Black holes

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Outline

- ⋆ Spherical black holes,
- ⋆ Weak field limit,
- ⋆ pressureless dust star collapse,
- ⋆ black holes and rotation,
- ⋆ orbits,
- ⋆ Penrose-Carter diagrams,
- ⋆ Penrose process and thermodynamics.

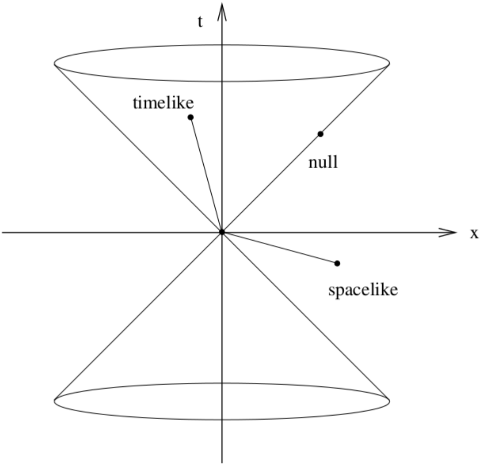
History of black holes

- ⋆ O. C. Rømer (1676) - from observations of the Jupiter moons from orbiting Earth → speed of light finite,
- ⋆ I. Newton (1686): gravitational force follows

$$F = -\frac{GMm}{r^2}$$

- ⋆ J. Michell (1783): *"All light emitted from such a body would be made to return towards it by its own proper gravity"*,
- ⋆ P.S. Laplace (1796): Exposition du système du monde ("dark stars")
- ⋆ A. Einstein (1905): Special relativity
- ⋆ A. Einstein (1915): General relativity (GR)
- ⋆ K. Schwarzschild (1916): First exact solution of GR - a black hole,
- ⋆ H. Reissner (1916), G. Nordström (1918): electrically charged black hole solution,
- ⋆ M. Kruskal & G. Szekeres (1960): Global structure of Schwarzschild,
- ⋆ R. Kerr (1963): rotating stationary black hole.

The light cone



Timelike $ds^2 < 0$; spacelike $ds^2 > 0$; null (light-like) $ds^2 = 0$.

The Schwarzschild solution

Motivated by the form of the Minkowski metric,

$$ds^2 = -dt^2 + dr^2 + r^2 \underbrace{\left(d\theta^2 + \sin^2\theta d\phi^2\right)}_{d\Omega^2},$$

let's chose a generally-enough form of a spherically-symmetric metric

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2 d\Omega^2.$$

To know the functions α and β, one must solve the Einstein equations ($\rightarrow$ connection coefficients $\rightarrow$ Riemann, Ricci tensors). The non-zero Christoffel symbols

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\beta}(g_{\beta\nu,\rho} + g_{\beta\rho,\nu} - g_{\nu\rho,\beta})$$

are

$$\Gamma^t_{tt} = \partial_t\alpha, \quad \Gamma^t_{tr} = \partial_r\alpha, \quad \Gamma^t_{rr} = e^{2(\beta-\alpha)}\partial_t\beta, \quad \Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha,$$

$$\Gamma^r_{tr} = \partial_t\beta, \quad \Gamma^r_{rr} = \partial_r\beta, \quad \Gamma^\theta_{r\theta} = \frac{1}{r}, \quad \Gamma^r_{\theta\theta} = -re^{-2\beta}, \quad \Gamma^\phi_{r\phi} = \frac{1}{r},$$

$$\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi_{\theta\phi} = \cot\theta.$$

The Schwarzschild solution

Ricci tensor non-zero components are:

$$R_{tt} = [\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta] + e^{2(\alpha-\beta)}[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha],$$
$$R_{rr} = -[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_r\beta] + e^{2(\beta-\alpha)}[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta],$$
$$R_{tr} = \frac{2}{r}\partial_t\beta, \quad R_{\theta\theta} = e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1, \quad R_{\phi\phi} = R_{\theta\theta}\sin^2\theta.$$

The solution is obtained by demanding $R_{\mu\nu} = 0$: trace-reversed version of the Einstein's equations is

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}T^\rho_\rho g_{\mu\nu} = 0, \quad (T_{\mu\nu} = 0 \text{ in vacuum}).$$

From $R_{tr} = 0 \;\rightarrow\; \partial_t\beta = 0$,
$\partial_t(R_{\theta\theta}) = 0$ and $\partial_t\beta = 0 \;\rightarrow\; \partial_t\partial_r\alpha = 0$.

That is $\begin{cases} \beta = \beta(r), \\ \alpha = f(r) + g(t). \end{cases}$

By redefining $dt \rightarrow e^{-g(t)}dt$,
$g(t) = 0$ so that $\alpha = f$.

The Schwarzschild solution

We therefore have a metric with components independent of the t coordinate:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2.$$

$\rightarrow$ stationarity, timelike Killing vector.
Another useful combination from the Ricci tensor is:

$$\underbrace{R_{rr} + R_{tt}}_{=0} e^{2(\beta-\alpha)} = \frac{2}{r}(\partial_r\alpha + \partial_r\beta) \rightarrow \alpha + \beta = \underbrace{const. = 0}_{coord.\ rescaling}$$

and

$$R_{\theta\theta} = e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1 = -e^{2\alpha}(2r\partial_r\alpha - 1) + 1 = 0,$$

$$\rightarrow \partial_r(re^{2\alpha}) = 1, \text{ that is } e^{2\alpha} = 1 + \frac{\mu}{r}.$$

The metric is then

$$ds^2 = -\left(1 + \frac{\mu}{r}\right)dt^2 + \left(1 + \frac{\mu}{r}\right)^{-1}dr^2 + r^2 d\Omega^2.$$

Weak field limit

In order to compare GR with the Newtonian theory, one must express GR in the limit of small velocities ($v/c \ll 1$) and time derivatives much smaller than spatial derivatives:

- relate the geodesic equation to Newton's law of motion,
- relate the Einstein equation to the Poisson equation.

Assume

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta}, \quad g^{\alpha\beta} = \eta^{\alpha\beta} - \epsilon h^{\alpha\beta} \quad (\text{because } g^{\mu\beta} g_{\nu\beta} = \delta^{\mu}_{\nu}).$$

The connection's Christoffel symbols are, in first order

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\beta}(g_{\beta\nu,\rho} + g_{\beta\rho,\nu} - g_{\nu\rho,\beta}) \approx \frac{1}{2}\epsilon\eta^{\mu\beta}(h_{\beta\nu,\rho} + h_{\beta\rho,\nu} - h_{\nu\rho,\beta})$$

The geodetic equation for slowly moving particle, for which $\tau \approx t$ and $dx^i/dt = \mathcal{O}(\epsilon)$:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\nu\rho}\frac{dx^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau} \to \frac{d^2 x^{\mu}}{dt^2} + \Gamma^{\mu}_{\nu\rho}\frac{dx^{\nu}}{dt}\frac{dx^{\rho}}{dt} \to \frac{d^2 x^{\mu}}{dt^2} + \Gamma^{\mu}_{tt}\frac{dx^{t}}{dt}\frac{dx^{t}}{dt} = 0.$$

