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On the advantage of using analysis of variance for period search

A. Schwarzenberg-Czerny *Warsaw University Observatory, Al. Ujazdowskie 4, 00-478 Warszawa, Poland*

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Summary. We recommend one way analysis of variance (AoV) as a method for detection of sharp periodic signals. Application of the method requires folding and binning data with a trial period. Among several methods of this type employed in astronomy, AoV has the advantage that its probability distribution is known for any number of observations, so that its usefulness for small samples is unquestionable. We compare the AoV test with other tests in use and demonstrate that for large samples it is at least as powerful as any of them. Examples of application of the AoV method for photometric observations are discussed. We discuss an error in the phase dispersion minimization (PDM) method, namely an incorrect probability distribution and significance criterion. We argue that the power of the Lafler and Kinman test is comparable to that of the AoV2 test, the AoV test with narrow bins, containing two observations each. However, the AoV2 test is less powerful than any AoV test with a reduced number of bins and so is the Lafler and Kinman test.

1 Introduction

At least two factors should be taken into account in the choice of the most suitable method for detection of a periodic signal in observations, namely the shape of the signal we are looking for and the distribution of observations in time.

Fourier methods are best suited for detection of sinusoidal signals. For observations distributed very non-uniformly in time, such as observations with large gaps, one has to use the clasical Fourier method (e.g. Deeming 1975) or its modification by Scargle (1982). The latter method has the advantage that its statistical properties are well understood. For more or less uniform distribution of observations one can use faster versions of the above methods, namely the FFT method and its modification by Press & Rybicki (1988). The performance of Fourier methods in the detection of non-sinusoidal periodic signals, such as for example sharp and narrow pulses, is poor. The resolution of these methods is limited by the broad profile of a sinusoid and the power associated with such a signal is spread in the periodogram among many harmonics, rendering them less detectable.

Several methods suitable for detection of non-harmonic oscillations in observations uniform in time were employed. We have no space here to review all of them. The maximum entropy method modified by Fahlman & Ulrych (1982) performs well. The standard procedure well suited for non-uniformly distributed observations is folding data with a trial period and grouping them into phase bins. Several such methods were invented and Stellingwerf (1978, hereafter S78) lists some of them. They are discussed in Section 4. Lafler & Kinman (1965, see also Stobie & Hawarden 1972) and Dworetsky (1983) proposed methods which avoid binning and thus may be suitable for small samples. In this paper we pay special attention to the classical one-way analysis of variance (AoV) as a period search method in application to data folded and binned.

All popular methods of period search have much in common. For each method a test statistic is defined. It is a real function of all observations and of a trial period (or its corresponding frequency). Since the observations are random variables because of their measurement errors, so is the test statistic. Period is not a random variable but a parameter of the statistic. For each period, and for all available observations, the test statistic yields a single number. The plot of values of the statistic for a range of periods is called a periodogram. Oscillations in the observations correspond to features in the periodogram called lines by analogy to spectroscopy. However, since similar features may arise due to noise in the data, the essential ingredient of each period search method is a criterion for statistical significance of the lines. The most desirable are cases in which the probability distribution of the statistic is known so that the criterion corresponds to testing the validity of a certain statistical hypothesis for a given value of the test statistic. In this paper we discuss those methods in which observations are folded and grouped into bins according to the phase of a trial period. Since in effect this is a classification of observations, we propose in Section 2 to use a standard statistical method for testing significance of the classification, namely the one-way analysis of variance. Several period search methods and the AoV method are compared in Section 4. Examples of use of the AoV method are given in Section 5.

2 Analysis of variance and related test statistics

Most textbooks on statistics discuss the AoV method. We summarize here the basic concepts of the method and results of its application to random data, because they constitute a convenient departure point for our discussion of periodic data and for comparison of period search methods. We follow Fisz (1963) and use his notation: n, $\bar{x}$ are the total number of observations and their average, r is the number of bins, n_i, $\bar{x}_i$ are the number of observations in the ith bin and their average and $x_{ij}=x(t_{ij})$ is the jth individual observation in the ith bin, obtained at time t_{ij}. With no loss of generality we put $\bar{x}=0$. Three statistics s_1^2, s_2^2 and s_0^2 are defined

$$(r-1)s_1^2=\sum_{i=1}^{r} n_i(\bar{x}_i-\bar{x})^2,$$

$$(n-r)s_2^2=\sum_{i=1}^{r}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2, \tag{1}$$

$$(n-1)s_0^2=\sum_{i=1}^{r}\sum_{j=1}^{n_i}(x_{ij}-\bar{x})^2.$$

They satisfy an algebraic identity

$$(n-1)s_0^2=(r-1)s_1^2+(n-r)s_2^2 \tag{2}$$

valid for any observed values of statistics, so that all three may not be independent. Here we have departed from the Fisz (1963) notation since we use s_0^2 instead of his s^2. We reserve the symbol s^2 for the generic name of all three statistics.

