partial}{\partial x^t} \int T_{tt} x_j x_k dV = \int \frac{\partial T_{ti}}{\partial x^i} x_j x_k dV.$$

Integrating the right side by parts: $\frac{\partial}{\partial x^t} \int T_{tt} x_j x_k dV = -\int (x_k T_{tj} + x_j T_{tk}) dV$

Finally: $\int T_{kj} dV = \frac{1}{2c^2}\frac{\partial^2}{\partial t^2} \int T_{tt} x_j x_k dV.$

The quadrupole formula

For $T^{tt} = \mu c^2$,

$$\bar{h}_{ij} = \frac{2G}{c^4 R}\frac{\partial^2}{\partial t^2}\int \mu x_i x_j dV, \quad \text{In TT gauge: } \bar{h}_{ij}^{TT} = \frac{2G}{c^4 R}\mathcal{P}_i^k \mathcal{P}_j^l \ddot{I}_{kl},$$

where $I_{kl} = \int \rho(x_k x_l - \frac{1}{3}x^2\delta_{kl})dV$, $\quad \mathcal{P}_i^k = \delta_i^k - n^i n_k$ (projection operator).

Similarly one can show from the conservation equation #2 that

$$\int T_{tj} dV = \frac{\partial}{\partial x^t}\int T_{tt} x_j dV$$

and that the power P radiated per solid angle in a direction $\mathbf{n}$ is

$$\frac{dP}{d\Omega} = R^2 n^i \Theta^{it}, \quad \text{where} \quad \Theta^{it} = \frac{c^4}{32\pi G}\left\langle \partial_t \bar{h}_\alpha^\beta \partial_i \bar{h}_\beta^\alpha \right\rangle.$$

The radiated GW power averaged over polarizations is

$$P = \frac{G}{5c^5}\dddot{I}_{ij}^2.$$

The quadrupole formula: estimates

For a source of mass M, dimension L and deviation from sphericity ϵ,

$$P = \frac{G}{5c^5}\dddot{I}_{ij}^2 \quad \text{with} \quad I \propto \epsilon M L^2 \quad \rightarrow \quad \dddot{I} \sim \omega^3 \epsilon M L^3,$$

where $\omega \sim 1/\tau$ (source characteristic frequency)

$$\rightarrow \; P \sim \frac{G}{c^5}\epsilon^2 \omega^6 M^2 L^4, \quad \text{with} \quad \frac{G}{c^5} = 3.6 \times 10^{50} \; erg/s.$$

Some estimates:

- ⋆ From Misner-Thorne-Wheeler: steel bar of $M \simeq 500$ *tonnes*, $L = 20\; m$, $\omega \sim 30\; rad/s$:

$$GM\omega/c^3 \sim 10^{-32}, \quad Lc^2/GM \sim 10^{25}, \quad P \sim 10^{-27}\; erg/s \sim 10^{-60}\; P_\odot^{EM}$$

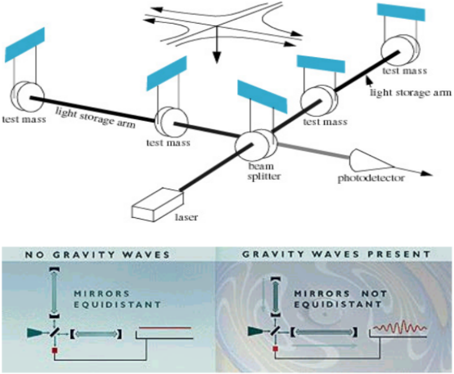
- ⋆ parametrization by Weber: $R_s = 2GM/c^2$, $\omega = (v/c)(c/L)$,

$$P = \frac{c^5}{G}\epsilon^2 \left(\frac{v}{c}\right)^6 \left(\frac{R_s}{L}\right) \quad \text{for } v \sim c,\; L \sim R_s \quad P \sim \frac{c^5}{G}\epsilon^2 \sim 10^{26}\; P_\odot^{EM}.$$

Detectors

Interferometer for GWs

- The concept is to compare the time it takes light to travel in two orthogonal directions transverse to the gravitational waves.
- The gravitational wave causes the time difference to vary by stretching one arm and compressing the other.
- The interference pattern is measured (or the fringe is split) to one part in 10^{10}, in order to obtain the required sensitivity.

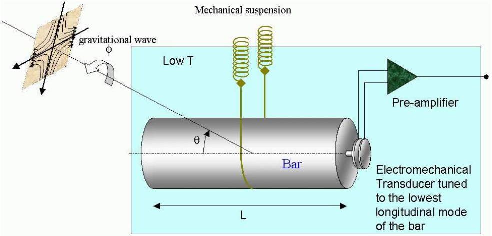


As seen by the detector, gravitational wave strain $h = \Delta L/L$, for $L \simeq$ km, $\Delta L < 10^{18}$ m (much smaller than the size of a proton).