Weak field limit

The spatial part of the geodetic equation (three-acceleration):

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{tt}\frac{dx^t}{dt}\frac{dx^t}{dt} = c^2\Gamma^i_{tt}, \quad \text{where} \quad dx^t/dt = c \quad (\text{'speed of time'}).$$

$$\Gamma^i_{tt} = \frac{1}{2}\epsilon\eta^{\mu\beta}(h_{\beta\nu,\rho} + h_{\beta\rho,\nu} - h_{\nu\rho,\beta}) \approx \frac{1}{2}\epsilon(\underbrace{h^i_{t,t} + h^i_{t,t}}_{small} - h_{tt}^{;i}) \approx -\frac{1}{2}\epsilon h_{tt}{}^{,i},$$

That is

$$\frac{d^2x^i}{dt^2} = \frac{c^2}{2}\epsilon h_{tt}{}^{,i} = \frac{c^2}{2}\epsilon\nabla^i h_{tt}, \quad \text{to be compared with} \quad \underbrace{\frac{d^2x^i}{dt^2} = -\nabla^i\Phi.}_{Newtonian\ equation\ of\ motion}$$

This means that one can identify the metric function g_{tt} with the Newtonian potential:

$$g_{tt} = \underbrace{\eta_{tt}}_{=-1} + \quad \epsilon h_{tt} = -\left(1 + \frac{2\Phi}{c^2}\right).$$

The Schwarzschild solution

Coming back to spherically-symmetric stationary metric:

$$ds^2 = -\left(1 + \frac{\mu}{r}\right) dt^2 + \left(1 + \frac{\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

The interpretation of the parameter μ in terms of physical quantities is done in the *weak field limit*. Far from the center,

$$g_{tt}(r \to \infty) = -\left(1 + \frac{\mu}{r}\right), \quad g_{rr}(r \to \infty) = \left(1 - \frac{\mu}{r}\right).$$

On the other hand, weak limit gives

$$g_{tt}(r \to \infty) = -\left(1 + \frac{2\Phi}{c^2}\right),$$

with the Newtonian potential $\Phi = -GM/r$. Therefore, the Schwarzschild metric finally reads:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

Spherically-symmetric pressureless collapse

Consider a collapse of a spherical star made of 'dust' (pressure $p = 0$). With $G = c = 1$ the outside metric is Schwarzschild vacuum solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

If radius of the star is $R(t)$, on the surface one has,

$$ds^2 = -\left(\left(1 - \frac{2M}{R}\right) - \left(1 - \frac{2M}{R}\right)^{-1}\left(\frac{dR}{dt}\right)^2\right) dt^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

and from symmetry the collapsing particles will infall in radial direction (follow radial timelike geodesics) $\rightarrow d\theta = d\phi = 0$:

$$\left(\frac{dt}{d\tau}\right)^2 \left(\left(1 - \frac{2M}{R}\right) - \left(1 - \frac{2M}{R}\right)^{-1}\left(\frac{dR}{dt}\right)^2\right) = -1$$

with $ds^2 = d\tau^2$ denoting the proper time.

Spherically-symmetric pressureless collapse

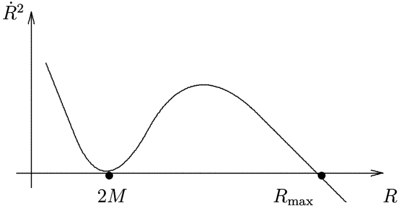
Schwarzschild spacetime admits one Killing vector, $\partial/\partial t$, responsible for time symmetries (conservation of energy).

$$\epsilon = -g_{t\mu}u^t = -g_{tt}\frac{dt}{d\tau} = \left(1 - \frac{2M}{R}\right)\frac{dt}{d\tau}$$

is specific energy of a particle, constant along the geodesic. This gives

$$\left(\frac{dR}{dt}\right)^2 = \dot{R}^2 = \frac{1}{\epsilon^2}\left(1 - \frac{2M}{R}\right)^2\left(\frac{2M}{R} - 1 + \epsilon^2\right)$$

(with $\epsilon < 1$ for bound particles).



For a collapse with $\dot{R}_{ini} = 0$ at $R_{max} = 2M/(1-\epsilon^2)$. R decreases approaching $R = 2M$ asymptotically (distant observer sees the collapse slowing down while it approaches $R = 2M$).

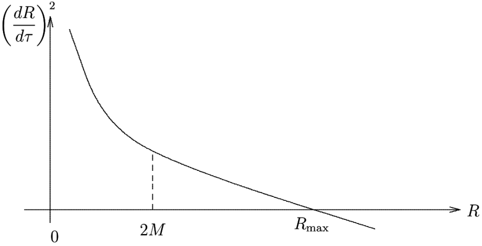
Spherically-symmetric pressureless collapse

What happens from the point of view of an infalling observer? Her clock measures the proper time along the radial geodesic, so one can rewrite

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \frac{1}{\epsilon}\left(1 - \frac{2M}{R}\right)\frac{d}{d\tau}$$

to obtain, from the previous expression

$$\left(\frac{dR}{d\tau}\right)^2 = \left(\frac{2M}{R} - 1 + \epsilon^2\right) = \left(\frac{R_{max}}{R} - 1\right)(1 - \epsilon^2).$$



Star collapses from R_{max} through $R = 2M$ in *finite proper time*. It falls to $R = 0$ in

$$t_{fall} = \frac{M\pi}{(1-\epsilon)^{3/2}}$$

What happens near $r = 2M$?

To probe the spacetime near $r = 2M$ the coordinates adapted to infalling observers should be used. Let's consider photons ($ds^2 = 0$).
Schwarzschild radial null geodesics are

$$dt^2 = \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} \equiv d\bar{r}^2 = r + 2M \ln\left|\frac{r - 2M}{2M}\right|,$$

with $\bar{r}$ is the Regge-Wheeler radial coordinate (made to be similar to time coordinate, $\bar{r} \in (-\infty, \infty)$). The Schwarzschild metric can be rewritten in the Eddington-Finkelstein ingoing coordinates

$$ds^2 = \left(1 - \frac{2M}{r}\right)(-dt^2 + d\bar{r}^2) + r^2 d\Omega^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2drdv + r^2 d\Omega^2,$$

with $v = t + \bar{r}$ a new ingoing radial null coordinate.

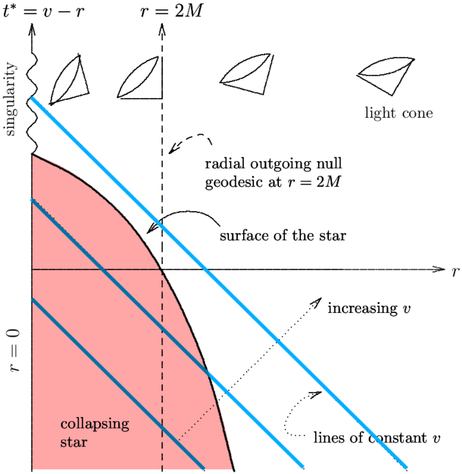
- ⋆ the metric coefficients related to dr are not singular at $r = 2M \rightarrow$ this singularity in Schwarschild metric is a *coordinate singularity*.