2.1 A PURE NOISE SIGNAL

Let us assume that the observations are Gaussian white noise with zero mean and variance chosen for unit of power

$$E[x_{ij}]=0, \qquad \mathrm{Var}[x_{ij}]=\sigma^2\equiv 1, \qquad \mathrm{Cov}[x_{ij}x_{kl}]=\delta_{ik}\delta_{jl}. \tag{3}$$

We call this assumption our null hypothesis H_0. If the hypothesis is valid then the probability distributions of $(r-1)s_1^2$, $(n-r)s_2^2$ and $(n-1)s_0^2$ are χ^2 with $r-1$, $n-r$ and $n-1$ degrees of freedom and s_1^2 and s_2^2 are independent. It follows from equation (2) that s_0^2 is not independent of s_1^2 and s_2^2. In fact, their correlation coefficients may be calculated using this equation. We summarize here properties of all three statistics

$$E[s_1^2]=E[s_2^2]=E[s_0^2]=\sigma^2\equiv 1$$

$$\mathrm{Var}[s_1^2]=\frac{2}{r-1}, \qquad \mathrm{Var}[s_2^2]=\frac{2}{n-r}, \qquad \mathrm{Var}[s_0^2]=\frac{2}{n-1} \tag{4}$$

$$\mathrm{Cov}[s_1^2s_2^2]=0, \qquad \mathrm{Cov}[s_1^2s_0^2]=\mathrm{Cov}[s_2^2{}_{,}s_0^2]=\frac{2}{n-1}.$$

The expected values of all s^2 demonstrate that the s^2 statistics are unbiased estimates of σ^2. In general we do not know the expected value of σ. So in order to verify H_0 we have to use the ratios $\Theta_1=s_1^2/s_0^2$, $\Theta_2=s_2^2/s_0^2$ and $\Theta_{\mathrm{AoV}}=s_1^2/s_2^2$ instead of the individual values of s^2. The Θ_{AoV} statistic is the standard AoV test statistic. Since it is the ratio of two independent χ^2 random variables it has Fisher–Snedecor F distribution with $r-1$ and $n-r$ degrees of freedom

$$E[\Theta_{\mathrm{AoV}}]=\frac{n-r}{n-r-2} \tag{5}$$

$$\mathrm{Var}[\Theta_{\mathrm{AoV}}]=\frac{2(n-r)^2(n-3)}{(r-1)(n-r-2)^2(n-r-4)}.$$

However, Θ_1 and Θ_2 are not ratios of independent random variables, so it is not guaranteed that their probability distributions are F distributions.

The Θ_1 and Θ_2 statistics or their functions are used in practice as test statistics. We shall investigate their distributions now while we defer any discussion of their relation to other statistics to Section 4. For our purpose it suffices to consider only an asymptotic case

$$n\to\infty, \qquad r\to\infty \quad \text{and} \quad r/n\to 0, \tag{6}$$

in which distributions of all s^2 are approximately normal by virtue of the central limit theorem, under certain conditions which are satisfied here. On the one hand, we shall demonstrate that our results are interesting whereas, on the other hand, calculation of the distribution in the general case may be difficult. Additionally, we assume that all bins contain an equal number of data n_i. The latter assumption is not particularly restrictive since we may skip some data and adjust bin borders accordingly.

It follows from equation (4) that in the asymptotic limit the relative scatter of the statistics $\sqrt{\mathrm{Var}[s^2]}/E[s^2]$ decreases to zero. So, we may expand any sufficiently smooth function of s^2 in a series around their expected values $E[s^2]$ (Eadie *et al.* 1971, their equation 2.43). In this way we obtain the expressions for expected values and variances of all Θ, in terms of known expected values, variances and covariances of s^2 (equation 4). We obtain the following formulae

$$E[\Theta_{\mathrm{AoV}}]=E[\Theta_1]=E[\Theta_2]=1$$

$$\mathrm{Var}[\Theta_{\mathrm{AoV}}]=\frac{2(n-1)}{(n-r)(r-1)} \qquad (7)$$

$$\mathrm{Var}[\Theta_1]=\frac{2(n-r)}{(n-1)(r-1)}$$

$$\mathrm{Var}[\Theta_2]=\frac{2(r-1)}{(n-r)(n-1)}.$$

It turns out that the limit of $\mathrm{Var}[\Theta_2]$ is not a limit for any Fisher–Snedecor distribution, since in its expression the numerator does not contain the sum of factors in the denominator, as it should. In consequence, we have proven that the probability distribution of the Θ_2 statistic is not F-type. Using similar considerations it may be proven that the probability distribution of Θ_1 in the limit $n \gg r$ does approach that of the Fisher–Snedecor variable with $r-1$ and $n-1$ degrees of freedom. In order to understand these results better, let us note that in the limit the correlation coefficient $\rho[s_2^2 s_0^2]$ approaches unity while the coefficient $\rho[s_1^2 s_0^2]$ approaches zero. Of course, vanishing of the correlation coefficient does not imply independence of s_1^2 and s_0^2 required for applicability of the F distribution, but nevertheless it is a necessary condition. We conclude this section by stating that for small r and n, Θ_{AoV} has the F probability distribution but two other statistics certainly do not have the F distribution.