Virgo detector (Cascina near Pisa, arm length - 3km)

Bar detectors

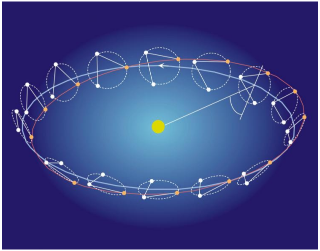
Passing GW resonates with the characteristic frequency of the bar (narrow frequency searches, typically $\sim 500\ Hz$):

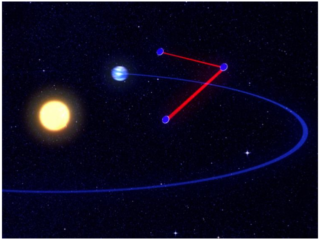


Amplitude of the vibrations is $\Delta L \sim hL$, response quite complicated.

Space-based interferometry

- ⋆ Local geometry of geodesic deviation is not enough to analize the free-falling bodies response to the passing gravitational wave, $L \geqslant \lambda_{GW}$. Proposed space-borne mission eLISA with $L \sim 10^6$ km and study GW in the range 0.1 $mHz < f_{GW} < 1$ Hz,
- ⋆ Each spacecraft carries a free-falling test mass,
- ⋆ Direct reflection from mirrors not possible due to loses ($\sim$ 1 photon/day detection rate) $\rightarrow$ transponder mode,
- ⋆ Time-delay interferometry, central spacecraft compares the phases of lasers from "arm spacecrafts" (absolute lengths of arms known up to 1 m).





Triangular formation with center 20° behind the Earth, each spacecraft on an individual orbit around the Sun.

Space-based interferometry: time measurements

Let's consider an interferometer with an arm in the x =direction. In a case for pure + polarization (TT gauge, wave travels to $z-$direction),

$$ds^2 = -dt^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 + dz^2,$$

the coordinate speed along x-axis is $(dx/dt)^2 = 1/(1 + h_+)$. A photon emitted at time t from $x = 0$ reaches the end of arm at $x = L$ at the coordinate time

$$t_1 = t + \int_0^L \sqrt{1 + h_+(t(x))}dx, \quad \text{(implicit knowledge of } t(x) \text{ needed).}$$

One must know the time to reach x in order to calculate h_+.
In linear approximation, h_+ small: expansion in powers of h_+ and assuming zero-order solution of a photon in flat space-time traveling with $c = 1$, $t(x) = t + x$:

$$t_1 = t + L + \frac{1}{2}\int_0^L h_+(t + x)dx.$$

Space-based interferometry: time measurements

Round-trip time (after reflection/transmission back) takes

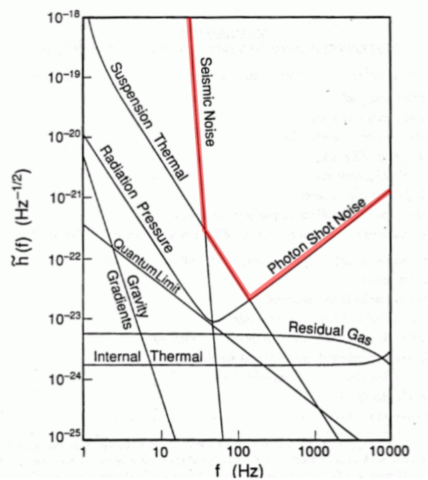
$$t_r = t + L + \frac{1}{2}\left(\int_0^L h_+(t+x)dx + \int_0^L h_+(t+x+L)dx\right).$$

The variation of t_r means changing GW background (L is fixed),

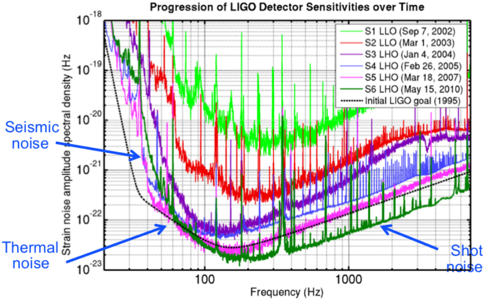
$$\frac{dt_r}{dt} = 1 + \frac{1}{2}\left(h_+(t+2L) - h_+(t)\right),$$

depending only on the wave amplitude h_+.

Sources of noise - the detector sensitivity



(1989 LIGO proposal)



GW detector output time series:

$$s(t) = F^{+}(t) \circ h_{+}(t) + F^{\times}(t) \circ h_{\times}(t) + n(t)$$

In Fourier domain, strain amplitude spectral density is $h(f) = \sqrt{S(f)} = \sqrt{\tilde{s}^{*}(f)\tilde{s}(f)}$,

where $\tilde{s}(f) = \int_{-\infty}^{\infty} e^{-2\pi i f t} s(t) dt$.