What happens near $r = 2M$? Finkelstein diagram

For $r \leqslant 2M$,

$$2drdv = -\left(\left(\frac{2M}{r} - 1\right) dv^2 + r^2 d\Omega^2 - ds^2\right) \leqslant 0 \text{ for } ds^2 \leqslant 0.$$

- ⋆ for all timelike or null worldlines $dr\,dv \leqslant 0$.
- ⋆ $dv > 0$ for future-directed worldlines, so $dr \leqslant 0$ with equality when $r = 2M$ (i.e., ingoing radial null geodesics - $d\Omega = 0$ - at $r = 2M$).



- ⋆ No future-directed timelike or null worldline can reach $r > 2M$ from $r \leqslant 2M$ - nothing physical (any **event**) can communicate from under the **event horizon**,
- ⋆ Coordinates change meaning: t becomes spacelike and r becomes timelike - singularity is no longer *where*, but *when*.

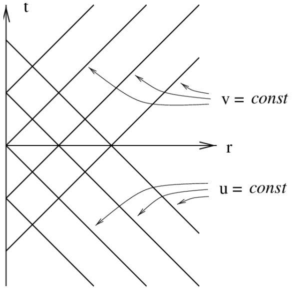
Penrose-Carter diagrams

The goal is to present the whole spacetime in a compact way. Let's start with the Minkowski spacetime:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \quad \text{with} \quad -\infty < t < \infty, \quad 0 \geqslant r < \infty.$$

By changing in to *null coordinates*

$$u = \frac{1}{2}(t+r), \quad v = \frac{1}{2}(t-r),$$
$$-\infty < u < +\infty,$$
$$-\infty < v < +\infty, v \leqslant u$$



the metric is

$$ds^2 = -2(dudv + dvdu) + (u-v)^2 d\Omega^2.$$

This metric is in turn transformed to coordinates $U(u), V(v)$ that take finite value at infinity, such as

$$U = \arctan(u), \;\; V = \arctan(v),$$
$$-\pi/2 < U < +\pi/2, \quad -\pi/2 < V < +\pi/2, \quad V \leqslant U.$$

Penrose-Carter diagrams

The Minkowski metric in terms of U and V is

$$ds^2 = \frac{1}{\cos^2 U \cos^2 V}\left(-2(dUdV + dVdU) + \sin^2(U-V)d\Omega^2\right).$$

In order to recover the timelike and spacelike character of the coordinates, there is another transformation

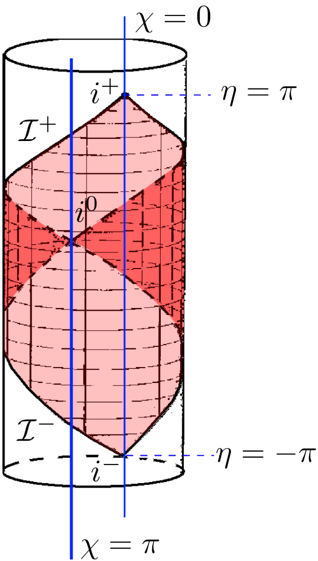
$$\underbrace{\eta = U + V}_{timelike}, \quad \underbrace{\chi = U - V}_{spacelike\ (radial)}, \quad \text{with } -\pi < \eta < \pi,\ 0 \leqslant \chi < \pi.$$

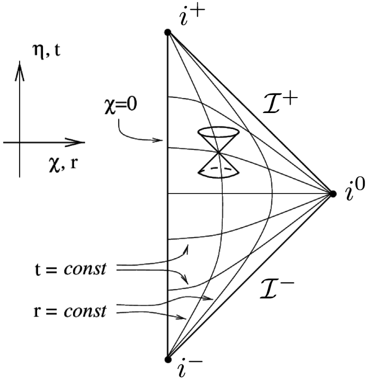
The metric is then expressed by an unphysical conformal metric

$$ds^2 = \omega^{-2}\left(-d\eta^2 + d\chi^2 + \sin^2\chi d\Omega^2\right), \quad \omega = \cos U \cos V = \frac{1}{2}(\cos\eta + \cos\chi),$$

where ω is the conformal factor.

Penrose-Carter diagram for Minkowski spacetime





- ⋆ i^+ future timelike infinity ($\eta = \pi,\ \chi = 0$),
- ⋆ i^0 spatial infinity ($\eta = 0,\ \chi = \pi$),
- ⋆ i^- past timelike infinity ($\eta = -\pi,\ \chi = 0$),
- ⋆ $\mathcal{I}^+$ future null infinity ($\eta = \pi - \chi,\ 0 < \chi < \pi$),
- ⋆ $\mathcal{I}^-$ past null infinity ($\eta = -\pi + \chi,\ 0 < \chi < \pi$).

Kruskal–Szekeres coordinates

M. Kruskal and G. Szekeres (1960) defined coordinates that cover the whole Schwarzschild manifold - t and r coordinates are replaced by, for $r > 2GM$,

$$V = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right),$$

$$U = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right),$$

for $r < 2GM$:

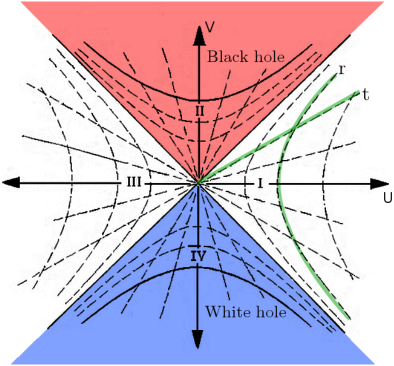
$$V = \left(1 - \frac{r}{2GM}\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right),$$

$$U = \left(1 - \frac{r}{2GM}\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right).$$

with $V^2 - U^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}$

the metric is

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-dV^2 + dU^2) + r^2 d\Omega^2.$$



Even horizon is defined by $V = \pm U$.

Kruskal–Szekeres coordinates

A null version of KS coordinates:

$$\tilde{U} = V - U, \quad \tilde{V} = V + U,$$

that with $\tilde{U}\tilde{V} = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}$

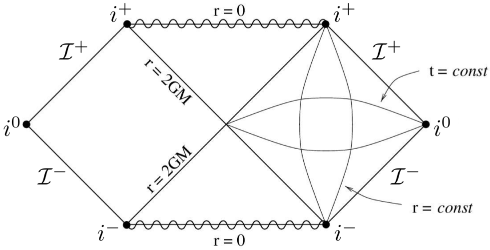
produces the metric

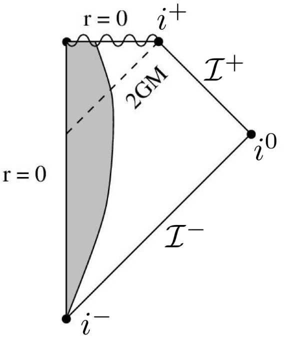
$$ds^2 = -\frac{32G^3M^3}{r} e^{-r/2GM}(d\tilde{U}d\tilde{V}) + r^2 d\Omega^2.$$

'Rescaling the infinities' to finite values

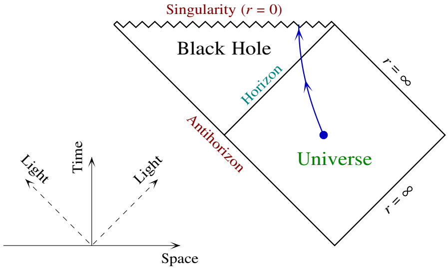
$$u = \arctan\left(\frac{\tilde{U}}{\sqrt{2GM}}\right), \quad v = \ldots$$

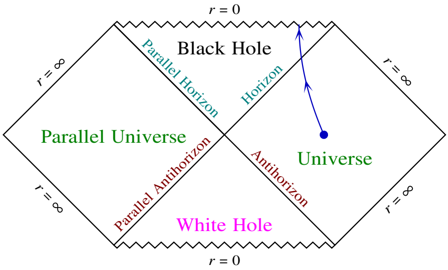
gives the conformal structure similar to previous Minkowski case.





