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Editor's Note: A Note on Some General Solutions of the Einstein Field Equations in a Spherically Symmetric World

by Paul Kustaanheimo and Bertil Qvist, *Societas Scientiarum Fennica. Commentationes Physico-Mathematicae* **XIII.16 (1948),** 1–11

Although the authors did not provide (and were probably not aware of) an invariant definition of the problem they considered, such a definition exists: they considered spherically symmetric perfect fluid solutions of Einstein's equations with zero shear. The zero shear appears unannounced in the first sentence of the paper together with the assumption that the coordinates are isotropic and comoving at the same time. The main result of the paper is the reduction of the Einstein equations in the nonstatic case to the single ordinary differential equation, eq. (12). Just because this result has such a neat invariant definition, it was bound to reappear in several contexts. In consequence, it was independently rediscovered by other authors 17 times altogether (see Ref. 1 for the full list).

The paper contains one more valuable result. Equation (12) can be solved in terms of elementary functions only for some special forms of the function $f(x)$, as later research showed, and a few specific solutions in terms of elliptic functions are known today [see again Ref. 1, and also Ref. 2 in which a serious effort was undertaken to find a general solution of eq. (12)]. Kustaanheimo and Qvist were the first to note that solutions expressible in elementary functions exist when $f(x) = (ax^2 + bx + c)^{-5/2}$. They classified and found all but one of these solutions (the missing case is when the trinomial $f^{-2/5}$ has no real roots).

Nevertheless, each of the cases was rediscovered later; the number of rediscoveries ranges from 1 to 8 for each conformally nonflat case ($f \neq 0$), and the total number, including simpler subcases, is 22 (see Ref. 1). The conformally flat case ($f = 0$) was rediscovered 21 times. The authors of

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the rediscoveries usually provided some physical discussion of the solutions, but unawareness of the general classification has been an obstacle to progress. It is hoped that this reprinting will channel future research efforts in a more fruitful direction.

Readers are warned that the problem of general conditions under which eq. (12) is solvable in terms of elementary functions also has its literature already, see e.g. Refs. 3 and 4.

— *Andrzej Krasiński, Associate Editor*

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REFERENCES

1. Krasiński, A. (1998). *Inhomogeneous Cosmological Models* (Cambridge University Press, Cambridge), in press.
2. Wyman, M. (1976). *Canad. Math. Bull.* 19, 343.
3. Stephani, H. (1983). *J. Phys. A*16, 35299.
4. Maharaj, S. D., Leach, P. G. L., and Maartens, R. (1996). *Gen. Rel. Grav.* 28, 35.

Bertil Qvist and Paul Edwin Kustaanheimo: brief biographies

Bertil Qvist was born on October 21, 1920, in Vaasa, Finland. He graduated from the University of Helsinki in 1945 with mathematics as major. He also studied astronomy and physics, achieving the highest mark. From 1945 to 1954 he held an assistantship in astronomy at the Helsinki University. During this time he worked on his doctoral dissertation in astronomy, receiving his Ph.D. in 1950. In 1954 he was appointed lecturer in mathematics at the Åbo Akademi, Turku, Finland. From 1962 to his retirement in 1983 he held a professorship of applied mathematics at the same university. He died on May 27, 1991.

In his doctoral dissertation "On the Integration of Stellar Models in Radiative Equilibrium" Qvist studied the dependence between temperature, pressure and density in theoretical stellar models based on the latest achievements in nuclear physics. The numerical integrations (performed

without a computer) on the differential equations involved were extremely tedious. In the early fifties Qvist returned to mathematics. Partly together with Paul Kustaanheimo he wrote some remarkable papers on finite geometries, of which "Some Remarks Concerning Curves of the Second Degree in a Finite Plane" (1952) may be the most important one. After the appointment to the chair of applied mathematics he mainly did research in the theory of stochastic processes.

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Paul Edwin Kustaanheimo was born on 12 April 1924 in Turku, Finland. He studied mathematics, astronomy and physics at the University of Helsinki in the years 1941–50. There he received his Ph.D. degree in mathematics in 1950. He worked at the Astronomical Observatory in Helsinki (1945–1957 and from 1969 as its director), as Professor of Mathematics at the University of Helsinki (1958–69) and at the Finnish Academy of Sciences (1971–76). He left Finland in protest over some of the developments in the Observatory that followed the political turmoil of 1968, and in the years 1976–89 he worked at the Technical University of Denmark in Lyngby, where he was responsible for teaching mathematics.

Apart from relativity, P. Kustaanheimo published papers mainly on celestial mechanics, observational astronomy (early in his career he worked as an observer for some time), basic mathematics (in particular finite geometries and Galois fields), spinor algebra, philosophy and history of science. He was also an author or co-author of a few textbooks on various subjects, and of several popular papers on physics and astronomy.

P. Kustaanheimo died on 24 August 1997 in his house near Lyngby.

— Juhana Kustaanheimo

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The Editor is grateful to Mrs. Lori Rasmussen from Lyngby for her help in trying to contact P. Kustaanheimo. Professor Kustaanheimo knew about our plan to republish his paper and gave us his permission to do so. He had been preparing his biography for us by himself, but, very unfortunately, he died before he could complete this task. The biography was written by his son.