Sources of noise of ground-based detectors

- ⋆ **Seismic noise**: important below 100 Hz, falls with frequency; multiple pendula with characteristic freq. ~ 1 Hz attenuating the ground vibrations etc.,
- ⋆ **Thermal noise**: vibrations of the mirrors and suspension pendulum. Their characteristic frequencies designed to be either small (< 1 Hz, pendulum) or large (> 1 kHz, mirrors) and high quality factors to narrow the resonances. Typically dominant at ~ 100 Hz,
- ⋆ **Photon shot noise**: due to quantization of laser light, number of particles that hit the mirror varies δN $\rightarrow$ random light intensity variations, and resulting lenght variations is

$$\delta L_{shot} \sim \frac{\lambda}{2\pi\sqrt{N}}$$

To measure freq. f one needs at least $2f$ measurements/s, so the relation between the number of photons N and the laser power P is

$$N = \frac{2fP\lambda}{hc}, \quad \text{for } \delta L_{shot} = \delta L_{GW} \;\rightarrow\; P = 600\ kW(!)$$

Solution: power recycling of laser light by reflecting it many times in the arm and coherently adding in phase.

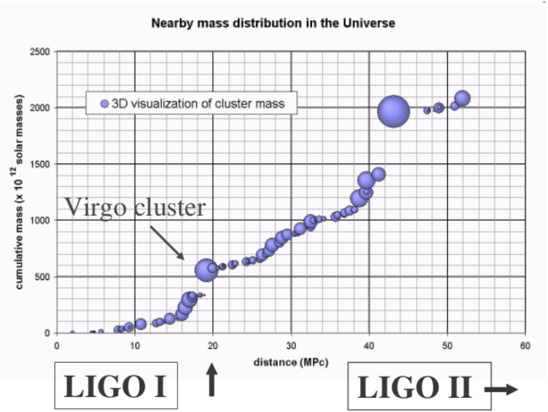
Types of sources

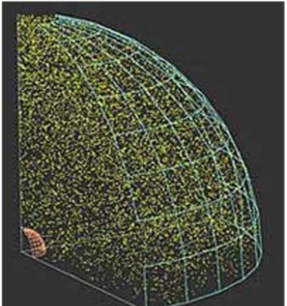
- **bursts**: short in duration, modulation due to the detector motion is negligible (SN explosions, collapses, inspiral of NS and stellar mass BHs etc.); more than one detector (3 for triangulation) needed to 'do astrophysics',
- **continuous waves**: long-lived and steady, motion of the detector modulates the phase and amplitude (binary systems, rotating NSs); in principle one detector pin-points the signal on the sky,
- **stochastic background**: cosmic origin of GW noise, an excess of power in certain range - can be studied only if the detector noise is well-understood; cross-correlation between detectors needed to confirm.

How many sources can we see?

Improve amplitude sensitivity by a factor of 10x, and…

$\Rightarrow$ Number of sources goes up 1000x!





Sensitivity inversly proportional to the distance (amplitude of the wave measured)

Initial and advanced detectors' rates

- Really need an 'Advanced' detector with about a factor of 10 greater sensitivity, broader bandwidth –
- Since gravitational waves are an amplitude phenomenon, x1000 more volume searched, plus yet greater reach due to bandwidth:

IFO	Source[a]	$\dot{N}_{\mathrm{low}}\ \mathrm{yr}^{-1}$	$\dot{N}_{\mathrm{re}}\ \mathrm{yr}^{-1}$	$\dot{N}_{\mathrm{high}}\ \mathrm{yr}^{-1}$	$\dot{N}_{\mathrm{max}}\ \mathrm{yr}^{-1}$
	NS–NS	2×10^{-4}	0.02	0.2	0.6
	NS–BH	7×10^{-5}	0.004	0.1	
Initial	BH–BH	2×10^{-4}	0.007	0.5	
	IMRI into IMBH			<0.001[b]	0.01[c]
	IMBH-IMBH			10^{-4}[d]	10^{-3}[e]
	NS–NS	0.4	40	400	1000
	NS–BH	0.2	10	300	
Advanced	BH–BH	0.4	20	1000	
	IMRI into IMBH			10[b]	300[c]
	IMBH-IMBH			0.1[d]	1[e]

- At ~40 events per year, the rate is much more attractive!