Penrose-Carter diagrams for Schwarzschild





Embedding of Schwarzschild spacetime

In order to visualize that the Schwarzschild spacetime is really curved, let's draw a 2-surface of $t = const.$ and $\theta = \pi/2$ - spatial slice of the line element

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2.$$

By comparing the Euclidean Cartesian with cylindrical $(x = r\cos\phi,\ y = r\sin\phi)$ coordinates

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\phi^2 + dz^2.$$

one obtains

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2 = \left(1 + \left(\frac{dz}{dr}\right)^2\right) dr^2 + r^2 d\phi^2.$$

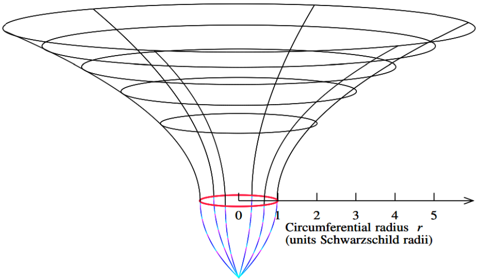
with $z(r)$, the *elevation function* that will visualize the actual shape of the surface embedded in the Euclidean space.

Embedding of Schwarzschild spacetime

Comparing the terms, one calculates the $z(r)$ function (a way to visualize how distorted the radial distances are):

$$1 + \left(\frac{dz}{dr}\right)^2 = \left(1 - \frac{2M}{r}\right)^{-1} \quad \rightarrow$$

$$z(r) = \int_0^r \frac{dr}{\sqrt{r/2M - 1}}.$$



Orbits in Schwarzschild spacetime

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

In general, every symmetry of the metric (symmetry of the action) corresponds to a specific Killing vector field, $\mathcal{L}_\xi g_{\mu\nu} = \xi_{\nu;\mu} + \xi_{\mu;\nu} = 0$. The Lie derivative of the metric g along ξ vanishes - ξ preserves g along its direction. From symmetry considerations we have the following constants of motion of an orbiting particle (λ a parameter along the path):

$$\text{Time translation (energy conservation)} : g_{t\mu} u^\mu = \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\lambda} = \epsilon,$$

$$\text{Spatial rotation (angular momentum conservation)} : g_{\phi\mu} u^\mu = \underbrace{r^2 \frac{d\phi}{d\lambda} = l}_{Kepler's\ law},$$

$$\text{Also, on any geodesic} : g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -\mathcal{E}$$

(for massive particles it is $\mathcal{E} = m^2$, for massless $\mathcal{E} = 0$).

Orbits in Schwarzschild spacetime

Expanding the $g_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda)$:

$$-\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2+\left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2+r^2\left(\frac{d\phi}{d\lambda}\right)^2=-\mathcal{E}.$$

If multiplied by $1-2GM/r$ and with eqs. for ϵ and l it can be rewritten as

$$\frac{1}{2}\left(\frac{dt}{d\lambda}\right)^2+V(r)=\frac{1}{2}\epsilon^2,\ \text{where } V(r)=\underbrace{\frac{1}{2}\mathcal{E}}_{const.}-\underbrace{\mathcal{E}\frac{GM}{r}}_{Grav.\ pot.}+\underbrace{\frac{l^2}{2r^2}}_{centrifugal}-\frac{GMl^2}{r^3}.$$

- ⋆ Equation of motion of a particle of energy $1/2\epsilon^2$ in a potential $V(r)$,
- ⋆ Last term in $V(r)$ - deviation from Newtionian result (which makes all the difference!)

Orbits in Newtonian 'spacetime'

The orbital movement depends on the $V(r)$ vs. $1/2\epsilon^2$ relation:

$$\frac{1}{2}\left(\frac{dt}{d\lambda}\right)^2 + V(r) = \frac{1}{2}\epsilon^2$$

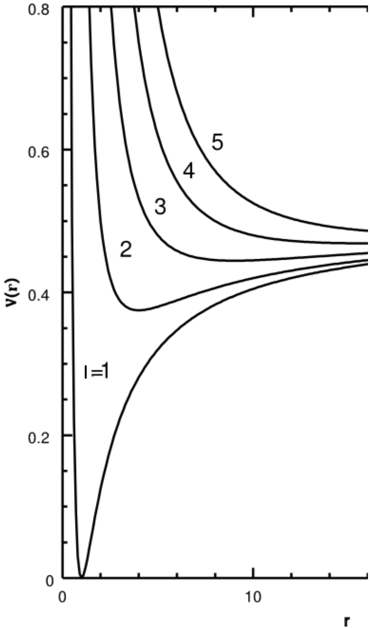
- ⋆ If $V(r) = 1/2\epsilon^2$ - turning point, particle starts to move the other way,
- ⋆ $r = const. \leftrightarrow dV/dr = 0.$

$$\frac{dV}{dr} = \mathcal{E}GMr^2 - l^2 r + \underbrace{2GMl^2}_{GR\ \ term} = 0.$$

In Newtonian gravity, circular orbits for

$$r = \frac{l^2}{GM\mathcal{E}}.$$

(no circular orbits for photons!)



Orbits in Schwarzschild spacetime: massless particles

$$V(r) = \frac{1}{2}\mathcal{E} - \mathcal{E}\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}.$$

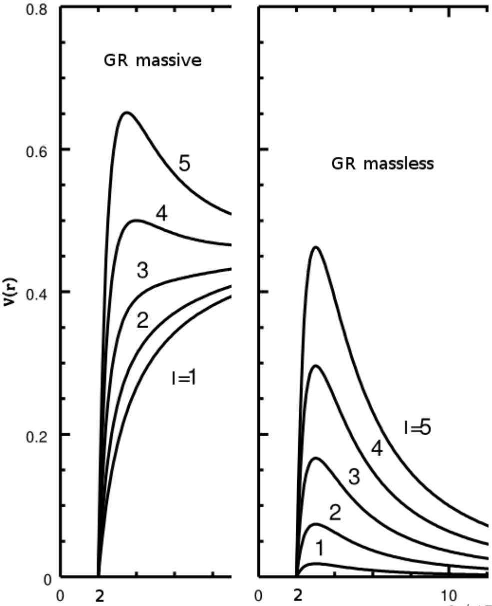
In GR, the additional term $-GMl^2/r^3$ is important for small r ($r \to \infty$ - Newtonian limit).

