Redshift propagation equations in the $\beta' \neq 0$ Szekeres models

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The set of differential equations obeyed by the redshift in the general $\beta' \neq 0$ Szekeres spacetimes is derived. Transversal components of the ray's momentum have to be taken into account, which leads to a set of 3 coupled differential equations. It is shown that in a general Szekeres model, and in a general Lemaître-Tolman (L-T) model, generic light rays do not have repeatable paths (RLPs): two rays sent from the same source at different times to the same observer pass through different sequences of intermediate matter particles. The only spacetimes in the Szekeres class in which *all* rays are RLPs are the Friedmann models. Among the proper Szekeres models, RLPs exist only in the axially symmetric subcases, and in each one the RLPs are the null geodesics that intersect each $t = $ constant space on the symmetry axis. In the special models with a 3-dimensional symmetry group (L-T among them), the only RLPs are radial geodesics. This shows that RLPs are very special and in the real Universe should not exist. We present several numerical examples which suggest that the rate of change of positions of objects in the sky, for the studied configuration, is 10^{-6}–10^{-7} arc sec per year. With the current accuracy of direction measurement, this drift would become observable after approximately 10 years of monitoring. More precise future observations will be able, in principle, to detect this effect, but there are basic problems with determining the reference direction that does not change.

I. THE MOTIVATION

The quasispherical Szekeres solutions have recently begun to be taken seriously as cosmological models [1–7]. For this application, one has to know the equations obeyed by the redshift. The corresponding equation for radial null geodesics in the Lemaître-Tolman (L-T) model [8,9] was derived long ago by Bondi [10], see also Ref. [11]. The generalization to the Szekeres geometry is nontrivial because in general there are no radial geodesics in the latter [12,13]. Consequently, the transversal components of the ray's momentum necessarily have to be taken into account, and a set of 3 coupled differential equations is obtained. These equations can then be applied to nonradial geodesics in the L-T model.

The purpose of this paper is to derive the redshift propagation equations in a general[1] Szekeres model of the $\beta' \neq 0$ family [11], so that they can be numerically solved and applied in various situations.

In Sec. II, the Szekeres models are introduced. In Sec. III it is pointed out that the Bondi redshift equation for radial null geodesics in the L-T model is in fact an approximation, the small parameter being the period of the electromagnetic wave. The same is true for the equations derived here. In Sec. IV, the general equations of null geodesics in Szekeres models are presented. In Sec. V, the set of redshift

equations for the Szekeres models is derived. In Sec. VI, conditions are discussed under which light rays between a given source and a given observer proceed through always the same intermediate matter particles; such rays are termed "repeatable light paths", RLPs. In Sec. VII, the equations of Secs. V and VI are applied to general null geodesics in the L-T model and in the associated plane- and hyperbolically symmetric models. It is shown there that in these models the only RLPs are the radial null geodesics. Sec. IX is a brief summary of the results.

II. THE SZEKERES SOLUTIONS

The Szekeres solutions [14,15] follow when the metric

$$ds^2 = dt^2 - e^{2\alpha(t,r,x,y)}dr^2 - e^{2\beta(t,r,x,y)}(dx^2 + dy^2), \quad (2.1)$$

is substituted in the Einstein equations with a dust source, assuming that the coordinates of (2.1) are comoving, so that the velocity field is $u^\mu = \delta^\mu{}_0$ (with $(x^0, x^1, x^2, x^3) = (t, r, x, y)$).

There are two families of Szekeres solutions, depending on whether $\beta,_r = 0$ or $\beta,_r \neq 0$. The first family is a simultaneous generalization of the Friedmann and Kantowski-Sachs [16] models. Since so far it has found no useful application in astrophysical cosmology, we shall not discuss it here (see Ref. [11]). After the Einstein equations are solved, the metric functions in the second family become

$$e^\beta = \Phi(t, r)e^{\nu(r,x,y)},$$
$$e^\alpha = h(r)\Phi(t, r)\beta,_r \equiv h(r)(\Phi,_r + \Phi\nu,_r), \quad (2.2)$$
$$e^{-\nu} = A(r)(x^2 + y^2) + 2B_1(r)x + 2B_2(r)y + C(r),$$

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[1]*General* means not only quasispherical. The generalization to cover the quasiplane and quasihyperbolic cases is immediate, so it would not make sense to leave it out.

where the function $\Phi(t, r)$ is a solution of the equation

$$\Phi_{,t}{}^2 = -k(r) + \frac{2M(r)}{\Phi} + \frac{1}{3}\Lambda\Phi^2; \qquad (2.3)$$

while $h(r)$, $k(r)$, $M(r)$, $A(r)$, $B_1(r)$, $B_2(r)$ and $C(r)$ are arbitrary functions obeying

$$g(r) \stackrel{\text{def}}{=} 4(AC - B_1{}^2 - B_2{}^2) = 1/h^2(r) + k(r). \qquad (2.4)$$

The mass density in energy units is

$$\kappa\rho = \frac{(2Me^{3\nu})_{,r}}{e^{2\beta}(e^\beta)_{,r}}; \qquad \kappa = 8\pi G/c^4. \qquad (2.5)$$

Whenever $(e^\beta)_{,r} = 0$ and $(2Me^{3\nu})_{,r} \neq 0$, a shell crossing singularity occurs. It is similar to the shell crossing singularity in the L-T models, but with a difference. In a quasispherical model a shell crossing may occur along a circle, or, in exceptional cases, at a single point, and not at a whole surface of constant t and r, as was the case in the L-T models.

As in the L-T model, the bang time function follows from (2.3):

$$\int_0^\Phi \frac{d\tilde{\Phi}}{\sqrt{-k + 2M/\tilde{\Phi} + \frac{1}{3}\Lambda\tilde{\Phi}^2}} = t - t_B(r), \qquad (2.6)$$

The solutions of the above equation for $\Lambda \neq 0$ involve elliptic functions and were first studied by Barrow and Stein-Schabes [17].

As seen from (2.1) and (2.2), the Szekeres models are covariant with the transformations $r = f(r')$, where $f(r')$ is an arbitrary function.

The Szekeres metric has in general no symmetry, but acquires a 3-dimensional symmetry group with 2-dimensional orbits when A, B_1, B_2 and C are all constant (that is, when $\nu_{,r} = 0$).

The sign of $g(r)$ determines the geometry of the 2-surfaces of constant t and r (and the symmetry of the constant A, B_1, B_2 and C limit). The geometry of these surfaces is spherical, planar or hyperbolic (pseudospherical) when $g > 0$, $g = 0$ or $g < 0$, respectively. With A, B_1, B_2 and C being functions of r, the surfaces $r = $ const within a single space $t = $ const may have different geometries, i.e. they can be spheres in one part of the space and the surfaces of constant negative curvature elsewhere, the curvature being zero at the boundary.

The sign of $k(r)$ determines the type of evolution; with $k > 0 = \Lambda$ the model expands away from an initial singularity and then recollapses to a final singularity, with $k < 0 = \Lambda$ the model is either ever-expanding or ever-collapsing, depending on the initial conditions; $k = 0$ is the intermediate case corresponding to the "flat" Friedmann model ($k = 0$ can also occur on a 3-surface as the boundary between a region with $k > 0$ and another one

with $k < 0$). The sign of $k(r)$ influences the sign of $g(r)$. Since $1/h^2$ in (2.4) must be non-negative,[2] we have the following: With $g > 0$ (spherical geometry), all three types of evolution are allowed, with $g = 0$ (plane geometry), k must be nonpositive (only parabolic or hyperbolic evolutions are allowed), and with $g < 0$ (hyperbolic geometry), k must be strictly negative, so only the hyperbolic evolution is allowed.

The Friedmann limit follows when $\Phi(t, r) = \Phi_1(r)S(t)$. No further specialization of the Szekeres functions is needed; the limiting Friedmann model is represented in the little-known Goode-Wainwright [18] coordinates, see also Ref. [19].

The Szekeres models are subdivided according to the sign of $g(r)$ into the quasispherical (with $g > 0$), quasi-plane ($g = 0$) and quasihyperbolic ($g < 0$). Despite suggestions to the contrary made in the literature, the geometry of the latter two classes has not been investigated at all and is not really understood; see Refs. [20,21] for recent work on their interpretation. Only the quasispherical model has been rather well investigated, and found useful application in astrophysical cosmology. However, including $g \leq 0$ in the redshift equations causes no complication, so we consider here an arbitrary g.

The quasispherical model may be imagined as a generalization of the L-T model in which the spheres of constant mass are made nonconcentric. The functions $A(r)$, $B_1(r)$ and $B_2(r)$ determine how the center of a sphere changes its position in a space $t = $ const when the radius of the sphere is increased or decreased [22]. Still, this is a rather simple geometry because all the arbitrary functions depend on one variable, r.

It is often convenient to reparametrize the Szekeres metric as follows [23]. Even if $A = 0$ initially, a transformation of the (x, y)-coordinates can restore $A \neq 0$, so we may assume $A \neq 0$ with no loss of generality [11]. Then let $g \neq 0$. Writing $A = \sqrt{|g|}/(2S)$, $B_1 = -\sqrt{|g|}P/(2S)$, $B_2 = -\sqrt{|g|}Q/(2S)$, $\varepsilon \stackrel{\text{def}}{=} g/|g|$, $k = |g|\tilde{k}$ and $\Phi = \sqrt{|g|}\tilde{\Phi}$, we can represent the metric (2.2) as[3]

$$e^{-\nu} = \sqrt{|g|}\mathcal{E},$$

$$\mathcal{E} \stackrel{\text{def}}{=} \frac{(x - P)^2}{2S} + \frac{(y - Q)^2}{2S} + \frac{\varepsilon S}{2}, \qquad (2.7)$$

$$ds^2 = dt^2 - \frac{(\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E})^2}{\varepsilon - k(r)}dr^2 - \frac{\Phi^2}{\mathcal{E}^2}(dx^2 + dy^2),$$

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[2] $1/h(r)$ can be zero at isolated points—it is then either a coordinate singularity or a neck or belly—but not on open intervals.

[3] The tildes were dropped in (2.7) and in all further text. The Φ in (2.7) is in fact $\tilde{\Phi}$ and the $k(r)$ is $\tilde{k}(r)$. The redefinitions imply, via (2.4), $C = \sqrt{|g|}[(P^2 + Q^2)/S + \varepsilon S]/2$, $h^2 = 1/[|g|(\varepsilon - \tilde{k})]$ and $M = \sqrt{|g|}^3\tilde{M}$. The M used from now on is in fact $\tilde{M}$.

where, so far, $\varepsilon = \pm 1$ ($+1$ for the quasispherical and -1 for the quasihyperbolic model). When $g = 0$, the transition from (2.2) to (2.7) is $A = 1/(2S)$, $B_1 = -P/(2S)$, $B_2 = -Q/(2S)$ and Φ is unchanged.[4] Then (2.7) applies with $\varepsilon = 0$, and the resulting model is quasiplane.

The parametrization introduced above makes several formulae simpler, mainly because the constraint (2.4) is identically fulfilled in it. However, this parametrization obscures the fact, evident in (2.1)–(2.4), that *the same* Szekeres model may be quasispherical in one part of the spacetime, and quasihyperbolic elsewhere, with the boundary between these two regions being quasiplane; see an explicit simple example in Ref. [20]. In most of the literature published so far, these models have been considered separately, but this was either for purposes of systematic research, or with a specific application in view that fixed the sign of $g(r)$.

Equation (2.3), is formally identical to the Friedmann equation, but with k and M depending on r, so each surface $r = $ const evolves independently of the others. The solutions $\Phi(t, r)$ are the same as the corresponding L-T solutions, and are unaffected by the dependence of the Szekeres metric on the (x, y) coordinates.

As defined by (2.2) and (2.3), the Szekeres models contain 8 functions of r, of which only 7 are arbitrary because of (2.4). The parametrization of (2.7) turns $g(r)$ to a constant parameter ε, thus reducing the number to 6. By a choice of r (still arbitrary up to now), we can fix one more function (for example, by defining $r' = M(r)$). Thus, the number of arbitrary functions that correspond to physical degrees of freedom is 5.