Binary coalescence time

From the Newtonian point of view, effective energy of a system is

$$E = \frac{1}{2}\mu v^2 - \frac{GM\mu}{r} = -\frac{GM\mu}{2r} \quad \rightarrow \quad r = -\frac{GM\mu}{2E},$$

where $M = m_1 + m_2$ and $\mu = m_1 m_2/(m_1 + m_2)$.

$$\dot{r} = \frac{dr}{dE}\frac{dE}{dt} = -\frac{64}{5}\frac{GM^2\mu}{r^3} \quad \rightarrow \quad r(t) = (r_0^4 - \frac{256}{5}GM^2\mu\Delta\tau_{coal})^{1/4}.$$

$$\text{If } r(t_{coal}) \ll r_0 \quad \rightarrow \quad \Delta\tau_{coal} = \frac{5c^5}{256}\frac{r_0^4}{GM^2\mu}$$

- ⋆ Virgo/LIGO stellar mass black hole binary: $M = 10\ M_\odot + 10\ M_\odot$, $r_0 \simeq 500\ km$, $f_{GW} \sim 40\ Hz$: $\Delta\tau_{coal} \sim 1\ s$,
- ⋆ eLISA supermassive black hole binary: $M = 10^6\ M_\odot + 10^6\ M_\odot$, $r_0 \simeq 10^8\ km$, $f_{GW} \sim 10^{-5}\ Hz$: $\Delta\tau_{coal} \sim 1\ yr$

Binary coalescence parameters

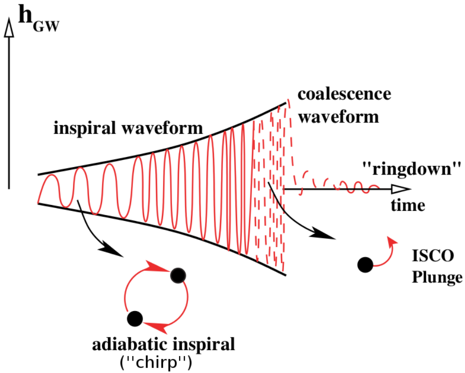
In mass quadrupole approximation (h_{ij}^{TT}),

$$h_{GW} \propto \omega^{2/3} \cos 2\Psi,$$

where for quasi-circular orbits (Kepler)

$$\omega^2 = \frac{GM}{r^3}$$

The signal is called *the chirp* - amplitude rises with frequency:



$$h \propto \frac{M_{chirp}^{5/3} f_{GW}^{2/3}}{r}$$

where $M_{chirp} = (m_1 m_2)^{3/5}/(m_1 + m_2)^{1/5}$. For $r = 100\ Mpc$, $f_{GW} \simeq 100\ Hz$, $M_{chirp} \sim 10$, $h \sim 10^{-21}$.

Binary system inspiral

Second example: due to the emission of GWs, binary system orbital period and separation decreases. In Newtonian terms, the orbital energy change is the power emitted in GWs

$$\frac{dE_{orb}}{dt} = -P, \quad \text{with} \quad E_{orb} = -\frac{Gm_1 m_2}{2R}, \quad \text{and} \quad \omega^2 = \frac{GM}{R^3},$$

For adiabatic, quasi-circular orbits ($\dot{\omega}/\omega \ll 1$),

$$\dot{R} = -\frac{2}{3} R\omega \left(\frac{\dot{\omega}}{\omega^2}\right), \quad \text{with} \quad \frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\nu \left(\frac{GM\omega}{c^3}\right)^{5/3},$$

where $\nu = \mu/M$ is the symmetric mass ratio; $M = m_1 + m_2$ and $\mu = m_1 m_2/(m_1 + m_2)$. If GW is purely $f = 2\omega$,

$$\dot{f}_{GW} = \frac{96}{5}\pi^{8/3} \left(\frac{GM_{chirp}}{c^3}\right)^{5/3} f_{GW}^{11/3} \rightarrow f_{GW} \simeq 130 \left(\frac{1.21 M_\odot}{M_{chirp}}\right)^{5/8} \left(\frac{1\ s}{\tau}\right)^{3/8} Hz$$

Coalescence times τ 17 min, 2 s, 1 ms for $f_{GW} = 10, 100, 10^3$ Hz.

Binary system inspiral

One can obtain the relation between radial separation and the GW frequency,

$$R \simeq 300 \left(\frac{M}{2.8 M_\odot}\right)^{1/3} \left(\frac{100\ Hz}{f_{GW}}\right)^{2/3} km\,.$$

and the number of GW cycles,

$$\mathcal{N}_{GW} = \frac{1}{\pi}\int_{t_i}^{t_f} \omega(t)\, dt = \frac{1}{\pi}\int_{\omega_i}^{\omega_f} \frac{\omega}{\dot{\omega}}\, d\omega.$$

For $\omega_{\rm f} \gg \omega_{\rm i}$, one gets

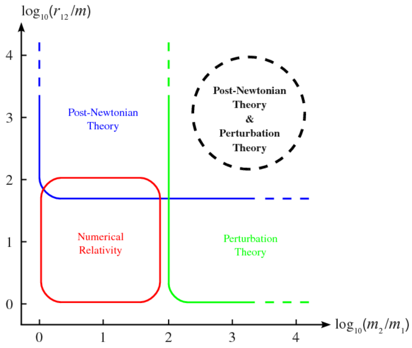
$$\mathcal{N}_{\rm GW} \simeq 10^4 \left(\frac{M_{chirp}}{1.21 M_\odot}\right)^{-5/3} \left(\frac{f_i}{10\ \ Hz}\right)^{-5/3} .$$