2.2 A PERIODIC SIGNAL WITH NOISE

In this section we consider properties of the Θ statistics under an alternative hypothesis H_1 Namely we assume that observations contain the sum of a white noise a and a periodic signal f:

$$x_{ij}=a_{ij}+Af_{ij}. \qquad (8)$$

The noise $a_{ij}=a(t_{ij})$ satisfies equation (3). The periodic signal has amplitude A and period P. The function f is an arbitrary periodic function with period P_0, normalized to unit amplitude. With no loss of generality we put $\bar{f}=0$.

Straightforward calculations yield the expected values of the s^2 statistics:

$$E[s_1^2]=1+A^2F_1^2$$

$$E[s_2^2]=1+A^2F_2^2 \qquad (9)$$

$$E[s_0^2]=1+A^2F_0^2,$$

where

$$F_1^2=\frac{1}{r-1}\sum_{i=1}^{r} n_i(\bar{f}_i-\bar{f})^2$$

$$F_2^2=\frac{1}{n-r}\sum_{i=1}^{r}\sum_{j=1}^{n_i}(f_{ij}-\bar{f}_i)^2 \qquad (10)$$

$$F_0^2=\frac{1}{n-r}\sum_{i=1}^{r}\sum_{j=1}^{n_i}(f_{ij}-\bar{f})^2$$

are coefficients dependent on the shape of the signal and independent of its amplitude. It follows from equation (9) that the asymptotic distributions of s^2 do depend on the shape of the periodic signal.

Again we consider the asymptotic case (equation 6). With such assumptions, and for a small A, the distributions of all statistics considered here also tend to the normal ones. Since the expected values of the s^2 statistics approach unity while their variances still remain small we can again expand functions of s^2 in a Taylor series. In this way we obtain approximate expected values of Θ

$$\begin{aligned} E[\Theta_{\mathrm{AoV}}] &= 1 + A^2(F_1^2 - F_2^2) + O(A^4) \\ E[\Theta_1] &= 1 + A^2(F_1^2 - F_0^2) + O(A^4) \\ E[\Theta_2] &= 1 + A^2(F_2^2 - F_0^2) + O(A^4). \end{aligned} \tag{11}$$

The coefficients F^2 do satisfy equation (2) (Fisz 1963), so we can eliminate one of them from equation (11)

$$\begin{aligned} E[\Theta_{\mathrm{AoV}}] &= 1 + A^2(F_0^2 - F_2^2)\frac{n-1}{r-1} + O(A^4) \\ E[\Theta_1] &= 1 + A^2(F_0^2 - F_2^2)\frac{n-r}{r-1} + O(A^4) \\ E[\Theta_2] &= 1 - A^2(F_0^2 - F_2^2) + O(A^4). \end{aligned} \tag{12}$$

We shall compare the power $1-\beta$ of various Θ statistics for testing randomness of observations (Eadie *et al.* 1971, their equation 10.2). In order to make our task simpler we derive an explicit expression for the power. Since in the asymptotic limit the probability distributions of Θ are normal, their power $1-\beta$ may be expressed in terms of the error function erf (Abramowitz & Stegun 1964)

$$1-\beta = \frac{1}{2}\left\{1 - \mathrm{erf}\left[\frac{\Theta^{\mathrm{cr}}(\alpha) - E[\Theta; A]}{\sqrt{2\,\mathrm{Var}[\Theta; A]}}\right]\right\}, \tag{13}$$

where $\Theta^{\mathrm{cr}}(\alpha)$ is the critical value of the Θ statistic for confidence level α. The critical value satisfies an equation

$$\alpha = \frac{1}{2}\left\{1 - \mathrm{erf}\left[\frac{\Theta^{\mathrm{cr}}(\alpha) - E[\Theta; 0]}{\sqrt{2\,\mathrm{Var}[\Theta; 0]}}\right]\right\}. \tag{14}$$

Note that after the semicolon we indicate explicitly the dependence of means and variances on amplitude A. For small amplitudes A we may expand the argument of the erf function in equation (13) in a Taylor series. After retaining two terms of the series and substituting equation (14) we obtain an explicit approximate expression for power of the test

$$1-\beta = \tfrac{1}{2}\{1 - \mathrm{erf}\,[\mathrm{erf}^{-1}(1-2\alpha) - A^2 S]\}, \tag{15}$$

where

$$S = \left|\frac{dE[\Theta; 0]}{dA^2}\right| \frac{1}{\sqrt{2\,\mathrm{Var}[\Theta; 0]}}. \tag{16}$$