In the following, we will represent the Szekeres solutions with $\beta_{,r} \neq 0$ in the parametisation introduced in (2.7). The formula for mass density in these variables is

$$\kappa\rho = \frac{2(M_{,r} - 3M\mathcal{E}_{,r}/\mathcal{E})}{\Phi^2(\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E})}. \tag{2.8}$$

The shear tensor is

$$\sigma^\alpha{}_\beta = \frac{1}{3}\left(\frac{\Phi_{,tr} - \Phi_{,t}\Phi_{,r}/\Phi}{\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E}}\right)\mathrm{diag}(0, 2, -1, -1), \tag{2.9}$$

and the scalar of expansion is

$$\theta = u^\alpha{}_{;\alpha} = \frac{2\Phi_{,t}}{\Phi} + \frac{\Phi_{,tr} - \Phi_{,t}\mathcal{E}_{,r}/\mathcal{E}}{\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E}}. \tag{2.10}$$

III. REMARKS ON THE BONDI REDSHIFT EQUATION IN THE L-T MODEL

The L-T model is a special case of the quasispherical Szekeres models that follows from (2.7) when $\varepsilon = +1$ and the functions P, Q, S are all constant. With a different

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[4]The implied changes in C and h are then $C = (P^2 + Q^2)/(2S)$, $h^2 = -1/k$; k and M remain unchanged.

representation of the coordinates on a sphere, the resulting metric is:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{R_{,r}{}^2}{1 + 2E(r)}\mathrm{d}r^2 - R^2(t, r)(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2), \tag{3.1}$$

and the equation of an incoming radial null geodesic is

$$\frac{\mathrm{d}t}{\mathrm{d}r} = -\frac{R_{,r}(t, r)}{\sqrt{1 + 2E}}. \tag{3.2}$$

Bondi's derivation [10] of the redshift equation for this geodesic is as follows. Take a light signal obeying (3.2), the equation of its trajectory (the solution of (3.2)) is

$$t = T(r) \tag{3.3}$$

Take a second light signal, emitted from the same radial coordinate r, but later (as measured by the time coordinate t) by τ. The equation of its trajectory is:

$$t = T(r) + \tau(r), \tag{3.4}$$

where $(T + \tau)$ obeys, from (3.2):

$$\frac{\mathrm{d}T}{\mathrm{d}r} + \frac{\mathrm{d}\tau}{\mathrm{d}r} = -\frac{R_{,r}(T(r) + \tau(r), r)}{\sqrt{1 + 2E(r)}}. \tag{3.5}$$

From the Taylor formula we have:

$$R_{,r}(T(r) + \tau(r), r) = R_{,r}(T(r), r) + \tau(r)R_{,tr}(T(r), r) + \mathcal{O}(\tau^2, r), \tag{3.6}$$

where the last term has the property $\mathcal{O}(\tau^2, r)/\tau \xrightarrow[\tau \to 0]{} 0$. Now, *assuming that τ is small*, we neglect the last term in (3.6) and obtain from (3.5), taking into account (3.2):

$$\frac{\mathrm{d}\tau}{\mathrm{d}r} = -\tau(r)\frac{R_{,tr}(T(r), r)}{\sqrt{1 + 2E(r)}}. \tag{3.7}$$

If τ is the period of an electromagnetic wave, then by definition:

$$\frac{\tau(r_{\mathrm{obs}})}{\tau(r_{\mathrm{em}})} = 1 + z(r_{\mathrm{em}}), \tag{3.8}$$

where the subscripts "obs" and "em" refer to the points of observation and emission, respectively, and z is the redshift. From (3.8), keeping the observer at a fixed position and letting r_{em} vary, we obtain $(\mathrm{d}\tau/\mathrm{d}r)/\tau = -(\mathrm{d}z/\mathrm{d}r)/(1 + z)$, and so in (3.7):

$$\frac{1}{1 + z}\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{R_{,tr}(T(r), r)}{\sqrt{1 + 2E(r)}}. \tag{3.9}$$

This is Bondi's radial redshift equation [10]. It does not describe the redshift propagation exactly. Neglecting the last term in (3.6) we have changed the exact equation into one that only approximates the actual variation of τ along the ray. The approximation is better the smaller the value of τ. Considering that τ is the period of an electromagnetic wave, and taking into account the period range of relevance

in observational astronomy (from gamma rays up to radio waves, the longest observed of which have the wavelength of the order of 15 m, thus the period of about 5×10^{-8} s), we see that, compared to cosmological time-scales, the periods are short indeed and the approximation is not bad. Moreover, as seen from (3.8), by following the rays back from the observation event into the past, we encounter ever smaller values of τ, so the approximation gets progressively better with increasing redshift (or, rather, gets progressively worse as the ray approaches us). Still, it is conceptually important to remember that (3.9) involves an approximation (this approximation is equivalent to the geometric optics approximation [11,24] that leads to the commonly used expression for the redshift $1 + z = (k_\alpha u^\alpha)_{em}/(k_\alpha u^\alpha)_{obs}$).

We shall apply the same approach to the redshift equations in the Szekeres models in Sec. V.

IV. EQUATIONS OF GENERAL NULL GEODESICS IN A SZEKERES SPACETIME

For reference, the equations of general null geodesics in a Szekeres model are copied from Ref. [12] in Appendix A.[5] They are written there in terms of an affine parameter s. For our present purpose it is more convenient to use the coordinate r as an independent parameter (which is nonaffine).

This is allowed, but with some caution. It is easily seen from (A1)–(A4) in Appendix A that a geodesic on which $dr/ds = 0$ over some open range of s has $dx/ds = dy/ds = 0$ in that range, and so is timelike. However, (A1)–(A4) do not guarantee that $dr/ds \neq 0$ at all points; isolated points at which $dr/ds = 0$ can exist. Examples that explain how this can happen are the nonradial geodesics in an L-T model, considered in Sec. VIII. Thus, r can be used as a parameter on null geodesics only on such segments where $ds/dr > 0$ or $ds/dr < 0$ throughout.

Several subexpressions in the equations of Appendix A are multiply repeated, therefore we introduce the following abbreviations:

$$\Phi_{,r} - \Phi \mathcal{E}_{,r}/\mathcal{E} \stackrel{\text{def}}{=} \Phi_1, \qquad (4.1)$$

$$\Phi_{,tr} - \Phi_{,t} \mathcal{E}_{,r}/\mathcal{E} \stackrel{\text{def}}{=} \Phi_{01}, \qquad (4.2)$$

$$\Phi_{,rr} - \Phi \mathcal{E}_{,rr}/\mathcal{E} \stackrel{\text{def}}{=} \Phi_{11}, \qquad (4.3)$$

$$\mathcal{E}_{,r} \mathcal{E}_{,x} - \mathcal{E} \mathcal{E}_{,xr} \stackrel{\text{def}}{=} E_{12}, \qquad (4.4)$$

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[5]It is shown in Ref. [12] (see also Ref. [13]) that in general there exists no analogue of a radial null geodesic. Radial geodesics exist only when the Szekeres model is axially symmetric; then their intersections with every space of constant time coordinate lie on the axis of symmetry.

$$\mathcal{E}_{,r} \mathcal{E}_{,y} - \mathcal{E} \mathcal{E}_{,yr} \stackrel{\text{def}}{=} E_{13}. \qquad (4.5)$$

In addition, the following replacement will appear useful:

$$\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 \stackrel{\text{def}}{=} \Sigma. \qquad (4.6)$$

We have, for any coordinate:

$$\frac{d^2 x^\alpha}{ds^2} = \left(\frac{dr}{ds}\right)^2 \frac{d^2 x^\alpha}{dr^2} + \frac{d^2 r}{ds^2} \frac{dx^\alpha}{dr}. \qquad (4.7)$$

Then, from (A2) we have:

$$\frac{d^2 r}{ds^2} = \left(\frac{dr}{ds}\right)^2 \left\{ -2\frac{\Phi_{01}}{\Phi_1}\frac{dt}{dr} - \left(\frac{\Phi_{11}}{\Phi_1} - \frac{\mathcal{E}_{,r}}{\mathcal{E}} + \frac{1}{2}\frac{k_{,r}}{\varepsilon - k}\right) \right.$$
$$\left. - 2\frac{\Phi}{\mathcal{E}^2}\frac{E_{12}}{\Phi_1}\frac{dx}{dr} - 2\frac{\Phi}{\mathcal{E}^2}\frac{E_{13}}{\Phi_1}\frac{dy}{dr} + \frac{\Phi}{\mathcal{E}^2}\frac{\varepsilon - k}{\Phi_1}\Sigma \right\}$$
$$\stackrel{\text{def}}{=} U(t, r, x, y)\left(\frac{dr}{ds}\right)^2. \qquad (4.8)$$

Consequently, (A1), (A3), and (A4) become, using (4.7):

$$\frac{d^2 t}{dr^2} + \frac{\Phi_1 \Phi_{01}}{\varepsilon - k} + \frac{\Phi \Phi_{,t}}{\mathcal{E}^2}\Sigma + U\frac{dt}{dr} = 0, \qquad (4.9)$$

$$\frac{d^2 x}{dr^2} + 2\frac{\Phi_{,t}}{\Phi}\frac{dt}{dr}\frac{dx}{dr} - \frac{1}{\Phi}\frac{\Phi_1}{\varepsilon - k}E_{12} + \frac{2\Phi_1}{\Phi}\frac{dx}{dr} - \frac{\mathcal{E}_{,x}}{\mathcal{E}}\left(\frac{dx}{dr}\right)^2$$
$$- 2\frac{\mathcal{E}_{,y}}{\mathcal{E}}\frac{dx}{dr}\frac{dy}{dr} + \frac{\mathcal{E}_{,x}}{\mathcal{E}}\left(\frac{dy}{dr}\right)^2 + U\frac{dx}{dr} = 0, \qquad (4.10)$$

$$\frac{d^2 y}{dr^2} + 2\frac{\Phi_{,t}}{\Phi}\frac{dt}{dr}\frac{dy}{dr} - \frac{1}{\Phi}\frac{\Phi_1}{\varepsilon - k}E_{13} + \frac{2\Phi_1}{\Phi}\frac{dy}{dr}$$
$$+ \frac{\mathcal{E}_{,y}}{\mathcal{E}}\left(\frac{dx}{dr}\right)^2 - 2\frac{\mathcal{E}_{,x}}{\mathcal{E}}\frac{dx}{dr}\frac{dy}{dr} - \frac{\mathcal{E}_{,y}}{\mathcal{E}}\left(\frac{dy}{dr}\right)^2 + U\frac{dy}{dr} = 0.$$
$$\qquad (4.11)$$

V. THE REDSHIFT EQUATIONS IN THE SZEKERES MODELS

Consider, in the Szekeres metric (2.7), two light signals, the second one following the first one after a short time-interval τ, both emitted by the same source and arriving at the same observer of coordinates (r, x, y). The equation of the trajectory of the first signal is

$$(t, x, y) = (T(r), X(r), Y(r)), \qquad (5.1)$$

the corresponding equation for the second signal is

$$(t, x, y) = (T(r) + \tau(r), X(r) + \zeta(r), Y(r) + \psi(r)). \quad (5.2)$$

This means that while the first ray intersects the hypersurface of a given constant value of the r-coordinate at the point $(t, x, y) = (T, X, Y)$, the second ray intersects the same hypersurface at the point $(t, x, y) = (T + \tau, X + \zeta, Y + \psi)$. Thus, in general, those two rays will not intersect the same succession of intermediate matter

worldlines on the way. Note that the coordinates we use throughout the paper are comoving, so both the source of light and the observer keep their spatial coordinates unchanged throughout history. Given this, and given that we consider a pair of rays emitted by the same source and received by the same observer, we have $(\zeta, \psi) = (0, 0)$ at the point of emission and at the point of reception. However, we have to allow that the second ray was emitted in a different direction than the first one, and is received from a different direction by the observer.[6] The directions of the two rays will be determined by $(dx/dr, dy/dr)$ and $(dx/dr + \xi(r), dy/dr + \eta(r))$, respectively, where $\xi = d\zeta/dr$, $\eta = d\psi/dr$. We will assume that $(d\tau/dr, \zeta, \psi, \xi, \eta)$ are small of the same order as τ, so we will neglect all terms nonlinear in any of them and terms involving their products.

Since $\zeta = \psi = 0$ at the observer, these quantities are not in fact observable. However, they have to be numerically monitored along the ray because, as will be seen below, they enter the equation for τ, which is connected to the redshift by (3.8).