Post-Newtonian expansions

In GR, the two-body problem is not fully solved (needed for accurate template banks for filter-matching detection statistics). Different approaches:

- ⋆ numerical relativity,
- ⋆ perturbation-based self-force approach (extreme ratio inspirals, $m_1/m_2 \ll 1$),
- ⋆ post-Newtonian expansion:
  - ⋆ 0th order - Newtonian gravity,
  - ⋆ nth PN order - corrections of order

$$\left(\frac{v}{c}\right)^{2n} \propto \left(\frac{Gm}{rc^2}\right)^n .$$



(Blanchet et al., Phys. Rev. D 81, 064004, 2010)

Post-Newtonian expansions

Expansion in small parameter, which can be

$$\left(\frac{v}{c}\right)^2 \sim |h_{\mu\nu}| \sim \left|\frac{\partial_0 h}{\partial_i h}\right|^2 \sim \left|\frac{T^{0i}}{T^{00}}\right| \sim \left|\frac{T^{ij}}{T^{00}}\right|$$

For the parameter $\dot{\omega}/\omega^2$,

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\,\nu\, v_\omega^{5/3} \sum_{k=0}^{7} \omega_{(k/2)\mathrm{PN}}\, v_\omega^{k/3} \propto \mathcal{O}\left(\frac{v}{c}\right)^5 ,$$

$$\begin{aligned}
\omega_{0\mathrm{PN}} &= 1\,, \\
\omega_{0.5\mathrm{PN}} &= 0\,, \\
\omega_{1\mathrm{PN}} &= -\frac{743}{336} - \frac{11}{4}\,\nu\,, \\
\omega_{1.5\mathrm{PN}} &= 4\pi + \left[-\frac{47}{3}\frac{S_\ell}{M^2} - \frac{25}{4}\frac{\delta m}{M}\frac{\Sigma_\ell}{M^2}\right], \\
\omega_{2\mathrm{PN}} &= \frac{34\,103}{18\,144} + \frac{13\,661}{2\,016}\,\nu + \frac{59}{18}\,\nu^2 - \frac{1}{48}\,\nu\,\chi_1\chi_2 \Big[247\,(\hat{\boldsymbol{S}}_1\cdot\hat{\boldsymbol{S}}_2) - \\
&\quad 721\,(\hat{\boldsymbol{\ell}}\cdot\hat{\boldsymbol{S}}_1)(\hat{\boldsymbol{\ell}}\cdot\hat{\boldsymbol{S}}_2)\Big]\,,
\end{aligned}$$

etc.

Post-Newtonian expansions

Post-Newtonian contributions to the number of GW cycles accumulated from $\omega_{\rm in} = \pi \times 10\,{\rm Hz}$ to $\omega_{\rm fin} = \omega^{\rm ISCO} = 1/(6^{3/2}\,M)$ for binaries detectable by LIGO and VIRGO. We denote $\kappa_i = \hat{\mathbf{S}}_i \cdot \hat{\boldsymbol{\ell}}$ and $\xi = \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$.

	$(10+10)M_\odot$	$(1.4+1.4)M_\odot$
Newtonian	601	16034
1PN	$+59.3$	$+441$
1.5PN	$-51.4 + 16.0\,\kappa_1\,\chi_1 + 16.0\,\kappa_2\,\chi_2$	$-211 + 65.7\,\kappa_1\,\chi_1 + 65.7\,\kappa_2\,\chi_2$
2PN	$+4.1 - 3.3\,\kappa_1\,\kappa_2\,\chi_1\,\chi_2 + 1.1\,\xi\,\chi_1\,\chi_2$	$+9.9 - 8.0\,\kappa_1\,\kappa_2\,\chi_1\,\chi_2 + 2.8\,\xi\,\chi_1\,\chi_2$
2.5PN	$-7.1 + 5.5\,\kappa_1\,\chi_1 + 5.5\,\kappa_2\,\chi_2$	$-11.7 + 9.0\,\kappa_1\,\chi_1 + 9.0\,\kappa_2\,\chi_2$
3PN	$+2.2$	$+2.6$
3.5PN	-0.8	-0.9

(this and previous slides from A. Buonanno lecture, arxiv:0709.4682)