- ⋆ at $r = 2GM$ $V(r) = 0$,
- ⋆ for massless particles ($\mathcal{E} = 0$), the derivative of the potential gives

  $$r = 3GM.$$

  (a maximum for every l).

→ photon circular (unstable) orbit at $3GM$.



Orbits in Schwarzschild spacetime: massive particles

$$V(r) = \frac{1}{2}\mathcal{E} - \mathcal{E}\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}.$$

For massive particles ($\mathcal{E} \neq 0$), $V(r) = 0$ at $r = 2GM$. Also

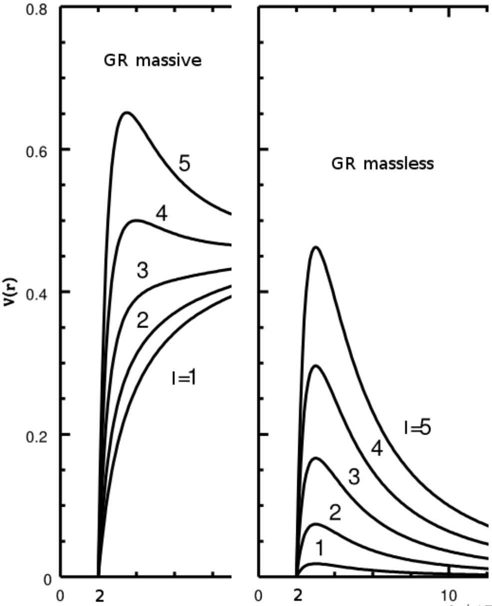
- ⋆ the circular orbits are at

$$r = \frac{l^2 \pm \sqrt{l^4 - 12G^2M^2l^2}}{2GM}$$

- ⋆ → for large l two orbits (one stable, one unstable). $l \to \infty$ gives limiting values

$$\underbrace{\frac{l^2}{GM}}_{stable} \quad \text{and} \quad \underbrace{3GM}_{unstable}.$$

→ approaching $3GM$ as for photons.



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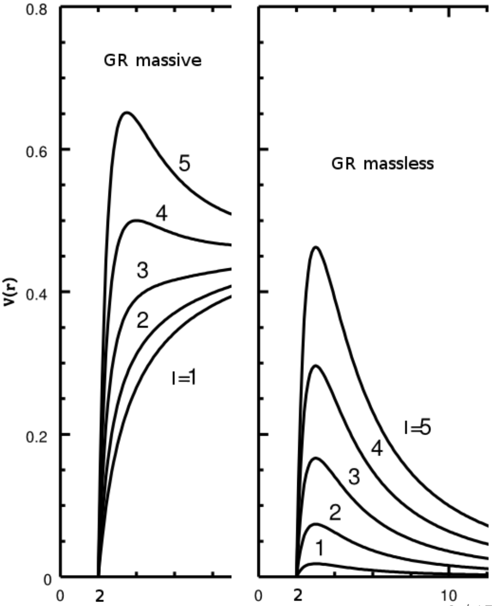
- ⋆ the circular orbits are at

$$r = \frac{l^2 \pm \sqrt{l^4 - 12G^2M^2l^2}}{2GM}$$

- ⋆ for small l two orbit concide for $l = \sqrt{12}GM$ at

$$r = 6GM \quad \text{(last stable orbit)}.$$

→ two regions of circular orbits: unstable $(3GM, 6GM)$ and stable $> 6GM$.



Electrically charged black holes

Assuming spherical symmetry, the general metric is again

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2 d\Omega^2,$$

and the spacetime is not vacuum, but filled with electromagnetic field $F_{\mu\nu}$

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\delta}F_{\nu}^{\delta} - -\frac{1}{4}g_{\mu\nu}F_{\delta\rho}F^{\delta\rho}\right).$$

From spherical symmetry, the only electric and magnetic components of $F_{\mu\nu}$ are the radial ones:

$$E_r = F_{tr} = f(r,t) = -F_{rt}, \quad \text{and}$$

$$B_r = g_{rr}\epsilon^{tr\mu\nu}F_{\mu\nu} = \frac{2g_{rr}}{\sqrt{|g|}}F_{\theta\phi} \;\rightarrow\; F_{\theta\phi} = -F_{\phi\theta} = h(r,t)\sin\theta.$$

$(|g| \propto r^4 \sin^2\theta)$. Then, the Maxwell equations together with the Einstein equations must be solved

$$g^{\mu\nu}\nabla_{\mu}F_{\nu\delta} = 0, \quad \nabla_{[\mu}F_{\nu\delta]} = 0, \quad R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}.$$

Reissner-Nordström metric

The solution is given by H. Reissner (1916) and G. Nordström (1918):

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2, \quad \text{where} \quad \Delta = 1 - \frac{2GM}{r} + \frac{G(p^2+q^2)}{r^2},$$

where p is the magnetic charge (equal to zero?), and q is the electric charge ($F_{rt} = -q^2/r$, $F_{\theta\phi} = p \sin\theta$). The horizon appears at r for which $\Delta = 0$:

$$r_\pm = GM \pm \sqrt{G^2M^2 - G(p^2+q^2)}$$

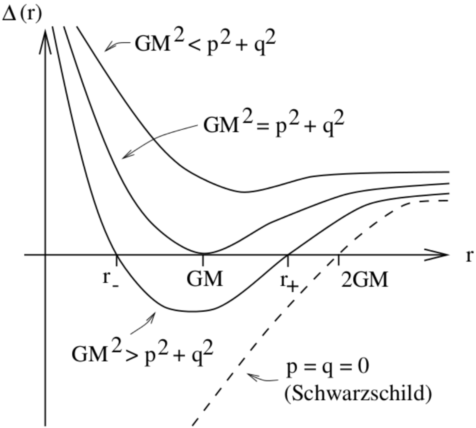
There are several possible cases:

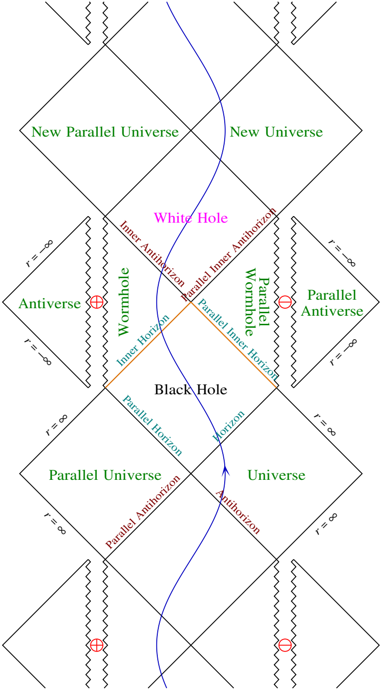
- $p^2 + q^2 > GM^2 \to \Delta > 0$ - no metric singularity until $r = 0$, no event horizon: **naked singularity** (related: cosmic censorship conjecture - gravitational collapse of physical matter can never produce a naked singularity).