The power of a test increases with the factor S. The presence of the periodic signal does not affect the expected values of the individual Θ statistics in the same direction, since the factors multiplying A^2 in equation (11) may differ in sign. We ignored that fact in order to simplify explanations. However, we account for it by taking the absolute value in equation (16). In order to calculate the power we substitute equations (12) and (7) into equation (16). The striking result is that in the limit of equation (6) and for small amplitude A the power of all three Θ tests is the same

$$S=(F_0^2-F_2^2)\sqrt{\frac{(n-r)(n-1)}{8(r-1)}}. \tag{17}$$

3 The AoV periodograms

Now, let us consider properties of the Θ statistics as functions of their parameter, namely the trial period P. We assume that the observations x_{ij} remain fixed. As a consequence, the parameters of the signal, namely its period P_0 and amplitude A, are also assumed to be fixed. For convenience of notation we shall skip most explicit references to those parameters. Under such assumptions the statistics are ordinary, non-stochastic functions of P. Plots of these functions of P are called periodograms. Any detectable periodic signal in the observations produces a feature in the periodogram at its corresponding period P_0. The feature is called a line by analogy with spectroscopy.

Let us note that s_0^2 and F_0^2 are independent of P and so are the left-hand side of equation (2) and their equivalent for F^2. It follows from the equations that s_1^2(or F_1^2) and $s_2^2(F_2^2)$ periodograms are reflections of each other and so are Θ_1 and Θ_2 periodograms. It follows from equation (12) that all Θ periodograms for small-amplitude signals are identical, except for scaling. All three Θ statistics depend on period P via the $F_2^2(P)$ function, therefore they all have the same spectral resolution.

As before we consider the asymptotic case (equation 6). Additionally, in order to avoid boundary effects, we assume that the observation interval $[-T, +T]$ covers many cycles

$$\frac{T}{P_0}\rightarrow\infty. \tag{18}$$

It is interesting to note that we do not request good coverage of each cycle by observations. On the contrary, the number of observations per cycle may tend to zero, as long as sufficient coverage of all phases is ensured by the whole sample. The expected continuum value of the statistic is near unity, even for finite signal amplitudes, provided that our asymptotic assumptions hold. We defer the proof for Section 3.1.

The depth of a spectral line is a function of the ratio $\Theta(P_0)/\Theta(P)\approx E[\Theta(P_0); A]$ (see equation 23), so that the results of Section 2, in particular equation (12), apply here. We approximate the profile of the line by a parabola in order to simplify the estimate of its width $\Delta_{1/2}$ (HWHI)

$$\Delta_{1/2}\approx\sqrt{\frac{F_2^2(P_0)-\ddot{F}_2^2(P_0)}{\ddot{F}_2^2(P_0)}}\overset{\delta\to 0}{\rightarrow}\frac{P_0}{T}\sqrt{\frac{3\langle f^2\rangle}{2\langle f'^2\rangle}}. \tag{19}$$

The truth of the latter limit value remains to be proven in Section 3.2.

We apply our results to two kinds of periodic signals frequently encountered in practice: (a) a sinusoidal signal and (b) a Gaussian narrow pulse

$$f_a = \sin \omega t \qquad \langle f_a^2 \rangle = \tfrac{1}{2} \qquad \langle f_a'^2 \rangle = \frac{\omega^2}{2} \tag{20}$$

$$f_b = \exp(t^2/2\sigma^2) - \sqrt{2}\varepsilon \qquad \langle f_b^2 \rangle = \varepsilon - 2\varepsilon^2 \qquad \langle f_b'^2 \rangle = \frac{\pi}{P_0^2 \varepsilon},$$

where $\varepsilon = \sqrt{\pi}\sigma/P_0$. Both powers (or S factors) and resolutions $\Delta_{1/2}/P_0$ of the AoV and related tests are functions of number of bins r or of bin width $\delta = P/2r$:

$$S_a = \frac{1}{4}\left(1 - \frac{16\pi^2}{3r^2}\right)\sqrt{\frac{(n-r)(n-1)}{2(r-1)}} \qquad \frac{\Delta_{1/2a}}{P_0} \approx \frac{\sqrt{3}P_0}{2\pi\sqrt{2}T} \tag{21}$$

$$S_b \approx \frac{1}{2}\left(\varepsilon - \frac{4\pi}{3r^2\varepsilon}\right)\sqrt{\frac{(n-r)(n-1)}{2(r-1)}} \qquad \frac{\Delta_{1/2}}{P_0} \approx \frac{\sqrt{3}P_0\varepsilon}{\sqrt{2\pi}T}.$$

For the sinusoidal signal the maximum power of Θ tests is achieved for a small number of bins r. However, very small bin numbers $r \le 3$ should be avoided since then the tests become excessively sensitive to the phase of the signal. For narrow pulses the maximum power is achieved for bins and pulses of comparable widths. Resolutions of the Θ tests and of the Fourier spectrum for a sinusoidal signal are comparable. However, for narrow pulses the gain in the resolution achieved by binning observations is substantial.