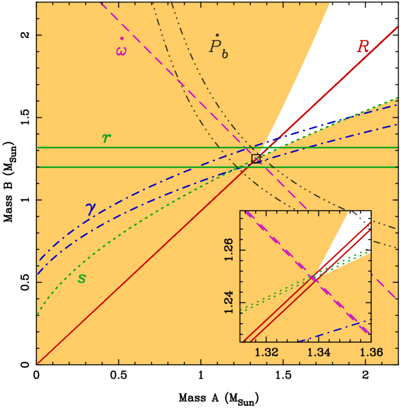
Neutron stars in relativistic binaries: PSR J0737-3039

Post-Keplerian parameters

- Periastron advance:
  $$\dot{\omega} = 3 \left(\frac{P_b}{2\pi}\right)^{-5/3} (T_\odot M)^{2/3} (1 - e^2)^{-1}$$
- Orbit decay:
  $$\dot{P}_b = -\frac{192\pi\, m_p m_c}{5 M^{1/3}} \left(\frac{P_b}{2\pi}\right)^{-5/3} \times \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right)(1 - e^2)^{-7/2}\, T_\odot^{5/3}$$
- Shapiro effect:
  $$r = T_\odot m_c,$$
  $$s = \frac{a_p \sin i}{c\, m_c} \left(\frac{P_b}{2\pi}\right)^{-2/3} T_\odot^{-1/3} M^{2/3}$$
- Gravitational redshift:
  $$\gamma = e \left(\frac{P_b}{2\pi}\right)^{1/3} T_\odot^{2/3} M^{-4/3} m_c (M + m_c)$$

where $T_\odot = G M_\odot / c^3$, $M = m_p + m_c$.

(All measurements compatible with GR so far)



PSR J0737-3039A/B:

- Pulsar A: $P = 22.7$ ms, pulsar B: $P = 2.77$ s,
- Orbital period $\simeq 2.4$ h,
- eccentricity $\simeq 0.08$,
- Orbit decay $\simeq 7$ mm/day.

Orbital decay $\dot{P}_b$ test for GR with the PSR J0348+0432

The most relativistic NS-white dwarf binary to date:

PSR J0348+432:

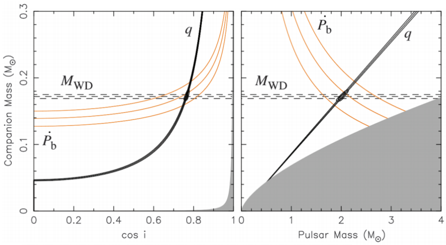
- ★ Pulsar mass: $2.01 \pm 0.04\ M_\odot$, WD mass: $0.172 \pm 0.003\ M_\odot$,
- ★ Orbital period $P_b \simeq 2.4$ h,
- ★ $\dot{P}_b = -2.73 \times 10^{-11}$ s/s
- ★ $\dot{P}_b/\dot{P}_b^{GR} = 1.05 \pm 0.18$

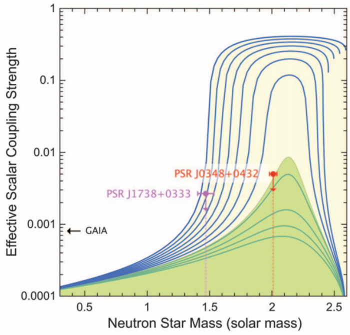
Testing scalar-tensor theories of gravity - dipolar term in $\dot{P}_b$:

$$\dot{P}_b^{dipolar} \simeq -\frac{4\pi^2 G}{c^3 P_b}\frac{m_p m_c}{m_p+m_c}(\alpha_p-\alpha_c)^2$$

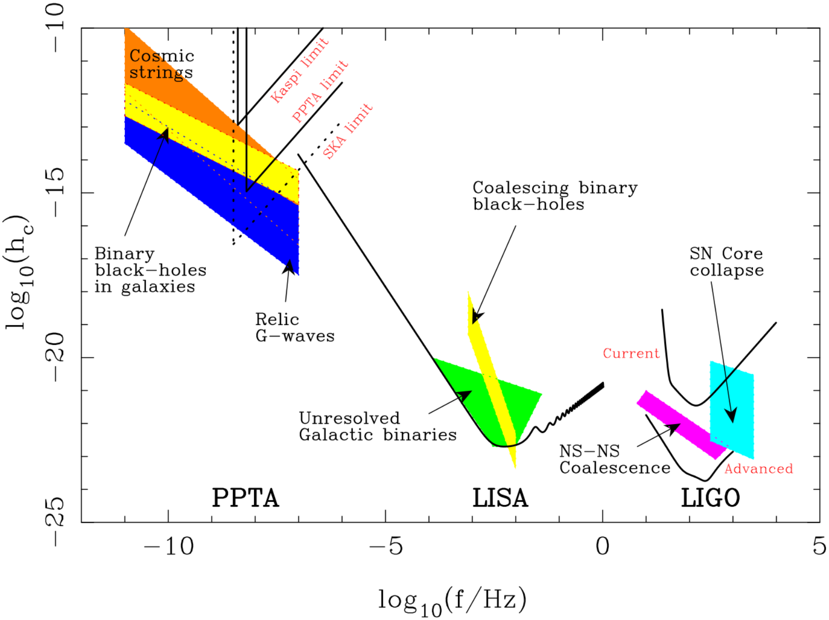
$|\alpha_p - \alpha_0| < 0.005$ based on the comparison with $\dot{P}_b^{GR}$