  $p^2 + q^2 > GM^2$ indeed unphysical - total mass-energy of the BH is smaller than the electromagnetic field contribution.

Reissner-Nordström metric

- $p^2 + q^2 < GM^2$ corresponds to real situation, $r_\pm$ are coordinate singularities.
  - $r \to r_+$ like in Schwarzschild case, for $r_- < r < r_+$ the radial coordinate changes character (from spacelike becomes timelike),
  - for $r < r_-$ spacelike again $\to$ not neccesary doomed to hit the $r = 0$ singularity!
  - $r = 0$ is timelike, as opposed to Schwarzschild spacelike singularity ($\to$ not necessarily in the future).
  - the in-falling observer can cross r_- again, and be forced in the direction of increasing r towards r_+.




New Parallel Universe
New Universe
White Hole
Inner Antihorizon
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$r = -\infty$
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Parallel Wormhole
Parallel Antiverse
Inner Horizon
Parallel Inner Horizon
Black Hole
$r = \infty$
Parallel Horizon
Horizon
Parallel Universe
Universe
Parallel Antihorizon
Antihorizon

Some facts

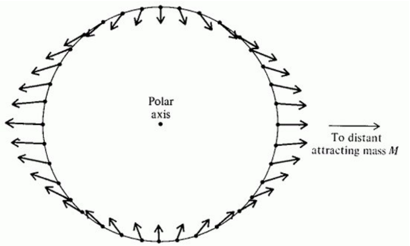
- ⋆ **Birkhoff's theorem**: any spherically symmetric vacuum solution is static → Schwarzschild. If electromagnetic fields are included (Einstein-Maxwell system) → Reissner-Nordström.
- ⋆ In order to study real singularities, a measure of curvature must be used (Riemann tensor). Interesting invariant is **Kretschmann scalar**

$$K = R^R_{\mu\nu\rho\delta}\mu\nu\rho\delta = \underbrace{\frac{48G^2M^2}{c^4r^6}}_{Schwarzschild\ \ value} .$$

Tidal force acting on a body m of size l:

$$F = \frac{GMm}{r^2}\frac{l}{r} \quad \propto \quad \underbrace{\frac{l}{M^2}}_{At\ \ the\ \ horizon}$$

(it's better to fall into a big black hole).



("spaghettification")

Rotating black holes

The solution for a rotating black hole is due to R. Kerr (1963). The metric in Boyer-Lindquist coordinates reads

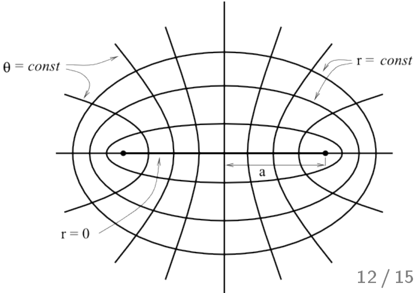
$$ds^2 = -dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + \frac{2GMr}{\rho^2}(a\sin^2\theta d\phi - dt)^2,$$

with

$$\underbrace{a = \frac{J}{Mc}}_{spin\ \ parameter} \in (0,1), \quad \Delta(r) = r^2 - 2GMr + a^2 \quad \text{and} \quad \rho^2(r,\theta) = r^2 + a^2\cos^2\theta.$$

(by changing $2GMr$ to $2GMr - (q^2 + p^2)/G$ - the **Kerr-Newman** metric).

- ⋆ $a \to 0$ reduces to the Schwarzschild metric,
- ⋆ $a = const.$, $M \to 0$ - flat space (metric expressed in ellipsoidal coordinates).



Kerr black hole singularities: horizons

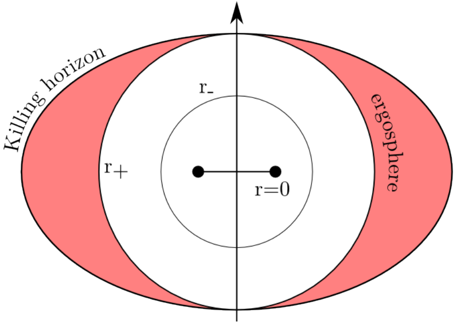
$$ds^2 = -dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + \frac{2GMr}{\rho^2}(a \sin^2\theta d\phi - dt)^2,$$

Singularities can appear at $\Delta = 0$ and $\rho = 0$.

- ⋆ $(GM)^2 \leqslant a^2$ cases correspond to naked singularities (super-spinar) and the *extremal* solution ($a = 1$),
- ⋆ for $(GM)^2 > a^2$, it yields two singular points

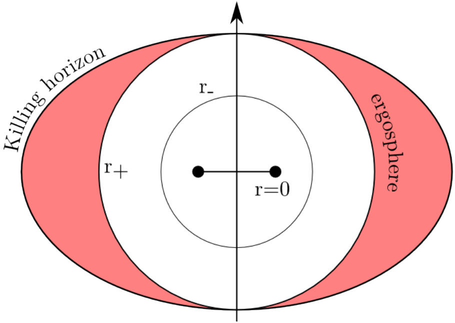
$$r_\pm = GM \pm \sqrt{G^2 M^2 - a^2}.$$

(null surfaces, event horizons). r_+ (outer horizon) corresponds to the Schwarzschild horizon, r_- is called the Cauchy horizon.



Kerr black hole singularities: static limit

Rotating solution admits two Killing vectors, $\xi^\mu = \partial_t$ and $\eta^\mu = \partial_\phi$, corresponding to energy conservation and axial symmetry (ξ^μ not orthogonal to $t = const.$ hypersurfaces → metric is stationary, not static.



- ⋆ In Schwarzschild time-symmetry Killing vector $\xi^\mu = \partial_t$ becomes null at the horizon and spacelike inside.

In Kerr,

$$\xi^\mu \xi_\mu = -\frac{1}{\rho^2}(\Delta - a^2 \sin^2\theta) \quad \text{does not vanish at } r_+ \quad (\xi^\mu \xi_\mu(r_+) \geqslant 0).$$

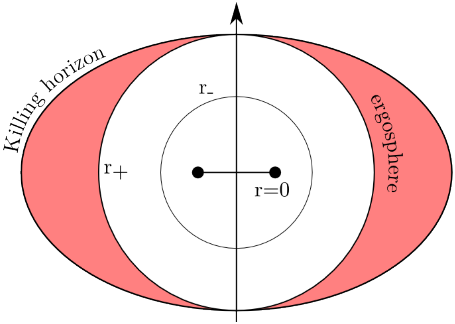
The surface $\xi^\mu \xi_\mu = 0$ is the *Killing horizon (static limit)*: $(r - GM)^2 = G^2 M^2 - a^2 \cos^2\theta$. Region between it and r_+ is the *ergosphere* - inertial observers forced to move with the spin of the BH ($d\phi/dt > 0$).