In writing out the equations of propagation of redshift, we will introduce the symbol Δ. It will denote the difference between the relevant expression taken at $(t + \tau, r, x + \zeta, y + \psi)$ and at (t, r, x, y), linearized in (τ, ζ, ψ), for example $\Phi_{01}(t + \tau, r, x + \zeta, y + \psi) - \Phi_{01}(t, r, x, y) \stackrel{\text{def}}{=} \Delta\Phi_{01} + \mathcal{O}(\tau^2, \tau\zeta, \tau\psi, \zeta^2, \ldots)$. We have:[7]

$$\Delta\Phi = \Phi_{,t}\,\tau, \qquad \Delta(\Phi_{,t}) = \Phi_{,tt}\,\tau, \qquad \Delta\frac{dt}{dr} = \frac{d\tau}{dr},$$

$$\Delta\frac{dx}{dr} = \xi, \qquad \Delta\frac{dy}{dr} = \eta, \tag{5.3}$$

$$\Delta\mathcal{E} = \mathcal{E}_{,x}\,\zeta + \mathcal{E}_{,y}\,\psi, \qquad \Delta\mathcal{E}_{,x} = \zeta/S, \qquad \Delta\mathcal{E}_{,y} = \psi/S, \tag{5.4}$$

$$\Delta\Phi_1 = \Phi_{01}\tau + \frac{\Phi E_{12}}{\mathcal{E}^2}\zeta + \frac{\Phi E_{13}}{\mathcal{E}^2}\psi, \tag{5.5}$$

$$\Delta\Phi_{01} = (\Phi_{,ttr} - \Phi_{tt}\mathcal{E}_{,r}/\mathcal{E})\tau + \frac{\Phi_{,t}E_{12}}{\mathcal{E}^2}\zeta + \frac{\Phi_{,t}E_{13}}{\mathcal{E}^2}\psi, \tag{5.6}$$

$$\Delta\Phi_{11} = (\Phi_{,trr} - \Phi_{t}\mathcal{E}_{,rr}/\mathcal{E})\tau + \frac{\Phi}{\mathcal{E}^2}(\mathcal{E}_{,rr}\mathcal{E}_{,x} - \mathcal{E}\mathcal{E}_{,rrx})\zeta$$
$$+ \frac{\Phi}{\mathcal{E}^2}(\mathcal{E}_{,rr}\mathcal{E}_{,y} - \mathcal{E}\mathcal{E}_{,rry})\psi. \tag{5.7}$$

In the next two equations account is taken of the fact that $\mathcal{E}_{,xy} \equiv 0$.

$$\Delta E_{12} = (\mathcal{E}_{,r}\mathcal{E}_{,xx} - \mathcal{E}\mathcal{E}_{,rxx})\zeta + (\mathcal{E}_{,ry}\mathcal{E}_{,x} - \mathcal{E}_{,y}\mathcal{E}_{,rx})\psi, \tag{5.8}$$

$$\Delta E_{13} = (\mathcal{E}_{,rx}\mathcal{E}_{,y} - \mathcal{E}_{,x}\mathcal{E}_{,ry})\zeta + (\mathcal{E}_{,r}\mathcal{E}_{,yy} - \mathcal{E}\mathcal{E}_{,ryy})\psi, \tag{5.9}$$

$$\Delta\Sigma = 2\frac{dx}{dr}\xi + 2\frac{dy}{dr}\eta, \tag{5.10}$$

$$\Delta U = 2\left(-\frac{\Delta\Phi_{01}}{\Phi_1} + \frac{\Phi_{01}\Delta\Phi_1}{\Phi_1^2}\right)\frac{dt}{dr} - 2\frac{\Phi_{01}}{\Phi_1}\frac{d\tau}{dr} - \frac{\Delta\Phi_{11}}{\Phi_1} + \frac{\Phi_{11}\Delta\Phi_1}{\Phi_1^2} + \frac{\Delta\mathcal{E}_{,r}}{\mathcal{E}} - \frac{\mathcal{E}_{,r}\Delta\mathcal{E}}{\mathcal{E}^2}$$
$$+ 2\left(-\frac{\Phi_{,t}E_{12}\tau}{\mathcal{E}^2\Phi_1} + 2\frac{\Phi\Delta\mathcal{E}E_{12}}{\mathcal{E}^3\Phi_1} - \frac{\Phi\Delta E_{12}}{\mathcal{E}^2\Phi_1} + \frac{\Phi E_{12}\Delta\Phi_1}{\mathcal{E}^2\Phi_1^2}\right)\frac{dx}{dr} - 2\frac{\Phi E_{12}\xi}{\mathcal{E}^2\Phi_1}$$
$$+ 2\left(-\frac{\Phi_{,t}E_{13}\tau}{\mathcal{E}^2\Phi_1} + 2\frac{\Phi\Delta\mathcal{E}E_{13}}{\mathcal{E}^3\Phi_1} - \frac{\Phi\Delta E_{13}}{\mathcal{E}^2\Phi_1} + \frac{\Phi E_{13}\Delta\Phi_1}{\mathcal{E}^2\Phi_1^2}\right)\frac{dy}{dr} - 2\frac{\Phi E_{13}\eta}{\mathcal{E}^2\Phi_1}$$
$$+ \frac{(\varepsilon - k)\Phi\Sigma}{\mathcal{E}^2\Phi_1}\left(\frac{\Phi_{,t}\tau}{\Phi} - 2\frac{\Delta\mathcal{E}}{\mathcal{E}} - \frac{\Delta\Phi_1}{\Phi_1} + \frac{\Delta\Sigma}{\Sigma}\right). \tag{5.11}$$

Applying the Δ-operation to (4.9)–(4.11) we obtain:

$$\frac{d^2\tau}{dr^2} + \frac{\Phi_{01}\Delta\Phi_1 + \Phi_1\Delta\Phi_{01}}{\varepsilon - k} + \frac{(\Phi_{,t}^2 + \Phi\Phi_{,tt})\Sigma\tau}{\mathcal{E}^2} - 2\frac{\Phi\Phi_{,t}\Delta\mathcal{E}\Sigma}{\mathcal{E}^3} + \frac{\Phi\Phi_{,t}\Delta\Sigma}{\mathcal{E}^2} + \Delta U\frac{dt}{dr} + U\frac{d\tau}{dr} = 0, \tag{5.12}$$

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[6]This means that in a general inhomogeneous and anisotropic Universe, the observed objects should drift across the sky. See a brief quantitative discussion of this effect in Sec. VIII.

[7]A quick way to calculate (5.3)–(5.16) is to take the differential of the corresponding quantity at constant r and replace $(dt, dx, dy, d(dx/dr), d(dy/dr))$ by $(\tau, \zeta, \psi, \xi, \eta)$.

$$\frac{d^2\zeta}{dr^2} + 2\left(\frac{\Phi_{,tt}}{\Phi} - \frac{\Phi_{,t}{}^2}{\Phi^2}\right)\frac{dt}{dr}\frac{dx}{dr}\tau + 2\frac{\Phi_{,t}}{\Phi}\frac{dx}{dr}\frac{d\tau}{dr} + 2\frac{\Phi_{,t}}{\Phi}\frac{dt}{dr}\xi - \frac{\Delta\Phi_1 E_{12}}{(\varepsilon-k)\Phi} + \frac{\Phi_{,t}\,\Phi_1 E_{12}\tau}{(\varepsilon-k)\Phi^2} - \frac{\Phi_1\Delta E_{12}}{(\varepsilon-k)\Phi}$$
$$+ 2\left(\frac{\Delta\Phi_1}{\Phi} - \frac{\Phi_1\Phi_{,t}\,\tau}{\Phi^2}\right)\frac{dx}{dr} + 2\frac{\Phi_1}{\Phi}\xi - \left(\frac{dx}{dr}\right)^2\left(\frac{\zeta}{S\mathcal{E}} - \frac{\mathcal{E}_{,x}\Delta\mathcal{E}}{\mathcal{E}^2}\right) - \frac{2\mathcal{E}_{,x}\,\xi}{\mathcal{E}}\frac{dx}{dr} - 2\frac{dx}{dr}\frac{dy}{dr}\left(\frac{\psi}{S\mathcal{E}} - \frac{\mathcal{E}_{,y}\Delta\mathcal{E}}{\mathcal{E}^2}\right)$$
$$- 2\frac{\mathcal{E}_{,y}}{\mathcal{E}}\left(\frac{dy}{dr}\xi + \frac{dx}{dr}\eta\right) + \left(\frac{dy}{dr}\right)^2\left(\frac{\zeta}{S\mathcal{E}} - \frac{\mathcal{E}_{,x}\Delta\mathcal{E}}{\mathcal{E}^2}\right) + \frac{2\mathcal{E}_{,x}\,\eta}{\mathcal{E}}\frac{dy}{dr} + \Delta U\frac{dx}{dr} + U\xi = 0, \tag{5.13}$$

$$\frac{d^2\psi}{dr^2} + 2\left(\frac{\Phi_{,tt}}{\Phi} - \frac{\Phi_{,t}{}^2}{\Phi^2}\right)\frac{dt}{dr}\frac{dy}{dr}\tau + 2\frac{\Phi_{,t}}{\Phi}\frac{dy}{dr}\frac{d\tau}{dr} + 2\frac{\Phi_{,t}}{\Phi}\frac{dt}{dr}\eta - \frac{\Delta\Phi_1 E_{13}}{(\varepsilon-k)\Phi} + \frac{\Phi_{,t}\,\Phi_1 E_{13}\tau}{(\varepsilon-k)\Phi^2} - \frac{\Phi_1\Delta E_{13}}{(\varepsilon-k)\Phi}$$
$$+ 2\left(\frac{\Delta\Phi_1}{\Phi} - \frac{\Phi_1\Phi_{,t}\,\tau}{\Phi^2}\right)\frac{dy}{dr} + 2\frac{\Phi_1}{\Phi}\eta + \left(\frac{dx}{dr}\right)^2\left(\frac{\psi}{S\mathcal{E}} - \frac{\mathcal{E}_{,y}\Delta\mathcal{E}}{\mathcal{E}^2}\right) + \frac{2\mathcal{E}_{,y}\,\xi}{\mathcal{E}}\frac{dx}{dr} - 2\frac{dx}{dr}\frac{dy}{dr}\left(\frac{\zeta}{S\mathcal{E}} - \frac{\mathcal{E}_{,x}\Delta\mathcal{E}}{\mathcal{E}^2}\right)$$
$$- 2\frac{\mathcal{E}_{,x}}{\mathcal{E}}\left(\frac{dy}{dr}\xi + \frac{dx}{dr}\eta\right) - \left(\frac{dy}{dr}\right)^2\left(\frac{\psi}{S\mathcal{E}} - \frac{\mathcal{E}_{,y}\Delta\mathcal{E}}{\mathcal{E}^2}\right) - \frac{2\mathcal{E}_{,y}\,\eta}{\mathcal{E}}\frac{dy}{dr} + \Delta U\frac{dy}{dr} + U\eta = 0, \tag{5.14}$$

In addition, we have the first integral of the geodesic Eqs. (4.9)–(4.11):

$$\left(\frac{dt}{dr}\right)^2 = \frac{(\Phi_1)^2}{\varepsilon-k} + \frac{\Phi^2}{\mathcal{E}^2}\left[\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2\right], \tag{5.15}$$

Applying the Δ-operation to this we get

$$\frac{d\tau}{dr}\frac{dt}{dr} = \frac{\Phi_1\Delta\Phi_1}{\varepsilon-k} + \left(\frac{\Phi\Phi_{,t}\,\tau}{\mathcal{E}^2} - \frac{\Phi^2\Delta\mathcal{E}}{\mathcal{E}^3}\right)\left[\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2\right]$$
$$+ \frac{\Phi^2}{\mathcal{E}^2}\left(\frac{dx}{dr}\xi + \frac{dy}{dr}\eta\right). \tag{5.16}$$

Note: $dt/dr < 0$ for an incoming ray.

VI. REPEATABLE LIGHT PATHS

As attested by (5.12)–(5.14), in a generic Szekeres model two light rays connecting a given source to a given observer at different instants of emission do not proceed through the same succession of intermediate matter particles. We will now investigate under what conditions this intermediate succession is the same. This property will be called repeatable light paths (RLP).