(linear term $\alpha_0 < 0.004$ from weak-field experiments)





Other detectors



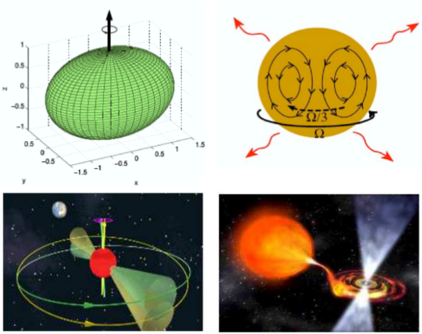
Continuous GWs from rotating neutron stars

Time-varying quadrupole moment needed:

- ⋆ Mountains (supported by elastic and/or magnetic stresses in the NS crust and/or core),
- ⋆ Oscillations (r-modes)
- ⋆ Free precession,
- ⋆ Accretion from the companion (deformations, thermal gradients, magnetic fields).

Main characteristics of such GWs:

- ⋆ periodic, $f_{\rm GW} \propto f_{\rm rot}$,
- ⋆ long-lived, $T > T_{\rm obs}$.



Estimated GW amplitude

Using the quadrupole formula, the amplitude is estimated as follows:

$$h_0 = 4 \times 10^{-25} \left(\frac{\epsilon}{10^{-6}}\right) \left(\frac{I}{10^{45}\ \mathrm{g\ cm^2}}\right) \left(\frac{f}{100\ \mathrm{Hz}}\right)^2 \left(\frac{100\ \mathrm{pc}}{d}\right)$$

where $\epsilon = (I_1 - I_2)/I$, I - moment of inertia.

Theoretical predictions for maximal possible deformations:

- ''Normal matter'', $\epsilon \leqslant 10^{-6} - 10^{-7}$ (Ushomirsky, Cutler & Bildsten 2000, Johnson-McDaniel & Owen 2012)
- Quark matter, $\epsilon \leqslant 10^{-4} - 10^{-5}$ (Owen 2005, Johnson-McDaniel & Owen 2012)

Spin-down limit for known pulsars

Limit on h_0, assuming that all rotational energy is lost in GWs

- ⋆ Change of rotational energy: $E_{\rm rot} = If^2,\ \dot{E}_{\rm rot} \propto I f \dot{f}$
- ⋆ GW luminosity: $\dot{E}_{\rm GW} \propto \epsilon^2 I^2 f^6$

$$\dot{E}_{\rm GW} = \dot{E}_{\rm rot} \quad \rightarrow \quad h_{\rm sd} = \frac{1}{d}\sqrt{\frac{5GI}{2c^3}\frac{|\dot{f}\,|}{f}} =$$

$$= 8 \times 10^{-24}\sqrt{\left(\frac{I}{10^{45}\ \mathrm{g\ cm^2}}\right)\left(\frac{|\dot{f}\,|}{10^{-10}\ \mathrm{Hz/s}}\right)\left(\frac{100\ \mathrm{Hz}}{f}\right)\left(\frac{100\ \mathrm{pc}}{d}\right)}\,.$$

$h_0 \leqslant h_{\rm sd} \rightarrow$ **upper limit on the deformation ϵ:**

$$\epsilon_{\rm sd} = 2 \times 10^{-5}\sqrt{\left(\frac{10^{45}\ \mathrm{g\ cm^2}}{I}\right)\left(\frac{100\ \mathrm{Hz}}{f}\right)^5\left(\frac{|\dot{f}\,|}{10^{-10}\ \mathrm{Hz/s}}\right)}\,.$$

Targeted searches

Spin-down limit has been beaten for Crab pulsar:

Epoch	$h_0^{95\%}$		Ellipticity		$h_0^{95\%}/h_0^{sd}$	
	Uniform	Restricted[a]	Uniform	Restricted[a]	Uniform	Restricted[a]
			Crab pulsar			
Model (1)[b]	2.6×10^{-25}	2.0×10^{-25}	1.4×10^{-4}	1.1×10^{-4}	0.18	0.14
Model (2)[c]	2.4×10^{-25}	1.9×10^{-25}	1.3×10^{-4}	9.9×10^{-5}	0.17	0.13
1.	4.9×10^{-25}	3.9×10^{-25}	2.6×10^{-4}	2.1×10^{-4}	0.34	0.27
2.	2.4×10^{-25}	1.9×10^{-25}	1.3×10^{-4}	1.0×10^{-4}	0.15	0.13