3.1 THE CONTINUUM

As long as the assumptions of equations (6) and (18) hold and times of observations are not correlated with the observed signal, the sums in equation (10) are good Monte Carlo or other approximations of integrals. Texts on numerical integration discuss convergence of such approximations.

In order to estimate the continuum level we consider F_2^2 for periods P sufficiently distant from any spectral lines or their harmonics. Following our assumptions we may replace the average of signal values in ith bin by a sum over the whole interval $[-T, +T]$ covered by observations:

$$\bar{f}_i \to \psi_i \equiv \frac{1}{2T}\int_{-T}^{T} \Xi_i(t) f(t)\, dt, \tag{22}$$

where $\Xi_i(t)$ is a function which assumes value 1 for t falling into ith bin and 0 otherwise. Since periods P and P_0 of Ξ and f are non-commensurate, these functions are orthogonal with respect to integration, i.e. $\bar{f}_i \to 0$, for large T and for $\bar{f} = 0$. It follows from equation (10) that in the limit $F_2^2 \to F_0^2$, so that equation (12) implies

$$E[\Theta(P); A] \to 1 \qquad \text{if} \qquad P \ne P_0. \tag{23}$$

This continuum value does not apply if the observations cover a short interval of time or if noise in the observations is correlated, since our assumptions hold no longer. In the first case, when a small number of cycles was observed, an interference with the boundaries produced similar effects to those in the Fourier analysis with no window trimming. In the latter case the departure of the continuum from its white noise level is the worst for the mean correlation length equal to the bin size. Let us assume that, on average, n_c consecutive observations fall into a single bin and that they are correlated. Then our effective number of independent data points is less by a factor of n_c and so we overestimate the correction factor for the number of degrees of freedom in Θ_{AoV} by the same factor (Section 5.2).

3.2 THE LINES

Now we are going to consider the profile of a spectral line defined by the shape of $F_2^2(P)$ near the line centre at P_0. We assume that P satisfies an inequality $\Delta \ll P_0$, where $\Delta \equiv P - P_0$ and that the other assumptions from Section 3.1 hold (equations 6 and 18). In particular we stress that no coverage of each cycle by observations is required. Large gaps in observations are permitted as long as all P_0 phases are covered during a single beat period of P and P_0. The intrinsic or natural profiles of the spectral lines are rarely observed in practice, in close analogy to spectroscopy. Profiles of lines in the periodogram are in most cases artefacts produced by observation and data analysis procedures.

We start our analysis from consideration of the behaviour of the sums (or their integral approximations) in equation (10) near P_0. It is convenient to expand a periodic function of time in terms of small variables, the halfwidth of a bin, $\delta = P/2r$, and the distance from the line centre Δ

$$f(t) \equiv f(\phi_i + k\Delta + \tau) \to f(\phi_i) + f'(\phi_i)\{k\Delta + \tau\} + \frac{f''(\phi_i)}{2}\{k\Delta + \tau\}^2 + \dots, \tag{24}$$

where ϕ_i is the phase in the centre of the phase bin corresponding to t, in units of time, $k = [t/P]$ and $\tau = t - \phi_i - kP$, $|\tau| < \delta$. We exploited the P_0 periodicity of the function f and primes indicate the time derivatives. The integral over the whole interval of observations may be replaced by the sum of the integrals over small intervals where the expansion (equation 24) holds so we obtain an operator equation

$$\frac{1}{2T}\int_{-T}^{+T} dt = \frac{1}{r}\sum_{i=1}^{r} \frac{1}{2[T/P_0]} \sum_{k=-[T/P_0]}^{[T/P_0]} \frac{1}{2\delta}\int_{-\delta}^{\delta} d\tau. \tag{25}$$

Explicit formulae for ψ_i (equation 22) and F_2^2 may be obtained using the above expansions of functions and integrals. The calculations are straightforward but tedious so we just give the end results, retaining only their most significant terms for $\Delta \to \delta \to 0$

$$\begin{aligned} &\psi_i \to f(\phi_i) && F_2^2 \to \tfrac{1}{3}\,\delta^2\langle f'^2\rangle \\ &\dot{\psi}_i \to 0 && \dot{F}_2^2 \to 0 \\ &\ddot{\psi}_i \to \frac{f''(\phi_i)\,T^2}{3P_0^2} && \ddot{F}_2^2 \to \frac{2T^2\langle f'^2\rangle}{3P_0^2} \end{aligned} \tag{26}$$

and

$$F_0^2 \to \langle f^2\rangle,$$

where $\langle\,\rangle$ indicates a value averaged over the P_0 period. Dots indicate the partial derivatives with respect to the trial period P.