Kerr black hole singularities: ring singularity

The true, central curvature singularity does not occur simply at $r = 0$, but $\rho = 0$:

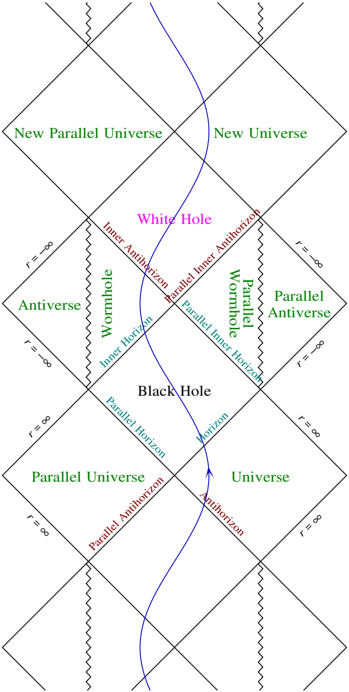
$$\rho^2 = r^2 + a^2 \cos^2\theta = 0$$
$$\rightarrow r = 0 \text{ and } \cos\theta = 0.$$

(a ring-like set of points). An observer who crosses the ring appears in a new Kerr spacetime with $r < 0 \rightarrow \Delta \neq 0 \rightarrow$ no horizons.



$\rightarrow$ Closed timelike curves. For $t, \theta = const.$ the line element in ϕ direction is

$$ds^2 = a^2 \left(1 + \frac{2GM}{r}\right) d\phi^2 < 0, \quad \text{for small } r < 0.$$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Parallel Antihorizon
Antihorizon
$r = \infty$
$r = \infty$

Orbital constants of motion in rotating spacetime

General orbits of particles (or photons) with 4-momentum p^μ are described by *four* constants of motion on the geodesic:

- ⋆ total energy $E = -p_t = -\xi_\mu p^\mu = g_{t\mu} p^\mu$,
- ⋆ component of angular momentum parallel to symmetry axis $L = p_\phi = \eta_\mu p^\mu = g_{\phi\mu} p^\mu$,
- ⋆ Carter constant: $Q = p_\theta^2 + \cos^2\theta (a^2(m^2 - E^2) + L^2/\sin^2\theta)$, separation constant from the Hamilton-Jacobi equations (in the equatorial plane $Q = 0$),
- ⋆ mass of the particle m.

How to measure the angular momentum of the hole and its influence on the moving particles? A photon emitted at r in ϕ direction in the equatorial plane has

$$ds^2 = 0 = g_{tt} dt^2 + 2 g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2,$$

which gives

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}.$$

Angular velocity of the hole

At the Killing horizon $g_{tt} = 0$ and

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} \rightarrow \frac{d\phi}{dt} = 0, \quad \text{and} \quad \frac{d\phi}{dt} = \frac{2a}{(2GM)^2 + a^2},$$

which is interpreted as the angular drag of retrograde and prograde photons. "Frame dragging" at the horizon r_+ (minimal angular velocity of the particle there) can be defined as the angular velocity of the horizon itself:

$$\Omega_H = \left(\frac{d\phi}{dt}\right)(r_+) = \frac{a}{r_+^2 + a^2}.$$

The coordinate angular velocity of a circular orbit is $(G = c = 1)$:

$$\Omega = \pm \frac{\sqrt{M}}{r^{3/2} \pm a\sqrt{M}}.$$

Circular orbits around the Kerr black hole

To summarize the characteristic distances and orbits in Kerr spacetime:

- ⋆ **Marginally-stable circular orbits (ISCO)**:

$$r_{ms} = M(3 + Z_2 \mp (3 - Z_1)(3 + Z_1 + 2Z_2))^{1/2}),$$

$$\text{with } Z_1 = 1 + \left(1 - \frac{a^2}{M^2}\right)^{1/3} \left(\left(1 + \frac{a}{M}\right)^{1/3} + \left(1 - \frac{a}{M}\right)^{1/3} \right),$$

$$Z_1 = (3a^2/M^2 + Z_1^2)^{1/2}.$$

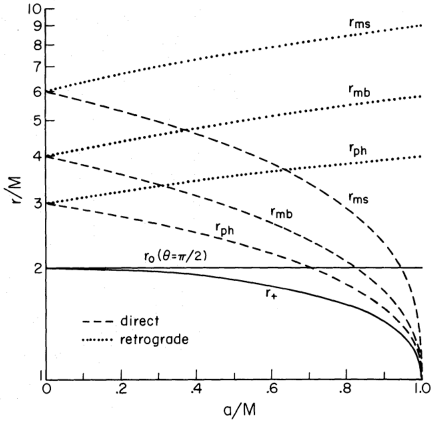
- ⋆ **Marginally-bound circular orbits**: limiting radius for marginal (''parabolic'') circular orbit with $\epsilon = E/m = 1$,

$$r_{mb} = 2M \mp a + 2\sqrt{M(M \mp a)}.$$

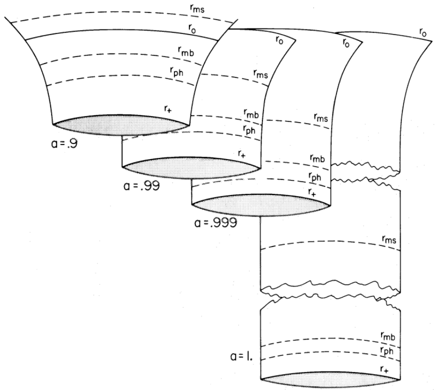
- ⋆ **Photon orbit**: in the limit of $E \to \infty$ the innermost boundary of the circular orbits for particles:

$$r_{ph} = 2M(1 + \cos(\frac{2}{3}\cos^{-1}(\mp a/M))).$$

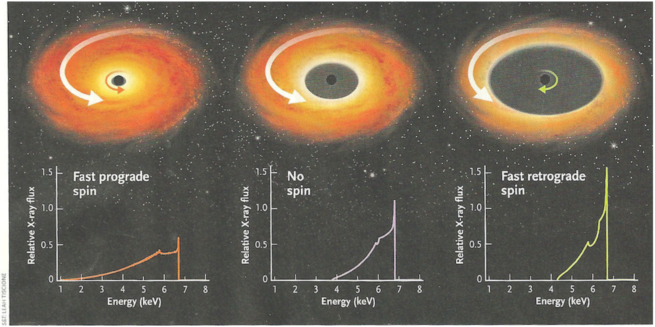
Circular orbits around the Kerr black hole



(from Bardeen et al., 1972)



Kerr vs. rotating star



- ⋆ The exterior metric of the Kerr metric differs from the rapidly rotation compact material star; they agree in the first order approximation - slow rotation:

$$r_{ms} = 6M \left(1 - \frac{J}{M^2} \left(\frac{2}{3} \right)^{3/2} \right) .$$

- ⋆ No 'natural' material source for Kerr metric (infinitesimally thin counter-rotating discs etc.)