For a RLP we have

$$\zeta = \psi = \xi = \eta = 0 \tag{6.1}$$

all along the ray. Then (5.12) decouples from (5.13) and (5.14) and just determines τ (and, with it, the redshift), if the null geodesic equations are solved first. Eqs. (5.13) and (5.14) become then:

$$2\left(\frac{\Phi_{,tt}}{\Phi} - \frac{\Phi_{,t}{}^2}{\Phi^2}\right)\frac{dt}{dr}\frac{dx}{dr}\tau + 2\frac{\Phi_{,t}}{\Phi}\frac{dx}{dr}\frac{d\tau}{dr} - \frac{\Delta\Phi_1 E_{12}}{(\varepsilon-k)\Phi} + \frac{\Phi_{,t}\,\Phi_1 E_{12}\tau}{(\varepsilon-k)\Phi^2} + 2\left(\frac{\Delta\Phi_1}{\Phi} - \frac{\Phi_1\Phi_{,t}\,\tau}{\Phi^2}\right)\frac{dx}{dr} + \Delta U\frac{dx}{dr} = 0, \tag{6.2}$$

$$2\left(\frac{\Phi_{,tt}}{\Phi} - \frac{\Phi_{,t}{}^2}{\Phi^2}\right)\frac{dt}{dr}\frac{dy}{dr}\tau + 2\frac{\Phi_{,t}}{\Phi}\frac{dy}{dr}\frac{d\tau}{dr} - \frac{\Delta\Phi_1 E_{13}}{(\varepsilon-k)\Phi} + \frac{\Phi_{,t}\,\Phi_1 E_{13}\tau}{(\varepsilon-k)\Phi^2} + 2\left(\frac{\Delta\Phi_1}{\Phi} - \frac{\Phi_1\Phi_{,t}\,\tau}{\Phi^2}\right)\frac{dy}{dr} + \Delta U\frac{dy}{dr} = 0. \tag{6.3}$$

These equations can be understood in 2 ways:

(1) As equations defining special Szekeres spacetimes in which *all* null geodesics are RLPs.
(2) As equations defining special null geodesics which are RLPs in subcases of the Szekeres spacetimes.

In the first interpretation, (6.2) and (6.3) should be identities in the components of dx^α/dr. They are polynomials of degree 3 in these components, and since $(dt/dr)^2$ does not appear in them, the constraint (5.15) plays no role—all powers of dx^α/dr that do appear are independent. Equating to zero the coefficient of $(dx/dr)^3$ in (6.2) (which arises inside ΔU, within Σ), and taking into account that $\Delta\mathcal{E} = \Delta\Sigma = 0$ when (6.1) holds, we get

$$\Psi \overset{\text{def}}{=} \Phi_{,tr} - \Phi_{,t}\,\Phi_{,r}/\Phi = 0. \tag{6.4}$$

The integral of this is $\Phi = S(t)f(r)$, where S and f are arbitrary functions. It is seen from (2.9) that this means zero shear, i.e. the Friedmann limit. With (6.4) fulfilled, (6.2) and (6.3) become identities, and (4.9)–(4.11) reduce to the equations of general null geodesics in a Friedmann spacetime.[8] With the observer placed at the origin, the geodesics become radial, $dx/dr = dy/dr = 0$, and then

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[8]We recall, however, that the Friedmann limit is represented in the Goode-Wainwright [18] coordinates (see the remark in para 4 after (2.6)). Consequently, all equations representing the Friedmann model will look unfamiliar.

(5.16) becomes equivalent to the ordinary Robertson-Walker redshift formula, $1 + z = S(t_o)/S(t_e)$. To verify this, some calculations are needed, in which Ref. [11] may prove helpful.

Thus, we have proven the following:

Corollary 1:

The only spacetimes in the Szekeres family in which all null geodesics have repeatable paths are the Friedmann models.[9]

In the second interpretation of (6.2) and (6.3), we consider 2 cases:

A. The general case: $dx/dr \neq 0 \neq dy/dr$ everywhere.

Then we multiply (6.2) by dy/dr, (6.3) by dx/dr and subtract the results. Disregarding the familiar case $\Psi = 0$ we get

$$E_{12} \frac{dy}{dr} - E_{13} \frac{dx}{dr} = 0. \qquad (6.5)$$

This, together with (6.2), (5.12), and (4.9)-(4.11) defines a certain subcase of the Szekeres model and a class of curves in it. Since both the subcase and the class will turn out to be empty, but the calculations proving it are rather elaborate, we present them in Appendix B.

B. The special cases: $dx/dr = 0$ or $dy/dr = 0$.

These two cases are equivalent under the coordinate transformation $(x, y) = (y', x')$, so we consider only the first one. Again disregarding $\Psi = 0$, we get from (6.2) $E_{12} = 0$. Then, (4.10) implies two possibilities:

1. Ba) $\mathcal{E}_{,x} = 0$.

This is possible only when P is constant, and then the geodesic lies in the subspace $x = P$. Equations (4.11) and (6.3) still have to be obeyed, while (4.10) and (6.2) are fulfilled identically. The simple coordinate transformation $x = x' + P$ has then the same effect as if $P = 0$ and $x = 0$ along the geodesic. We show in Appendix E that in this case, apart from the axially symmetric subcase mentioned below, RLPs may exist only when the Szekeres metric has a 3-dimensional symmetry group. Such spacetimes are considered in Sec. VII.

2. Bb) $dy/dr = 0$.

The case $dx/dr = dy/dr = 0$, $\varepsilon = +1$ was investigated in detail in Ref. [12]. It turned out that this can happen only when the Szekeres spacetime is axially symmetric, and then along only one subfamily of null geodesics—those that intersect each $t = $ constant space on the symmetry axis. We show in Appendix F that this result applies also

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[9]This is one more piece of evidence of how exceptional the Robertson-Walker class of models is.

with $\varepsilon \leq 0$, and that other RLPs may exist only with higher symmetries.

VII. RLPS IN THE G_3/S_2 MODELS

The symbol G_3/S_2 denotes such models that have 3-dimensional symmetry groups acting on 2-dimensional orbits [11]. They result from the general $\beta' \neq 0$ Szekeres family when the functions (P, Q, S) are all constant. The symmetry of the model is then spherical when $\varepsilon = +1$ (this is the L-T model), pseudospherical (also called hyperbolic) when $\varepsilon = -1$ and plane when $\varepsilon = 0$.

Using the G_3 symmetry, the origin of the (x, y) coordinates at $(x, y) = (P, Q)$ can be moved to any location on the S_2 surfaces. So let us consider the S_2 on which the first light ray is emitted, and let us choose the origin of (x, y) at the position of the emitter. Thus, in (4.9)–(4.11) the initial point of the earlier null geodesic will have the coordinates $(x, y) = (P, Q)$, and, at this point, $\mathcal{E}_{,x} = \mathcal{E}_{,y} = 0$. In addition, the isotropy subgroup of G_3, existing in each case at every point of the manifold, allows us to rotate the (x, y) coordinates, with no loss of generality, so that the initial value of dy/dr for our chosen geodesic is zero, i.e. so that the ray is initially tangent to the $y = $ constant subspace. Equation (4.11) shows that with such initial conditions (and with $\mathcal{E}_{,y} = 0$ at the initial point) we have $d^2y/dr^2 = 0$ initially, and so $d^2y/dr^2 = 0 = dy/dr$ all along the geodesic.

With coordinates chosen in such a way, Eqs. (4.11) and (6.3) are fulfilled identically. However, (6.2) is not an identity and reduces to:

$$\frac{dx}{dr}\left[2\left(\frac{\Phi_{,tt}}{\Phi} - \frac{\Phi_{,t}^2}{\Phi^2}\right)\frac{dt}{dr}\tau + 2\frac{\Phi_{,t}}{\Phi}\frac{d\tau}{dr}\right.$$
$$+ 2\left(\frac{\Phi_{,tr}}{\Phi} - \frac{\Phi_{,t}\Phi_{,r}}{\Phi^2}\right)\tau + 2\left(-\frac{\Phi_{,ttr}}{\Phi_{,r}} + \frac{\Phi_{,tr}^2}{\Phi_{,r}^2}\right)\frac{dt}{dr}\tau$$
$$- 2\frac{\Phi_{,tr}}{\Phi_{,r}}\frac{d\tau}{dr} - \frac{\Phi_{,trr}}{\Phi_{,r}}\tau + \frac{\Phi_{,rr}\Phi_{,tr}}{\Phi_{,r}^2}\tau$$
$$+ \frac{(\varepsilon - k)\tau}{\mathcal{E}^2}\left(\frac{dx}{dr}\right)^2\left(\frac{\Phi_{,t}}{\Phi_{,r}} - \frac{\Phi\Phi_{,tr}}{\Phi_{,r}^2}\right)\right] \stackrel{\text{def}}{=} \frac{dx}{dr}\chi = 0. \quad (7.1)$$

One solution of this is $dx/dr = 0$, which together with $dy/dr = 0$ defines a radial null geodesic. Then, (4.10) and (4.11) are fulfilled identically, while (5.15) and (5.16), together with (4.1), (4.2), and (5.5) reproduce the Bondi equation (3.7) when $\varepsilon = +1$. So, we found that in the G_3/S_2 models all radial null geodesics are RLPs.[10]

There would exist other RLPs in these models if χ in (7.1) were zero along any null geodesic—possibly in some subcases of the models. It is shown in Appendix G that this

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[10]The null geodesics with $dx/dr = dy/dr = 0$ can properly be called radial only in the L-T model, where $\varepsilon = +1$. What this condition means in the other two cases is not clear, so the term "radial" is used here only as a brief label.

does not happen, so the radial null geodesics are *the only* RLPs in these models.

VIII. NUMERICAL EXAMPLES OF NON-RLPS IN THE L-T MODEL

For illustration, we first consider a configuration that is not realistic, but shows the non-RLP effect in a clearly visible way. It is an L-T model specified by the following functions: $t_B = 0$ and $\rho(t_0, r) = \rho_0[1 + \delta - \delta \exp(-r^2/\sigma^2)]$, (where t_0 is the current instant, r is defined as $R(t_0, r)$, and ρ_0 is the density at the origin and equals $0.3 \times (3H_0^2)/(8\pi G)$, where $H_0 = 72$ km s^{-1} Mpc^{-1} and G is the gravitational constant). This model is the so-called giant void model discussed in detail in [25], with the best-fit parameters: $\delta = 4.05$ and $\sigma = 2.96$ Gpc. We use this model to study the configuration presented in Fig. 1, where, for the middle curve, the angle between the radial direction and the incoming geodesic is $\gamma = 0.22\pi$. We consider 3 light paths. The first one corresponds to photons received by the observer 5×10^9 years ago, the second one corresponds to photons received at the current instant, and the third one corresponds to photons which will be received in 5×10^9 years in the future. Figure 1 shows these 3 geodesics projected on the space $t = $ now along the flow lines of the matter source in the L-T model. Since in each case the light paths are different, the profile of matter density along each projected light ray is different. This feature is presented in the inset in Fig. 1. Even though the

FIG. 2. A schematic view of the considered configurations. The observer (O) is located at R_0 and observes a galaxy (*), the angle between the direction towards the galaxy and towards the origin is γ. Because of the non-RLP effect the angle at which the galaxy is observed at some other time instant is different.

density variation along the light path is of small amplitude, the effect is clearly visible. The average rate of change of the position of the source in the sky, seen by the observer, is $\sim 10^{-7}$ arc sec per year.

Now we will study a more realistic configuration. The parameters of the L-T model will be the same as above, but the placement of the observer and of the source will be different, see Fig. 2. The observer (O) is located at R_0 ($R_0 = R(t_0, r)$ is the present-day areal distance) and observes a galaxy (*), the angle between the direction towards the galaxy and towards the origin is γ. We study 3 configurations: (1) $R_0 = 3$ Gpc, (2) $R_0 = 1$ Gpc, (3) $R_0 = 1$ Gpc but with $\delta = 10$. All 3 cases have $d = 1$ Gyr (≈ 306.6 Mpc). For each case (for a given γ) we find a null geodesic that joins the observer and the galaxy. We then calculate the rate of change γ, which is equivalent to the change of the position of the galaxy in the sky. A detailed description of the algorithm is presented in Appendix H. The results are presented in Fig. 3.