4 Comparison of methods for period search

The results of Section 2 enable the comparison of various methods of period search relying on folding and binning of observations. S78 describes two statistics, namely the phase dispersion minimization (PDM) statistic, and Whittaker & Robinson (1926) Θ_{WR} statistic. The PDM statistic is a modification of Lafler & Kinman (1965) statistic. We shall compare all three statistics with the AoV statistic.

4.1 THE PHASE DISPERSION MINIMIZATION (PDM) STATISTIC

For simplicity we use only a single bin pattern (or 'a cover'), although the PDM statistic was used frequently with multiple bin covers. One can easily see that the PDM statistic is identical to the Θ_2 statistic in Section 2. Thus, its distribution for H_0 is not at all the F distribution, even in the asymptotic limit of equation 6. In the limit the true variance of the PDM statistic is by a factor of $1/2n$ less than the one for its distribution given in S78. The cause of the discrepancy is the fact that the PDM statistic is the ratio of *correlated* statistics, unlike the Fisher–Snedecor F statistic. The erroneous distribution renders this test insensitive to small but significant signals. In fact, the more observations are available, the less sensitive is the erroneous significance criterion. In Section 5.1 we demonstrate that also for a small number of observations, loss of sensitivity occurs. The effect was in fact noted (S78) and left unexplained. In principle, one could use the asymptotic normal distribution found in Section 2 in order to correct the significance criterion. As we demonstrated, such a test would be no more powerful than the AoV test. However, it would still fail for small samples since in this case its distribution is unknown and the correlation may result in accepting spurious periods as real.

4.2 WHITTAKER AND ROBINSON STATISTIC

Not quite the same situation applies to the Θ_{WR} statistic. Assuming that bins were selected so that they contain an equal number of points n_i = constant, equation (6) in S78 may be written in our notation as

$$F(r-1, n-1) \approx \frac{r-3}{(r-1)n_i}\left(1 - 1 + \frac{s_1^2}{n_i s_0^2}\right). \tag{27}$$

The right-hand side of the equation does converge in the limit of equation (6) to Θ_1, which approaches the $F(r-1, n-1)$ distribution (section 2). So for a rich sample the criterion given in S78 for Θ_{WR} yields correct results. Nevertheless, use of the Θ_{WR} statistic for small samples is not recommended since its distribution then remains unknown.

4.3 LAFLER AND KINMAN STATISTIC

Lafler & Kinman (1965) introduced a statistic which we designate Θ_{LK}. The AoV statistic for bins containing two observations each, called here the AoV2 statistic, is directly comparable to it. In order to demonstrate this, we define a covariance coefficient $C = \Sigma_{i=1}^{n} x_i x_{i+1}/(n-1)$. The sums in both statistics can be expanded and expressed in terms of s_0^2 and C, namely

$$\Theta_{AoV2} = \frac{s!!_0^2 + C!!}{s!!_0^2 - C!!} \qquad \Theta_{LK} = 2\,\frac{s_0^2 - C}{s_0^2}. \tag{28}$$

We indicated the sums of even terms with !! and exploited an approximation $s_0^2/2 \approx s!!_0^2$. There are two cases to be considered. For small amplitude A the covariance C is also small and $\Theta_{LK} \propto \Theta_{AoV2}^{-1/2}$. For large A the covariance $C \to s_0^2$ and $\Theta_{LK} \propto \Theta_{AoV2}^{-1}$. We assumed that the number of observations is so large and the bins sufficiently narrow that $C/s_0^2 \approx C!/s!!_0^2$ holds.

One consequence of the fact that the two statistics are functions of each other, in the large sample limit, is their equal sensitivity and resolution. However, we demonstrated in Section 3.2 that the AoV test with many observations per bin is more sensitive than the AoV2 test. So, in ordinary cases the AoV statistic is more sensitive than Θ_{LK}. An exception holds for pulses so narrow that their width is comparable to $1/n$ (Section 3.2). However, we have little chance of seeing more than one such pulse.

5 Examples

5.1 PHOTOMETRY OF BK Cen

For a direct comparison of the AoV and PDM methods let us consider $n = 49$ photometric observations of the double mode Cepheid BK Centauri by C. J. van Houten, published by Leotta-Janin (1967) and treated previously with the PDM method (S78). In Fig. 1, we present the AoV periodogram computed for $r = 5$ bins of equal extent in phase.