Penrose process

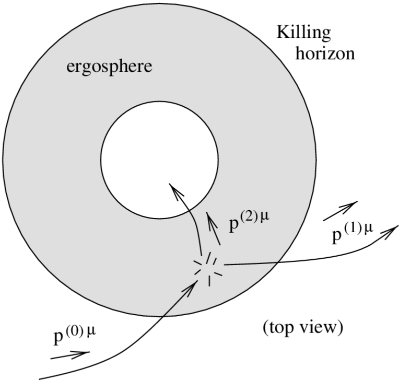
Inside the ergosphere ξ_μ becomes spacelike. There can exist particles with

$$E = -\xi_\mu p^\mu < 0.$$

Imagine particle with $p^\mu_{(0)}$ disintegrating in the ergosphere into two other particles:

$$p^\mu_{(0)} = p^\mu_{(1)} + p^\mu_{(2)}, \quad /\xi_\mu$$
$$\rightarrow E_{(0)} = E_{(1)} + E_{(2)}.$$

If arranged in such a way that $E_{(2)} < 0$, then $E_{(1)} > E_{(0)}$ - production of energy.



Irreducible mass

Penrose process is the extraction of energy from the kinetic (rotational) energy of the black hole. Let's define an additional Killing vector

$$\chi_\mu = \xi_\mu + \Omega_H \eta_\mu,$$

null and tangent to the outer horizon r_+. Particle (2) falls under the horizon if

$$p^\mu_{(2)}\chi_\mu = \underbrace{p^\mu_{(2)}\xi_\mu}_{-E_{(2)}} + \Omega_H \underbrace{p^\mu_{(2)}\eta_\mu}_{L_{(2)}} < 0 \quad \rightarrow \quad L_{(2)} < \frac{E_{(2)}}{\Omega_H} < 0 \quad \text{since} \quad E_{(2)} < 0.$$

The black hole mass M and angular momentum $J = Ma$ are decreased by

$$\delta M = E_{(2)}, \quad \delta J = L_{(2)} \quad \text{so that} \quad \delta J < \frac{\delta M}{\Omega_H}.$$

Although the energy is extracted, the horizon area A is not decreasing (!). By integrating over the horizon surface:

$$A = 4\pi(r_+^2 + a^2).$$

Irreducible mass

How does it work? Let's define an *irreducible mass* of the black hole as follows:

$$M_{irr}^2 = \frac{A}{16\pi G^2} = \frac{1}{4G^2}(r_+^2 + a^2) = \frac{1}{2}\left(M^2 + \sqrt{M^4 - (J/G)^2}\right).$$

A change in M_{irr} is related with δM and δJ

$$\delta M_{\rm irr} = \frac{a}{4G\sqrt{G^2M^2 - a^2}M_{\rm irr}} \underbrace{\left(\Omega_H^{-1}\delta M - \delta J\right)}_{>0} > 0,$$

hence $\delta M_{\rm irr}$ cannot decrease. The maximum amount of energy that can be extracted with a Penrose process is

$$M - M_{\rm irr} = M - \frac{1}{\sqrt{2}}\left(M^2 + \sqrt{M^4 - (J/G)^2}\right)^{1/2}.$$

for $a = 1$ Kerr BH it is approx. 30% of total mass-energy.

Thermodynamics of black holes

The analogous relation for the horizon area A is

$$\delta A = 8\pi G \frac{a}{\Omega_H \sqrt{G^2 M^2 - a^2}} (\delta M - \Omega_H \delta_J),$$

which is usually rewritten as

$$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega_H \delta J.$$

and where the *surface gravity* κ of the black hole is introduced:

$$\kappa = \frac{\sqrt{G^2 M^2 - a^2}}{2GM(GM + \sqrt{G^2 M^2 - a^2})}$$

(κ is acceleration of a ZAMO at the horizon; $\kappa = 0$ corresponds to extremal black holes). Curious relation to classical thermodynamics:

$$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega_H \delta J \leftrightarrow dU = T dS - p dV + \ldots$$

- ⋆ $\Omega_H \delta J$ related to "work" term $-pdV$,
- ⋆ the area A never decreases as the entropy S - surface gravity $\kappa / 8\pi G \sim T$.

Thermodynamics of black holes

The laws of black hole thermodynamics are (in comparison to classical):

- **0th**: in equilibrium the temperature the bodies in contact have the same temperature (temperature constant through the system),
- **1st**: the change of energy is related to the change of entropy and work as in $dU = TdS - pdV$,
- **2nd**: the entropy S of an isolated system cannot decrease,
- **3rd**: the entropy of any pure substance in thermodynamic equilibrium approaches zero as the temperature approaches zero.

- **0th**: for stationary black holes $\kappa 8\pi GT$ is constant everywhere on the horizon,
- **1st**: the change of black hole mass is related to the change of the horizon area and angular momentum as in $\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J$,
- **2nd**: the area A of the horizon never decreases.
- **3rd**: it is impossible to achieve $\kappa = 0$ in any physical process.

Blandford–Znajek process: EM analogue of Penrose process

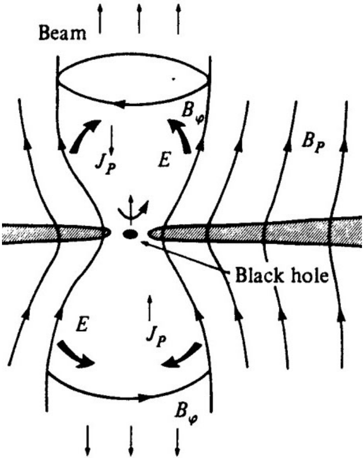
Accertion disk with polar magnetic field, penetrating the ergosphere, dragged along and extracting the rotational energy. Power at light cylinder r_c:

$$P = B^2 \left(\frac{r}{r_c}\right)^4 r_c c = \frac{B^2 r^4 \omega^2}{c}$$

Total amount of energy is

$$E = 0.3Mc^2 \simeq 5 \times 10^{53} \frac{M}{M_\odot} \; erg$$

Poynting flux $\vec{E} \times \vec{B} \rightarrow$ production of the jet.



Key open issues:

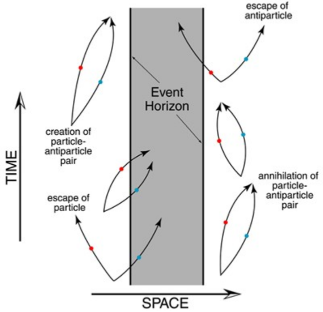
- ⋆ origin (generation mechanism?) of the magnetic field,
- ⋆ magnetic field dynamics in accretion disks,
- ⋆ mass loading/accretion disk outflows.

Hawking radiation

As a result of quantum particle creation effects, black hole radiates particles with a perfect black body spectrum of temperature proportional to surface gravity:

$$T = \frac{\kappa}{2\pi}$$

For Schwarzschild:



$$T = \frac{1}{8\pi M} = \left[\frac{\hbar c^3}{8\pi G M k_B}\right] \sim 6 \times 10^{-8} \frac{M_\odot}{M}\ K.$$

Related:

- ⋆ Unruh effect - an accelereated observer detects black-body radiation of a vacuum field,

$$T = \frac{\hbar}{2\pi c k_B} \times \text{acceleration},$$

- ⋆ Rindler coordinates and horizon observed by accelerated observers in Minkowski spacetime.

Further reading...

- *"Lecture Notes on General Relativity"*, Sean Carroll,
- *"Black holes"*, P.K. Townsend `arXiv:gr-qc/9707012`,
- Penrose diagrams:
  `http://jila.colorado.edu/~ajsh/insidebh/penrose.html`