As seen, the rate of change of the position of the source in the sky depends on the angle γ. The amplitude of the change is of the order $\sim 10^{-7}$ arc sec per year for case (2) and $\sim 10^{-6}$ arc sec per year for cases (1) and (3). Given Gaia[11] accuracy of position measurement, $5-20 \times 10^{-6}$ arc sec, we would need to wait at least a few years to detect the change of position due to non-RLP effects. However, this estimate assumes that we have a reference direction that does not change. This will be a difficult practical problem, since cosmological observations are done under the assumption that our

FIG. 1. Three nonradial null geodesics in an L-T model, projected on the space $t = $ now along the flow lines of the L-T dust. Each geodesic runs between the same observer and the same emitter, which, at present, lie at the same distance of 3.5 Gpc from the center. The solid line represents the ray that the observer receives at the current instant, $\gamma = 0.22\pi$, the dashed line represents the ray that was received 5×10^9 years ago, $\gamma = 0.228\pi$, and the dotted line represents the ray that will be received 5×10^9 years in the future, $\gamma = 0.214\pi$. As seen, nonradial null geodesics in the L-T model do not have the RLP property. The inset shows the density profile (ρ_0 is the density at the origin) evaluated at the current instant along these three different paths.

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[11]http://sci.esa.int/science-e/www/area/index.cfm?fareaid=26

FIG. 3.   The rate of change of position in the sky ($\dot{\gamma}$) due to the non-RLP effect, expressed as a change of an angle in arc sec per year $\times 10^7$. The solid line presents case (1) where $R_0 = 3$ Gpc, the dashed line presents case (2) where $R_0 = 1$ Gpc, and the dotted line presents case (3) where $R_0 = 1$ Gpc and $\delta = 10$.

Universe is precisely represented in large scales by the Robertson-Walker class of models, in which there is no such drift. We would have to identify a direction that does not change with time even in an inhomogeneous model or measure a relative change of position between various objects.

IX. SUMMARY

By a method analogous to that of Bondi [10] we have derived the equations to be obeyed by the redshift in a general Szekeres $\beta' \neq 0$ spacetime, (5.12)–(5.14). The null geodesic equations parametrized by r, which must be solved together with (5.12)–(5.14), are given by (4.9)–(4.11). Although the physically most interesting quantity is the longitudinal redshift determined by τ, the other two components, ζ and ψ, must be numerically monitored along the ray because the equations that determine (τ, ζ, ψ) are coupled.

We have shown that, in general, two light rays sent from the same source at different times to the same observer do not proceed through the same succession of intermediate matter particles; we refer to this property by saying that the light paths are not repeatable. In a toy model, with the present spatial distance between the light source B and the observer being of the order of 1.5 Gpc, the estimated rate of the drift of B across the sky would be $\approx 7 \times 10^{-8}$ arc sec per year. In a more realistic configuration, this number is $\approx 10^{-6}$ arc sec per year. The Gaia is expected to have the precision of position determination 5–20×10^{-6} arc sec .

We have derived the equations defining repeatable light paths (RLPs), (6.2) and (6.3); they must hold together with (4.9)–(4.11) and (5.12). We have shown that all null geodesics are RLPs only in the Friedmann models. The only other cases in which RLPs exist are the following:

(i) The axially symmetric Szekeres models, in which the RLPs are the null geodesics intersecting every space of constant time on the axis of symmetry.
(ii) The radial null geodesics in the G_3/S_2 subcases (i.e. in the spacetimes that have 3-dimensional symmetry groups).

APPENDIX A: EQUATIONS OF NULL GEODESICS IN A SZEKERES SPACETIME IN AN AFFINE PARAMETRIZATION

For convenience of the readers, the equations of null geodesics in a Szekeres spacetime are copied here from Ref. [12]. They are given in an affine parametrization.

$$\frac{d^2 t}{ds^2} + \frac{\Phi_{,tr} - \Phi_{,t}\mathcal{E}_{,r}/\mathcal{E}}{\varepsilon - k}(\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E})\left(\frac{dr}{ds}\right)^2$$
$$+ \frac{\Phi\Phi_{,t}}{\mathcal{E}^2}\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right] = 0, \tag{A1}$$

$$\frac{d^2 r}{ds^2} + 2\frac{\Phi_{,tr} - \Phi_{,t}\mathcal{E}_{,r}/\mathcal{E}}{\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E}}\frac{dt}{ds}\frac{dr}{ds} + \left(\frac{\Phi_{,rr} - \Phi\mathcal{E}_{,rr}/\mathcal{E}}{\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E}}\right.$$
$$\left. - \frac{\mathcal{E}_{,r}}{\mathcal{E}} + \frac{1}{2}\frac{k_{,r}}{\varepsilon - k}\right)\left(\frac{dr}{ds}\right)^2 + 2\frac{\Phi}{\mathcal{E}^2}\frac{\mathcal{E}_{,r}\mathcal{E}_{,x} - \mathcal{E}\mathcal{E}_{,xr}}{\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E}}\frac{dr}{ds}\frac{dx}{ds}$$
$$+ 2\frac{\Phi}{\mathcal{E}^2}\frac{(\mathcal{E}_{,r}\mathcal{E}_{,y} - \mathcal{E}\mathcal{E}_{,yr})}{\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E}}\frac{dr}{ds}\frac{dy}{ds} - \frac{\Phi}{\mathcal{E}^2}\frac{\varepsilon - k}{\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E}}$$
$$\times\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right] = 0, \tag{A2}$$

$$\frac{d^2 x}{ds^2} + 2\frac{\Phi_{,t}}{\Phi}\frac{dt}{ds}\frac{dx}{ds} - \frac{1}{\Phi}\frac{\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E}}{\varepsilon - k}(\mathcal{E}_{,r}\mathcal{E}_{,x} - \mathcal{E}\mathcal{E}_{,xr})\left(\frac{dr}{ds}\right)^2$$
$$+ \frac{2}{\Phi}\left(\Phi_{,r} - \Phi\frac{\mathcal{E}_{,r}}{\mathcal{E}}\right)\frac{dr}{ds}\frac{dx}{ds} - \frac{\mathcal{E}_{,x}}{\mathcal{E}}\left(\frac{dx}{ds}\right)^2$$
$$- 2\frac{\mathcal{E}_{,y}}{\mathcal{E}}\frac{dx}{ds}\frac{dy}{ds} + \frac{\mathcal{E}_{,x}}{\mathcal{E}}\left(\frac{dy}{ds}\right)^2 = 0, \tag{A3}$$

$$\frac{d^2 y}{ds^2} + 2\frac{\Phi_{,t}}{\Phi}\frac{dt}{ds}\frac{dy}{ds} - \frac{1}{\Phi}\frac{\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E}}{\varepsilon - k}(\mathcal{E}_{,r}\mathcal{E}_{,y} - \mathcal{E}\mathcal{E}_{,yr})\left(\frac{dr}{ds}\right)^2$$
$$+ \frac{2}{\Phi}\left(\Phi_{,r} - \Phi\frac{\mathcal{E}_{,r}}{\mathcal{E}}\right)\frac{dr}{ds}\frac{dy}{ds} + \frac{\mathcal{E}_{,y}}{\mathcal{E}}\left(\frac{dx}{ds}\right)^2$$
$$- 2\frac{\mathcal{E}_{,x}}{\mathcal{E}}\frac{dx}{ds}\frac{dy}{ds} - \frac{\mathcal{E}_{,y}}{\mathcal{E}}\left(\frac{dy}{ds}\right)^2 = 0. \tag{A4}$$

APPENDIX B: SOLUTIONS OF (6.5).

Since (6.5) should hold along certain null geodesics, its derivative by r along those geodesics must be zero. This derivative, denoted by $\mathcal{D}/dr$, of any quantity χ defined along the geodesic, $\chi(t(r), r, x(r), y(r))$, is:

$$\frac{\mathcal{D}\chi}{dr} = \frac{\partial \chi}{\partial t}\frac{dt}{dr} + \frac{\partial \chi}{\partial r} + \frac{\partial \chi}{\partial x}\frac{dx}{dr} + \frac{\partial \chi}{\partial y}\frac{dy}{dr}. \quad (B1)$$

Calculating $\mathcal{D}/dr$ of (6.5) we get:

$$\left(E_{12,r} + E_{12,x}\frac{dx}{dr} + E_{12,y}\frac{dy}{dr}\right)\frac{dy}{dr}$$
$$- \left(E_{13,r} + E_{13,x}\frac{dx}{dr} + E_{13,y}\frac{dy}{dr}\right)\frac{dx}{dr}$$
$$+ E_{12}\frac{d^2 y}{dr^2} - E_{13}\frac{d^2 x}{dr^2} = 0. \quad (B2)$$

The expression in the last line can be calculated from (4.10) and (4.11) using (6.5); it is:

$$E_{12}\frac{d^2 y}{dr^2} - E_{13}\frac{d^2 x}{dr^2} = (\mathcal{E}_{,y}\mathcal{E}_{,rx} - \mathcal{E}_{,x}\mathcal{E}_{,ry})\left[\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2\right]. \quad (B3)$$

Substituting (B3), (4.4), and (4.5) in (B2), and taking into account the identities $\mathcal{E}_{,xy} = \mathcal{E}_{,xx} - \mathcal{E}_{,yy} = 0$ we get:

$$E_{12,r}\frac{dy}{dr} - E_{13,r}\frac{dx}{dr} = 0. \quad (B4)$$

This should hold simultaneously with (6.5). Since we assumed $dx/dr \neq 0 \neq dy/dr$, (6.5) and (B4) imply:

$$E_{12}E_{13,r} - E_{13}E_{12,r} = 0. \quad (B5)$$

When (4.4) and (4.5) are substituted in (B5), $\mathcal{E}$ factors out, and the other factor is:

$$\mathcal{E}_{,r}(\mathcal{E}_{,y}\mathcal{E}_{,rrx} - \mathcal{E}_{,x}\mathcal{E}_{,rry}) + \mathcal{E}_{,rr}(\mathcal{E}_{,x}\mathcal{E}_{,ry} - \mathcal{E}_{,y}\mathcal{E}_{,rx})$$
$$+ \mathcal{E}(\mathcal{E}_{,rx}\mathcal{E}_{,rry} - \mathcal{E}_{,ry}\mathcal{E}_{,rrx}) = 0. \quad (B6)$$

This simplifies to a polynomial of second degree in x and y, which should vanish identically.[12] Using the algebraic program ORTOCARTAN [26,27] we find that the coefficient of $(x^2 + y^2)$ is

$$P_{,rr}Q_{,r} - P_{,r}Q_{,rr} = 0. \quad (B7)$$

One of the solutions of this is $P_{,r} = 0$; then no limitation for Q follows. This case we consider separately below.

When $P_{,r} \neq 0$, (B7) implies

$$Q = C_0 P + D_0, \quad (B8)$$

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[12]Should there exist any point $\mathcal{P}_n$ in the spacetime at which the polynomial would be nonzero, this would mean that the determinant of the set {(6.5), (B4)} is nonzero at $\mathcal{P}_n$, which in turn would mean $dx/dr = dy/dr = 0$ at $\mathcal{P}_n$, contrary to our initial assumption.

where C_0 and D_0 are arbitrary constants. When this is substituted in (B6), the coefficient of y implies:

$$\varepsilon(SS_{,r}P_{,rr} - S_{,r}{}^2 P_{,r} - SS_{,rr}P_{,r}) - (1 + C_0{}^2)P_{,r}{}^3 = 0, \quad (B9)$$

and this guarantees that the whole of (B6) is fulfilled.

The case $\varepsilon = 0$ is seen to be incompatible with $P_{,r} \neq 0$. This means that no RLPs exist in the $\varepsilon = 0$ models with $P_{,r} \neq 0$. Further calculations apply only to $\varepsilon = \pm 1$.

In integrating (B9) we can assume $S_{,r} \neq 0$ because $S_{,r} = 0$ immediately implies $P_{,r} = 0$, which we have left for a separate investigation. Therefore we can introduce $S(r)$ as the new independent variable in (B9), which then becomes:

$$\varepsilon(SP_{,SS} - P_{,S}) - (1 + C_0{}^2)P_{,S}{}^3 = 0. \quad (B10)$$

Since the case $P =$ constant was left for later, we assume $P_{,S} \neq 0$, and then (B10) is easily integrated with the result:

$$\varepsilon S^2 + (1 + C_0{}^2)P^2 = C_3 P + D_3, \quad (B11)$$

where C_3 and D_3 are new arbitrary constants.

When $\varepsilon = +1$, Eqs. (B8) and (B11) are equivalent to those that were shown in Ref. [12] (Sec. 3.3.1) to be sufficient conditions for the Szekeres metric to be axially symmetric. However, this equivalence is nontrivial, and the extension of the proof to $\varepsilon = 0, -1$ is not automatic, so we have to elaborate on this subject.