The continuum level corresponds to $\Theta_{AoV} = 1$, as expected for a random noise. However, the signal-to-noise ratio for these observations is 50 or more, so that they are clearly *non-*

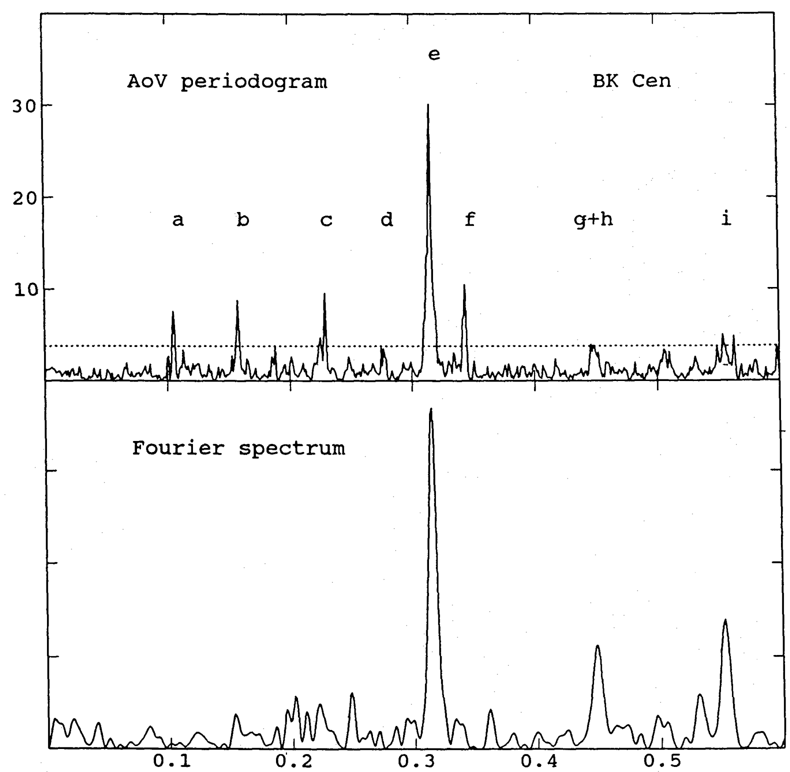

Figure 1. The analysis of variance (AoV) and Fourier periodograms for the photometric observations of the double-mode Cepheid BK Cen (Leotta-Janin 1967). Frequency is in cycles per day and power in arbitrary units. The annotation of the lines follows the PDM analysis by Stellingwerf (1978). The dotted line indicates the critical value of the AoV statistic for the significance level 0.05, uncorrected for bandwidth (see text). Comparison with the PDM results indicates the superior sensitivity of the AoV criterion.

random. The explanation of the paradox relies on the large mean interval between consecutive observations of 137/49 d, compared to the basic period. In effect, the consecutive observations are uncorrelated. It results immediately from the definition of $\Theta_{\rm AoV}$ that it depends primarily on the correlation coefficients, so that we naturally obtain the continuum value near unity for observations which are strongly dependent.

Several lines appear in the periodogram. Their annotation follows S78. The critical value of the F statistic for $r-1=4$ and $n-r=44$ degrees of freedom at 0.05 significance level is 3.78 and is indicated in Fig. 1 by the dotted line. Since we are looking for lines at locations known in advance, i.e. at combinations of the basic and first overtone frequencies, no correction for the total bandwidth is required. One can see that almost all lines marked are statistically significant contrary to what is indicated by the PDM significance test (S78). The advantage in using the AoV method is clear in this case. In a general case of search for lines at random locations, all probabilities should be corrected to account for the larger probability of detecting spurious lines in a wider frequency band (e.g. Scargle 1982). Each line occupies at least a $2\Delta_{1/2}$ band in the periodogram. This band should be increased if sub-harmonics are present. Therefore a periodogram is equivalent roughly to $T/\Delta_{1/2}$ independent observations of the statistic $\Theta_{\rm AoV}$.

5.2 PHOTOMETRY OF V603 Aql

The old nova V603 Aquilae (1918) was recently observed photometrically by Udalski & Schwarzenberg-Czerny (1989), who obtained nine runs lasting several hours each and