$<2\%$ of energy loss due to GW emission

ApJ, 713, 671, 2010: LIGO S5 data, Bayesian analysis

and Vela pulsar:

Analysis method	95% upper limit for h_0
Heterodyne, restricted priors	$(2.1\pm0.1)\times10^{-24}$
Heterodyne, unrestricted priors	$(2.4\pm0.1)\times10^{-24}$
$\mathcal{G}$-statistic	$(2.2\pm0.1)\times10^{-24}$
$\mathcal{F}$-statistic	$(2.4\pm0.1)\times10^{-24}$
MF on signal Fourier components, 2 d.o.f.	$(1.9\pm0.1)\times10^{-24}$
MF on signal Fourier components, 4 d.o.f.	$(2.2\pm0.1)\times10^{-24}$

$<35\%$ of energy loss due to GW emission; ellipticity $\epsilon<1.2\times10^{-3}$

ApJ, 737, 93, 2011: Virgo VSR2 data, Bayesian analysis, matched filtering

Indirect spin down limits

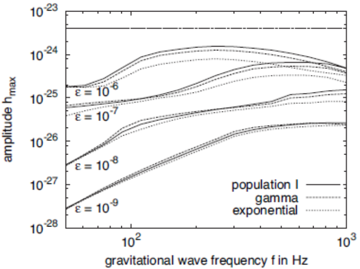
- For stars with unknown f, but known age $\tau = f/(4|\dot{f}|)$, h and ϵ estimated by assuming all energy lost in GWs:

$$h_{\rm isd} = 2 \times 10^{-23} \sqrt{\left(\frac{I}{10^{45}\ \rm g\ cm^2}\right)\left(\frac{1000\ \rm yr}{\tau}\right)}\left(\frac{100\ \rm pc}{d}\right),$$

- In accreting systems like Sco X-1, f unknown - accretion torque balanced by the GW emission (Papaloizou & Pringle 1978, Bildsten 1999, Chakrabarty et al., 2003); h related to flux in X-rays:

$$h_{acc} \approx 5 \times 10^{-27} \sqrt{\left(\frac{300\ \rm Hz}{f}\right)\left(\frac{F_x}{10^{-8}\ \rm erg\ cm^{-2}\ s^{-1}}\right)}$$

- Signal from hypothetical population of gravitars. Blandford limit (uniform distribution in the galactic disk) - $h \approx 4 \times 10^{-24}$, independent of ϵ and f. (more detailed study by Knispel & Allen 2008).



GWs from non-axisymmetric collapse

$$h_{GW} \sim 2 \times 10^{-17} \sqrt{\eta_{eff}} \left(\frac{1\ ms}{\tau}\right)^{1/2} \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{10\ kpc}{r}\right) \left(\frac{1\ kHz}{f_{GW}}\right),$$

where τ is the duration. Efficiency in case of core-collapse supernovae is estimated to be quite low,

$$\eta_{eff} = \frac{\Delta E}{Mc^2} \sim 10^{-7} - 10^{-10}.$$

Comparison with other kinds of radiation for SN at 20 kpc:

- ⋆ GW: $\sim 400 \left(\frac{1\ kHz}{f_{GW}}\right)^2 \left(\frac{h}{10^{-21}}\right)^2 \frac{erg}{cm^2\ s}$ in ms,
- ⋆ neutrinos: $10^5 \frac{erg}{cm^2\ s}$ in $\sim 10\ s$,
- ⋆ Optical: $\sim 10^{-4} \frac{erg}{cm^2\ s}$ in a week.

Electromagnetic vs gravitational waves: comparison

Electromagnetic waves:

- ⋆ radiation by accelerating charges (time changing dipole),
- ⋆ incoherent superposition of emission from electrons, atoms and molecules,
- ⋆ direct information about thermodynamics,
- ⋆ wavelenghts small compared to the source,
- ⋆ strong interaction with matter (absorption, scattering...)

Gravitational waves:

- ⋆ radiation by accelerating masses (time changing quadrupole),
- ⋆ coherent superposition of emission from moving masses,
- ⋆ direct information about the dynamics,
- ⋆ wavelengths large compared to the source,
- ⋆ small interaction with matter.

Further reading...

- *"Lecture Notes on General Relativity"*, Sean Carroll,
- *"Gravitational waves"*, A. Buonanno, `arXiv:0709.4682`,
- *"Gravitational waves, sources and detectors"*, B. F. Schutz, F. Ricci, *arXiv:1005.4735*,
- Living Reviews in Relativity: *"Gravitational-Wave Data Analysis. Formalism and Sample Applications: The Gaussian Case"*, P. Jaranowski & A. Królak,