For this purpose, we note the following properties of the general Szekeres metrics (2.7):

(1) The metric (2.7) does not change in form under the coordinate transformation:

$$(x, y) = (x' + x_0, y' + y_0), \quad (B12)$$

where (x_0, y_0) are arbitrary constants. This changes (P, Q) to

$$(\tilde{P}, \tilde{Q}) = (P - x_0, Q - y_0). \quad (B13)$$

(2) The metric (2.7) does not change in form when (x, y) are transformed by a general orthogonal transformation:

$$x = \frac{ax' + by'}{\sqrt{a^2 + b^2}}, \qquad y = \frac{-bx' + ay'}{\sqrt{a^2 + b^2}}, \quad (B14)$$

which implies the change of (P, Q) to:

$$\tilde{P} = \frac{aP - bQ}{\sqrt{a^2 + b^2}}, \qquad \tilde{Q} = \frac{bP + aQ}{\sqrt{a^2 + b^2}}. \quad (B15)$$

(3) The metric (2.7) does not change in form under the discrete transformations:

$$(x, y) = (y', x'), \qquad (x, y) = (-x', y'),$$
$$(x, y) = (x', -y'), \tag{B16}$$

which induce, respectively

$$(\tilde{P}, \tilde{Q}) = (Q, P), \qquad (\tilde{P}, \tilde{Q}) = (-P, Q),$$
$$(\tilde{P}, \tilde{Q}) = (P, -Q). \tag{B17}$$

(4) The metric (2.7) does not change in form when (x, y) are transformed by a conformal symmetry of a Euclidean 2-plane—a 2-dimensional Haantjes transformation by the terminology of Ref. [11]. It has the form:

$$x = \frac{x' + \lambda_1(x'^2 + y'^2)}{T},$$
$$y = \frac{y' + \lambda_2(x'^2 + y'^2)}{T}, \tag{B18}$$
$$T \stackrel{\text{def}}{=} 1 + 2\lambda_1 x' + 2\lambda_2 y'$$
$$+ (\lambda_1{}^2 + \lambda_2{}^2)(x'^2 + y'^2),$$

where λ_1 and λ_2 are arbitrary constants—the group parameters. This group is Abelian, the inverse transformation to (B18) being of the same form, but with parameters $(-\lambda_1, -\lambda_2)$. The characteristic properties of (B18), useful in calculations, are:

$$x^2 + y^2 = \frac{x'^2 + y'^2}{T},$$
$$dx^2 + dy^2 = \frac{dx'^2 + dy'^2}{T^2}. \tag{B19}$$

Under (B18) and (B19), (P, Q, S) change, respectively, to:

$$\tilde{P} = \frac{1}{U}[P - \lambda_1(P^2 + Q^2 + \varepsilon S^2)],$$
$$\tilde{Q} = \frac{1}{U}[Q - \lambda_2(P^2 + Q^2 + \varepsilon S^2)], \qquad \tilde{S} = S/U,$$
$$U \stackrel{\text{def}}{=} 1 - 2\lambda_1 P - 2\lambda_2 Q$$
$$+ (\lambda_1{}^2 + \lambda_2{}^2)(P^2 + Q^2 + \varepsilon S^2). \tag{B20}$$

Let $\tilde{\mathcal{E}}$ denote $\mathcal{E}$ with (x, y, P, Q, S) replaced by $(x', y', \tilde{P}, \tilde{Q}, \tilde{S})$. Then calculation shows that

$$\mathcal{E} = \tilde{\mathcal{E}}/T, \tag{B21}$$

and since T does not depend on r, it follows that the g_{rr} component in (2.7) is also covariant with (B18) and (B19).

Now we will use the properties listed above to interpret the consequences of (B8) and (B11) for the metric (2.7).

The D_0 in (B8) can be set to zero by (B12) with $(x_0, y_0) = (0, D_0)$. The C_0 in (B8) can be set to zero by (B15) with

$b = -aC_0$; the result of these two transformations is $Q = 0$. Finally, the C_3 in (B11) (with $C_0 = 0$ taken into account) can be set to zero by (B12) with $(x_0, y_0) = (-C_3/2, 0)$. Thus we can assume $D_0 = C_0 = C_3 = 0$ with no loss of generality.

We carry out a combination of (B12) with (B18):

$$x = x_0 + \frac{x' + \lambda_1(x'^2 + y'^2)}{T}, \qquad y = y', \tag{B22}$$

and get the following generalization of (B20) with $\lambda_2 = 0$:

$$\tilde{P} = \frac{1}{\mathcal{U}}\{P - x_0 - \lambda_1[(P - x_0)^2 + Q^2 + \varepsilon S^2]\},$$
$$(\tilde{Q}, \tilde{S}) = (Q, S)/\mathcal{U},$$
$$\mathcal{U} \stackrel{\text{def}}{=} 1 - 2\lambda_1(P - x_0) + \lambda_1{}^2[(P - x_0)^2 + Q^2 + \varepsilon S^2]. \tag{B23}$$

Using (B8) and (B11) with $D_0 = C_0 = C_3 = 0$, the above becomes

$$\tilde{P} = \frac{1}{\mathcal{U}}[P - x_0 - \lambda_1(-2x_0 P + D_3 + x_0{}^2)], \quad \tilde{Q} = 0,$$
$$\mathcal{U} \stackrel{\text{def}}{=} 1 - 2\lambda_1(P - x_0) + \lambda_1{}^2(-2x_0 P + D_3 + x_0{}^2). \tag{B24}$$

Now it can be seen that if the constants (x_0, λ_1), so far arbitrary, obey:

$$1 + 2\lambda_1 x_0 = 0, \qquad x_0 + \lambda_1(D_3 + x_0{}^2) = 0, \tag{B25}$$

then $\tilde{P} = \tilde{Q} = 0$, and in the (x', y') coordinates the Szekeres metric is explicitly axially symmetric. However, two things must be noted:

(1) The set (B25) has no solutions when $D_3 \leq 0$.
(2) With $P = Q = 0$, Eq. (B9) is fulfilled identically, and (B11) no longer follows, thus there is no limitation on S.

Looking at (B11) with $C_0 = C_3 = 0$ we see that $D_3 < 0$ cannot occur when $\varepsilon = +1$ or $\varepsilon = 0$. The case $D_3 = 0$, although possible with $\varepsilon = +1$ or $\varepsilon = 0$, need not be considered with these two values of ε, for the following reasons: With $\varepsilon = +1$ this would imply $S = 0$, which is an impossibility in (2.7), and with $\varepsilon = 0 = C_0 = C_3$, (B11) implies $P = 0$. With $Q = 0$ now being considered, $P = \varepsilon = 0$ guarantees that S may be set to 1 by a suitable reparametrization of the other metric functions [20]. Consequently, with $P = Q = 0$, the $\varepsilon = 0$ Szekeres metric is already plane symmetric even with nonconstant S, and the Szekeres metrics with 3-dimensional symmetry groups are considered in Sec. VII.

So, finally, $D_3 \leq 0$ must be considered only for $\varepsilon = -1$. Since these calculations are lengthy and very complicated, we have moved them to the separate Appendix C.

We now come back to (B7) to consider the case $P_{,r} = 0$. By a transformation of x this can be reduced to $P = 0$. Then, the whole of (B6) becomes:

$$x[\varepsilon(S,_r{}^2 Q,_r - SS,_r Q,_{rr} + SS,_{rr} Q,_r) + Q,_r{}^3] = 0. \quad \text{(B26)}$$

This is equivalent to the subcase $C_0 = 0$ of (B9) under the coordinate transformation $(x, y) = (\tilde{y}, \tilde{x})$ and the associated renaming $(P, Q) = (\tilde{Q}, \tilde{P})$. This case was included in the consideration above.

Thus, apart from the special cases $D_3 \leq 0$ to be considered further on, RLPs with $dx/dr \neq 0 \neq dy/dr$ may possibly exist only when the Szekeres metric is reducible, by a coordinate transformation, to one with $P = Q = 0$. In this case, (6.5) becomes:

$$\varepsilon \frac{S,_r}{S}\left(x\frac{dy}{dr} - y\frac{dx}{dr}\right) = 0. \quad \text{(B27)}$$

But with $\varepsilon = 0$ and $P = Q = 0$ now being considered, the quasiplane Szekeres metric is plane symmetric even with nonconstant S; see the paragraph following (B25). The Szekeres metrics with 3-dimensional symmetry groups are considered in Sec. VII, so we need not consider $\varepsilon = 0$ here.

When $S,_r = 0 = P,_r = Q,_r$, all Szekeres metrics acquire a 3-dimensional symmetry group and are considered in Sec. VII. Thus, we need not consider $S,_r = 0$ in (B27).

What remains of (B27) is $x dy/dr - y dx/dr = 0$. One solution of this is $x = 0$ along the null geodesic. The other solution is $y = G_0 x$ along the geodesic, where G_0 is a constant. However, we are now considering the axially symmetric Szekeres solutions in which $P = Q = 0$. In this case, a rotation (B14) can be chosen so that $y = G_0 x$ is transformed to $x' = 0$. So, apart from the special case $\varepsilon = -1$, $D_3 \leq 0$, for the other Szekeres solutions the following result applies:

Corollary 2:

The Szekeres spacetimes in which RLPs exist with $dx/dr \neq 0 \neq dy/dr$ either are the Friedmann models (in which all null geodesics are RLPs) or are inhomogeneous and axially symmetric or have a 3-dimensional symmetry group. In the first two cases, coordinates may be chosen so that $x = 0$ along the hypothetic RLP and $P = Q = 0$ in the metric. The second case is considered in Appendix F. The third case is considered in Sec. VII and in Appendix G.

APPENDIX C: THE SPECIAL METRIC WITH $Q = 0$, $\varepsilon = -1$ AND $D_3 \leq 0$.

We consider here the special case $D_3 \leq 0$ that arose in solving (B25). The Szekeres model in question has

$$\mathcal{E} = \frac{x^2 - 2Px + y^2 + D_3}{2S},$$
$$E_{12} = \frac{P,_r}{2S^2}(y^2 - x^2 + D_3), \quad \text{(C1)}$$
$$E_{13} = -\frac{P,_r}{S^2}xy.$$

The solution of (6.5) is then either $P,_r = 0$, which belongs to the axially symmetric case considered in Appendix E, or

$$x^2 + y^2 - Cy - D_3 = 0, \quad \text{(C2)}$$

where C is the arbitrary constant that arises while integrating (6.5). By writing the above as $x^2 + (y - C/2)^2 = D_3 + C^2/4$ we note that the following must hold:

$$D_3 + C^2/4 > 0. \quad \text{(C3)}$$

(With this quantity being negative, (C2) has no solutions, i.e. there are no RLPs. When it is zero, the only solution of (C2) is $(x, y) = (0, C/2)$, what is possible only in the axially symmetric case of Appendix E.)

Note that (C3) implies $C \neq 0$, since $D_3 \leq 0$.

Taking the second derivative of (C2) by r and substituting in it the expressions for $x,_{rr}$ and $y,_{rr}$ from (4.10) and (4.11), we obtain an identity. This means that (C2) is consistent with the geodesic equations (4.9)–(4.11) and defines a special class of null geodesics. We will verify in the following that this class does not contain any RLPs.