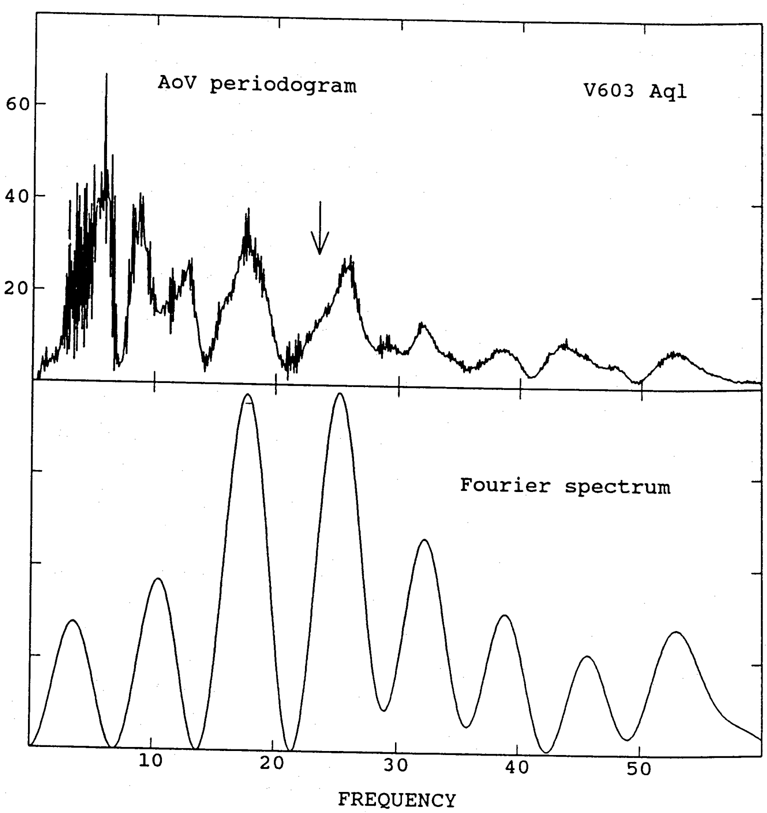

Figure 2. The same as Fig. 1 for one of nine runs of observations of Nova V603 Aquilae obtained by Udalski & Schwarzenberg-Czerny (1989). The run lasted 3.9 h and the main 3.5-h period was removed from the data. An arrow indicates the 61.4-min oscillation period discovered by these authors in optical and X-ray data. Since in the AoV periodograms no harmonic artefacts are produced, the first high-frequency feature (at 25 cycles d^{-1}) ought to be real, indicating detection of the 61-min oscillation in this single run (see text for details).

spanning 30 d. They discovered an oscillation with a 61-min period and 0.02-mag amplitude. Their discovery was confirmed by subsequent analysis of the *Einstein* satellite X-ray 'light' curve. Its power spectrum revealed a single strong line corresponding to the period 61 min. The complicating factor is that the oscillation is masked by a 0.2-mag amplitude sinusoidal variation with a 3.5-h period and by random flickering of similar amplitude. It is interesting to note whether a single night of observation may suffice for detection of the oscillation, in such a difficult case of $S/N \approx 0.1$. In Fig. 2 we present the AoV periodogram computed for $n = 609$ observations obtained during 3.9 h on 1988 August 14 by Udalski & Schwarzenberg-Czerny. Although the main 3.5-h period was removed from the data, some residual features at low frequencies clearly remained.

The mean continuum level is as much as $\Theta_{\rm AoV} \approx 5$. Such a high value of the continuum was obtained despite the presence in the data of relatively large and random flickering. The cause is the large time-scale of the flickering, typically several minutes, compared to the 20-s separation of observations. In effect, consecutive observations of essentially random signals are strongly correlated. In such circumstances the null hypothesis H_0 is not valid and neither does the Fisher–Snedecor significance criterion apply. In Section 3.1 a partial remedy for correlation of observations was discussed. Namely, one can adopt the value of the mean continuum level for the mean number of consecutive correlated observations $n_c = 5$. Then, one should divide all values of $\Theta_{\rm AoV}$ by n_c in order to obtain statistically meaningful values. In the observation domain such a procedure corresponds to taking mean values of observations over the correlation length. They are independent by design so that the results of Section 2 apply again.

Let us note a particular property of the periodogram for folded and binned data, namely, that no spurious harmonic features of the main component are produced by the AoV analysis. Instead we may get some sub-harmonics. Thus, the highest frequency feature in the AoV periodogram, in our case at 25 $\rm d^{-1}$, ought to be real. The peak value of the $\Theta_{\rm AoV}/n_c$ statistic is 5. The critical Fisher–Snedecor value for $r-1=4$ and $n/n_c - \rm r = 117$ degrees of freedom is $\Theta_{\rm cr} = 2.45$. It appears that the peak is statistically significant, which means that the 61-min oscillation may be detected during a single night.

6 Conclusions

On the one hand analysis of variance (AoV) found numerous applications in experimental science, so its use should not require any special justification. On the other hand, astronomers who search for periods in their observations tend to avoid its use. In this paper we have argued that AoV is ideally suited for the purpose of detection of sharp pulses by folding with trial periods. As we have shown in Section 2.1 for large samples the AoV significance test is asymptotically as powerful as other tests in use. For small samples the main advantage of the AoV statistic over the competition is its exactly known probability distribution. Its probability distribution for small and large samples and for H_0 hypothesis is the Fisher–Snedecor F distribution. Computation of the AoV statistics requires no more operations than computation of the other statistics, except that one has to be careful not to repeat the computation of some sums.

In Section 4 several statistics for testing significance of periods were compared. Compared to $\Theta_{\rm AoV}$ the other statistics considered are neither more powerful nor easier to compute. However, they all suffer from the lack of known distribution for small samples. The original significance criterion for the PDM method by S78 is incorrect. The Lafler and Kinman statistic corresponds to the AoV2 using bins containing two observations each, with loss of power of the test compared to the AoV test with a small number of bins. The AoV statistic does not suffer from any of these disadvantages.

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