We note the following auxiliary formulae. In Eqs. (C4)–(C11) asterisks mark those equations that hold only along the null geodesics obeying (C2), those without the asterisk are general. After (C11) all further calculations are done only along these geodesics, so we omit the asterisks for better readability.

$$(*) \quad \mathcal{E} = \frac{-2Px + Cy + 2D_3}{2S}, \quad \text{(C4)}$$

$$\mathcal{E},_x = \frac{x - P}{S}, \qquad \mathcal{E},_y = \frac{y}{S}, \qquad \mathcal{E},_r = -\frac{xP,_r}{S} - \frac{S,_r}{S}\mathcal{E}, \quad \text{(C5)}$$

$$\mathcal{E},_{rx} = -\frac{P,_r}{S} - \frac{S,_r}{S^2}(x - P), \qquad \mathcal{E},_{ry} = -\frac{yS,_r}{S^2}, \quad \text{(C6)}$$

$$(*) \quad E_{12} = \frac{P,_r}{2S^2}(2y^2 - Cy), \quad \text{(C7)}$$

$$(*) \quad y,_r = -\frac{2x}{2y - C}x,_r, \quad \text{(C8)}$$

$$(*) \quad x,_r{}^2 + y,_r{}^2 = \frac{4D_3 + C^2}{(2y - C)^2}x,_r{}^2. \quad \text{(C9)}$$

$$(*) \quad 2x,_r{}^2 E_{12} + 2x,_r y,_r E_{13} = \frac{P,_r}{S^2}y(2y - C)(x,_r{}^2 + y,_r{}^2). \quad \text{(C10)}$$

From (5.15) we have

$$\frac{x,_r{}^2 + y,_r{}^2}{\mathcal{E}^2} = \frac{t,_r{}^2}{\Phi^2} - \frac{\Phi_1{}^2}{(\varepsilon - k)\Phi^2}. \quad \text{(C11)}$$

In order to write the equations in a compact way it is convenient to use the symbol Ψ introduced in (6.4). Recall:

this is a factor of shear, and when it vanishes, the Szekeres model reduces to Friedmann. Thus, in searching for RLPs in nontrivial models we will assume $\Psi \neq 0$. Using (C11) in (5.16) adapted to RLPs we obtain

$$\tau_{,r} t_{,r} = \frac{\Phi_1 \Phi_{01} \tau}{\varepsilon - k} + \frac{\Phi_{,t} t_{,r}{}^2 \tau}{\Phi} - \frac{\Phi_{,t} \Phi_1{}^2 \tau}{(\varepsilon - k)\Phi}$$

$$\equiv \frac{\Phi_1 \Psi \tau}{\varepsilon - k} + \frac{\Phi_{,t} t_{,r}{}^2 \tau}{\Phi}. \qquad (C12)$$

We also have:

$$\Phi_{01} - \Phi_{,t} \Phi_1/\Phi \equiv \Phi_{,tr} - \Phi_{,t} \Phi_{,r}/\Phi = \Psi, \qquad (C13)$$

$$\Phi_{,ttr} - \Phi_{,tt} \Phi_{,r}/\Phi = \Psi_{,t} + \frac{\Phi_{,t}}{\Phi} \Psi, \qquad (C14)$$

$$\Phi_{,trr} - \Phi_{,t} \Phi_{,rr}/\Phi = \Psi_{,r} + \frac{\Phi_{,r}}{\Phi} \Psi. \qquad (C15)$$

Assuming $\Psi \neq 0$ we now multiply (6.2) by $\Phi \Phi_1{}^2 t_{,r} /[(\varepsilon - k)\Psi]$, use (C7)–(C15), cancel τ that factors out, and write the result in the form:

$$x_{,r} (t_{,r}{}^3 + c_2 t_{,r}{}^2 + c_3 t_{,r} + c_4) = B_1 t_{,r}{}^3 + B_3 t_{,r}, \qquad (C16)$$

where

$$c_2 \stackrel{\text{def}}{=} \frac{2\Phi(\Phi_1 \Psi_{,t} - \Psi^2)}{(\varepsilon - k)\Psi}, \qquad (C17)$$

$$c_3 \stackrel{\text{def}}{=} \frac{-3\Phi_1{}^2 + \Phi \Phi_1 \Psi_{,r}/\Psi - \Phi \Phi_{,rr}}{\varepsilon - k}$$
$$+ \frac{\Phi^2 \mathcal{E}_{,rr}/\mathcal{E} + \Phi_1 \Phi_{,r}}{\varepsilon - k}, \qquad (C18)$$

$$c_4 \stackrel{\text{def}}{=} \frac{2\Phi \Phi_1{}^2 \Psi}{(\varepsilon - k)^2}, \qquad (C19)$$

$$B_1 \stackrel{\text{def}}{=} \frac{P_{,r} y(2y - C)}{(\varepsilon - k)S^2}, \qquad (C20)$$

$$B_3 \stackrel{\text{def}}{=} -\frac{3\Phi_1{}^2}{2(\varepsilon - k)} B_1. \qquad (C21)$$

Then, using (C1), (C7), and (C11) we can rewrite (4.9) in the form:

$$t_{,rr} = c_5 t_{,r}{}^3 + c_6 t_{,r}{}^2 + c_7 t_{,r} + c_8 + A x_{,r} t_{,r}, \qquad (C22)$$

where

$$c_5 \stackrel{\text{def}}{=} -\frac{\varepsilon - k}{\Phi \Phi_1}, \qquad (C23)$$

$$c_6 \stackrel{\text{def}}{=} 2\frac{\Psi}{\Phi_1} + \frac{\Phi_{,t}}{\Phi}, \qquad (C24)$$

$$c_7 \stackrel{\text{def}}{=} \frac{\Phi_{,rr} - \Phi \mathcal{E}_{,rr}/\mathcal{E}}{\Phi_1} - \frac{\mathcal{E}_{,r}}{\mathcal{E}} + \frac{k_{,r}}{2(\varepsilon - k)} + \frac{\Phi_1}{\Phi}, \qquad (C25)$$

$$c_8 \stackrel{\text{def}}{=} -\frac{\Psi \Phi_1}{\varepsilon - k}, \qquad (C26)$$

$$A \stackrel{\text{def}}{=} \frac{(4D_3 + C^2)P_{,r} y\Phi}{(2y - C)S^2 \mathcal{E}^2 \Phi_1}. \qquad (C27)$$

Combining (C9) and (C11) we get:

$$x_{,r}{}^2 = \frac{(2y - C)^2 \mathcal{E}^2}{(4D_3 + C^2)\Phi^2} \left(t_{,r}{}^2 - \frac{\Phi_1{}^2}{\varepsilon - k}\right). \qquad (C28)$$

Equations (C16) and (C28) determine dt/dr along the hypothetic RLP. Formally, a solution for dt/dr of these equations always exists, but it must be consistent with the geodesic equations, and this is what we will investigate next. Namely, every solution of these equations must be preserved along the null geodesics. To see whether it is, we first transform this set into a single polynomial equation for dt/dr.

We square (C16) and use (C28) in the result. We thus obtain an 8th degree polynomial in $t_{,r}$, whose coefficient at $t_{,r}{}^8$ is

$$\alpha_1 = \frac{(2y - C)^2 \mathcal{E}^2}{(4D_3 + C^2)\Phi^2}. \qquad (C29)$$

It is seen that it cannot vanish except when $y = C/2$, but this defines a "radial" geodesic that exists only in the axially symmetric case [12]. Thus we divide the 8th degree polynomial by α_1 and obtain the following equation

$$t_{,r}{}^8 + 2c_2 t_{,r}{}^7 + a_3 t_{,r}{}^6 + a_4 t_{,r}{}^5 + a_5 t_{,r}{}^4 + a_6 t_{,r}{}^3$$
$$+ a_7 t_{,r}{}^2 + a_8 t_{,r} + a_9 = 0, \qquad (C30)$$

where:

$$a_3 \stackrel{\text{def}}{=} 2c_3 + c_2{}^2 - \frac{\Phi_1{}^2}{\varepsilon - k} - \frac{(4D_3 + C^2)\Phi^2 P_{,r}{}^2 y^2}{(\varepsilon - k)^2 S^4 \mathcal{E}^2}, \qquad (C31)$$

$$a_4 \stackrel{\text{def}}{=} 2c_4 + 2c_2 c_3 - 2c_2 \frac{\Phi_1{}^2}{\varepsilon - k}, \qquad (C32)$$

$$a_5 \stackrel{\text{def}}{=} 2c_2 c_4 + c_3{}^2 - (2c_3 + c_2{}^2)\frac{\Phi_1^2}{\varepsilon - k}$$
$$+ 3\frac{\Phi_1{}^2}{(\varepsilon - k)^3} \frac{(4D_3 + C^2)\Phi^2 P_{,r}{}^2 y^2}{S^4 \mathcal{E}^2}, \qquad (C33)$$

$$a_6 \stackrel{\text{def}}{=} 2c_3 c_4 - (2c_4 + 2c_2 c_3)\frac{\Phi_1{}^2}{\varepsilon - k}, \qquad (C34)$$

$$a_7 \stackrel{\text{def}}{=} c_4{}^2 - (2c_2 c_4 + c_3{}^2)\frac{\Phi_1{}^2}{\varepsilon - k}$$
$$- \frac{9\Phi_1{}^4}{4(\varepsilon - k)^4} \frac{(4D_3 + C^2)\Phi^2 P_{,r}{}^2 y^2}{S^4 \mathcal{E}^2}, \qquad (C35)$$

$$a_8 \stackrel{\text{def}}{=} -2c_3c_4\frac{\Phi_1{}^2}{\varepsilon - k}, \tag{C36}$$

$$a_9 \stackrel{\text{def}}{=} -c_4{}^2\frac{\Phi_1{}^2}{\varepsilon - k}. \tag{C37}$$

Now we differentiate (C30) along the null geodesic by the rule (B1), and use (C30) to eliminate $t,_r{}^{10}$, $t,_r{}^9$ and $t,_r{}^8$ from the result. In this way we obtain:

$$b_1t,_r{}^7 + b_2t,_r{}^6 + b_3t,_r{}^5 + b_4t,_r{}^4 + b_5t,_r{}^3 + b_6t,_r{}^2$$
$$+ b_7t,_r + b_8 + x,_r(\beta_1t,_r{}^7 + \beta_2t,_r{}^6 + \beta_3t,_r{}^5$$
$$+ \beta_4t,_r{}^4 + \beta_5t,_r{}^3 + \beta_6t,_r{}^2 + \beta_7t,_r + \beta_8) = 0, \tag{C38}$$

where

$$b_1 \stackrel{\text{def}}{=} 8c_8 + 2c_{2,r} + a_{3,t} - 2a_3c_6 - 3a_4c_5 - 2c_2c_7$$
$$- 4c_2c_{2,t} + 6a_3c_2c_5 + 4c_2{}^2c_6 - 8c_2{}^3c_5, \tag{C39}$$

$$b_2 \stackrel{\text{def}}{=} a_{3,r} + a_{4,t} + 14c_2c_8 - 2a_3c_7 - 3a_4c_6 - 4a_5c_5$$
$$- 2a_3c_{2,t} + 2a_4c_2c_5 + 2a_3c_2c_6 + 2a_3{}^2c_5 - 4a_3c_2{}^2c_5, \tag{C40}$$

$$b_3 \stackrel{\text{def}}{=} a_{4,r} + a_{5,t} + 6a_3c_8 - 3a_4c_7 - 4a_5c_6 - 5a_6c_5$$
$$- 2a_4c_{2,t} + 2a_5c_2c_5 + 2a_4c_2c_6 + 2a_3a_4c_5$$
$$- 4a_4c_2{}^2c_5, \tag{C41}$$

$$b_4 \stackrel{\text{def}}{=} a_{5,r} + a_{6,t} + 5a_4c_8 - 4a_5c_7 - 5a_6c_6 - 6a_7c_5$$
$$- 2a_5c_{2,t} + 2a_6c_2c_5 + 2a_5c_2c_6 + 2a_3a_5c_5$$
$$- 4a_5c_2{}^2c_5, \tag{C42}$$

$$b_5 \stackrel{\text{def}}{=} a_{6,r} + a_{7,t} + 4a_5c_8 - 5a_6c_7 - 6a_7c_6 - 7a_8c_5$$
$$- 2a_6c_{2,t} + 2a_7c_2c_5 + 2a_6c_2c_6 + 2a_3a_6c_5$$
$$- 4a_6c_2{}^2c_5, \tag{C43}$$

$$b_6 \stackrel{\text{def}}{=} a_{7,r} + a_{8,t} + 3a_6c_8 - 6a_7c_7 - 7a_8c_6 - 8a_9c_5$$
$$- 2a_7c_{2,t} + 2a_8c_2c_5 + 2a_7c_2c_6 + 2a_3a_7c_5$$
$$- 4a_7c_2{}^2c_5, \tag{C44}$$

$$b_7 \stackrel{\text{def}}{=} a_{8,r} + a_{9,t} + 2a_7c_8 - 7a_8c_7 - 8a_9c_6 - 2a_8c_{2,t}$$
$$+ 2a_9c_2c_5 + 2a_8c_2c_6 + 2a_3a_8c_5 - 4a_8c_2{}^2c_5, \tag{C45}$$

$$b_8 \stackrel{\text{def}}{=} a_{9,r} + a_8c_8 - 8a_9c_7 - 2a_9c_{2,t} + 2a_9c_2c_6$$
$$+ 2a_3a_9c_5 - 4a_9c_2{}^2c_5, \tag{C46}$$

$$\beta_1 \stackrel{\text{def}}{=} 2\left(c_{2,x} - \frac{2x}{2y - C}c_{2,y}\right) - 2c_2A, \tag{C47}$$

$$\beta_2 \stackrel{\text{def}}{=} a_{3,x} - \frac{2x}{2y - C}a_{3,y} - 2a_3A, \tag{C48}$$

$$\beta_3 \stackrel{\text{def}}{=} a_{4,x} - \frac{2x}{2y - C}a_{4,y} - 3a_4A, \tag{C49}$$

$$\beta_4 \stackrel{\text{def}}{=} a_{5,x} - \frac{2x}{2y - C}a_{5,y} - 4a_5A, \tag{C50}$$

$$\beta_5 \stackrel{\text{def}}{=} a_{6,x} - \frac{2x}{2y - C}a_{6,y} - 5a_6A, \tag{C51}$$

$$\beta_6 \stackrel{\text{def}}{=} a_{7,x} - \frac{2x}{2y - C}a_{7,y} - 6a_7A, \tag{C52}$$

$$\beta_7 \stackrel{\text{def}}{=} a_{8,x} - \frac{2x}{2y - C}a_{8,y} - 7a_8A, \tag{C53}$$

$$\beta_8 \stackrel{\text{def}}{=} a_{9,x} - \frac{2x}{2y - C}a_{9,y} - 8a_9A. \tag{C54}$$

We provisionally assume that the coefficient of $x,_r$ in (C16) is nonzero. (We will later come back to this point and investigate what happens when it is zero.) Then we determine $x,_r$ from (C16) and substitute the result in (C38). After multiplying out to get a polynomial in $t,_r$, we again use (C30) to eliminate $t,_r{}^{10}$ and $t,_r{}^9$ (but not $t,_r{}^8$). Then we assume that the coefficient of $t,_r{}^8$, denoted d_1, is nonzero (we will check the case $d_1 = 0$ later), and divide the equation by d_1. In this way we obtain:

$$t,_r{}^8 + \delta_2t,_r{}^7 + \delta_3t,_r{}^6 + \delta_4t,_r{}^5 + \delta_5t,_r{}^4 + \delta_6t,_r{}^3$$
$$+ \delta_7t,_r{}^2 + \delta_8t,_r + \delta_9 = 0, \tag{C55}$$

where $\delta_i \stackrel{\text{def}}{=} d_i/d_1$, $i = 2, \ldots, 9$, and

$$d_1 \stackrel{\text{def}}{=} b_3 - b_2c_2 + b_1c_3 + \beta_1B_3 + \beta_3B_1 - a_3b_1 - a_3\beta_1B_1$$
$$- 2\beta_2B_1c_2 + 2b_1c_2{}^2 + 4\beta_1B_1c_2{}^2, \tag{C56}$$

$$d_2 \stackrel{\text{def}}{=} b_4 + b_1c_4 + b_3c_2 + b_2c_3 + \beta_2B_3 + \beta_4B_1 - a_4b_1$$
$$- a_3b_2 - a_4\beta_1B_1 - a_3\beta_2B_1 + a_3b_1c_2 + 2a_3\beta_1B_1c_2, \tag{C57}$$

$$d_3 \stackrel{\text{def}}{=} b_5 + b_2c_4 + b_4c_2 + b_3c_3 + \beta_3B_3 + \beta_5B_1 - a_5b_1$$
$$- a_4b_2 - a_5\beta_1B_1 - a_4\beta_2B_1 + a_4b_1c_2$$
$$+ 2a_4\beta_1B_1c_2, \tag{C58}$$

$$d_4 \overset{\text{def}}{=} b_6 + b_3 c_4 + b_5 c_2 + b_4 c_3 + \beta_4 B_3 + \beta_6 B_1 - a_6 b_1$$
$$- a_5 b_2 - a_6 \beta_1 B_1 - a_5 \beta_2 B_1 + a_5 b_1 c_2$$
$$+ 2 a_5 \beta_1 B_1 c_2, \tag{C59}$$

$$d_5 \overset{\text{def}}{=} b_7 + b_4 c_4 + b_6 c_2 + b_5 c_3 + \beta_5 B_3 + \beta_7 B_1 - a_7 b_1$$
$$- a_6 b_2 - a_7 \beta_1 B_1 - a_6 \beta_2 B_1 + a_6 b_1 c_2$$
$$+ 2 a_6 \beta_1 B_1 c_2, \tag{C60}$$

$$d_6 \overset{\text{def}}{=} b_8 + b_5 c_4 + b_7 c_2 + b_6 c_3 + \beta_6 B_3 + \beta_8 B_1$$
$$- a_8 b_1 - a_7 b_2 - a_8 \beta_1 B_1 - a_7 \beta_2 B_1 + a_7 b_1 c_2$$
$$+ 2 a_7 \beta_1 B_1 c_2, \tag{C61}$$

$$d_7 \overset{\text{def}}{=} b_6 c_4 + b_8 c_2 + b_7 c_3 + \beta_7 B_3 - a_9 b_1 - a_8 b_2$$
$$- a_9 \beta_1 B_1 - a_8 \beta_2 B_1 + a_8 b_1 c_2 + 2 a_8 \beta_1 B_1 c_2, \tag{C62}$$

$$d_8 \overset{\text{def}}{=} b_7 c_4 + b_8 c_3 + \beta_8 B_3 - a_9 b_2 - a_9 \beta_2 B_1$$
$$+ a_9 b_1 c_2 + 2 a_9 \beta_1 B_1 c_2, \tag{C63}$$

$$d_9 \overset{\text{def}}{=} b_8 c_4. \tag{C64}$$

Every solution of (C30) is a candidate RLP, and every RLP must obey (C30). Equation (C55) is the condition that (C30) is preserved along null geodesics. Thus, every solution of (C30) must also be a solution of (C55). Since (C30) and (C55) are of the same degree in t, r, it follows that both must have the same set of zeros. Consequently, their coefficients must be the same. After we make sure that they are the same, we may next investigate which zeros define RLPs. Thus, the following equations are necessary conditions for the existence of RLPs:

$$2 c_2 = \delta_2, \qquad a_i = \delta_i, \qquad i = 3, \dots 9,$$
$$\Longleftrightarrow 2 c_2 d_1 - d_2 = 0, \qquad d_1 a_i - d_i = 0. \tag{C65}$$

By far the simplest condition, as seen from (C64), is the one with $i = 9$. Even so, further calculations are so complicated and involve intermediate equations so large that they could be done only using the computer algebra system ORTOCARTAN [26,27], and we only describe how they were done.

First we observe that the functions $(\Phi_1, y, \mathcal{E})$ are linearly independent, and in the resulting final equation can be used as independent variables. Although the proof is a simple exercise, it requires careful inspection of special cases that we had earlier excluded for separate investigation, so we give it in the separate Appendix D.

The condition (C65) corresponding to $i = 9$ is

$$d_1 a_9 - d_9 = 0. \tag{C66}$$

In this, one has to do the whole cascade of substitutions, listed in (C17)–(C64). In the result, we use (C5) and (C6). However, we use the last of (C5) only to eliminate $\mathcal{E}_{,rr}$. For $\mathcal{E}_{,r}$ we substitute from (4.1), i.e.

$$\mathcal{E}_{,r} = \mathcal{E}(\Phi_{,r} - \Phi_1)/\Phi, \tag{C67}$$

in order to express $\mathcal{E}_{,r}$ through Φ_1.

We then use (C15) to express $\Phi_{,trr}$ through Ψ, and (6.4) to express $\Phi_{,tr}$ through Ψ. In the result we use (C2) to eliminate x^2 and (C4) to express x through $\mathcal{E}$. From the resulting equation we can factor out $\Phi_1{}^6$, and we must multiply it by $\mathcal{E}^4$ to get rid of negative powers of $\mathcal{E}$. The final equation thus obtained has on its left-hand side a polynomial of 4th degree in $\mathcal{E}$, of 4th degree in y and of 6th degree in Φ_1 (recall, we determined that $(\Phi_1, y, \mathcal{E})$ are independent variables).[13] So, if this polynomial is to be zero, the coefficients of all powers of the independent variables must vanish separately.

Now we take this large polynomial as data for a second program, in which $(\Phi_1, y, \mathcal{E})$ are treated as independent variables, no longer as functions. In it, we determine the coefficient of y^4 and substitute $\mathcal{E} = 0$ to find the term independent of $\mathcal{E}$. The resulting equation is:

$$16(4 D_3 + C^2)^2 \Phi^6 \Psi \Psi_{,t} P_{,r}{}^4 / [S^8 (\varepsilon - k)^9] = 0. \tag{C68}$$

In this equation we can discard several alternatives: $\Phi = 0$ obviously, $4 D_3 + C^2 = 0$ because of (C3), $\Psi = 0$ because it defines the Friedmann limit and $P_{,r} = 0$ because, in view of (B11) and the paragraph above (B22), it leads to the G_3/S_2 symmetric cases considered in Appendix G. The only case to consider is thus $\Psi_{,t} = 0$.

In order to investigate it, we substitute $\Psi_{,t} = 0$ in the large main polynomial, and in the resulting somewhat smaller polynomial we take the coefficient of y^4. The equation that results is:

$$-16(4 D_3 + C^2)(4 P^2 + C^2) \mathcal{E} \Phi^5 \Psi^3 P_{,r}{}^3 / [P S^7 (\varepsilon - k)^9] = 0. \tag{C69}$$

The factors $\mathcal{E}$ and Φ obviously cannot vanish, and the reason zero values of $(4 D_3 + C^2)$, Ψ and $P_{,r}$ are discarded was explained above. Thus, the only case left is $4 P^2 + C^2 = 0$, which means $P = 0 = C$. But this is just a special case of $P_{,r} = 0$ discarded above. Thus, (C68) does not include any case that would define any new RLP, apart from those considered elsewhere.

We go back to (C55) to consider the case $d_1 = 0$. The calculation is almost the same as we did for (C66), with only minor differences: this time the expression is somewhat simpler, and $\Phi_1{}^6$ does not factor out. We employ the algebraic program to calculate $\mathcal{E}^4 d_1$, with the same cascade of substitutions as before, take the coefficient of y^4 at $\mathcal{E} = 0$, and obtain an equation almost identical to (C68):

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[13]The whole equation would take 1830 print lines on paper.

$$4(4D_3 + C^2)^2\Phi^4\Psi_{,t}P_{,r}{}^4/[S^8(\varepsilon - k)^4\Psi] = 0. \quad \text{(C70)}$$

As explained above, only $\Psi_{,t}$ could possibly be zero, so we substitute $\Psi_{,t} = 0$ in the main large polynomial, and in the resulting expression take the coefficient of y^4. The result is almost the same as (C69)

$$-4(4D_3 + C^2)(4P^2 + C^2)\mathcal{E}\Phi^3\Psi P_{,r}{}^3/[PS^7(\varepsilon - k)^4] = 0, \quad \text{(C71)}$$

and again does not include any case that would define a new RLP.

Finally, we go back to (C54), where we assumed that the coefficient of $x_{,r}$ in (C16) was nonzero, and investigate what happens when it is zero. Then

$$B_1 t_{,r}{}^3 + B_3 t_{,r} = 0, \quad \text{(C72)}$$

and one of the solutions of this is $t_{,r} = 0$. This we immediately discard because it defines a spacelike curve, while our RLPs must be null geodesics. In consequence of (C21), another solution of (C72) is $B_1 = 0$. But this implies $P_{,r} = 0$ or $y = 0$ or $y = C/2$. The first case leads to the G_3/S_2 solutions considered in Appendix G, the other two to the axially symmetric solutions of Appendix F. So the only possibility left to fulfil (C72) is

$$t_{,r} = \pm\sqrt{\frac{3}{2(\varepsilon - k)}}\Phi_1. \quad \text{(C73)}$$

Putting this into the coefficient of $x_{,r}$ in (C16) we get:

$$\left[\frac{3}{2(\varepsilon - k)}\right]^{3/2}\Phi_1\left\{\left[\mp 1 + \frac{2^{3/2}\Phi\Psi_{,t}}{\sqrt{3(\varepsilon - k)}\Psi}\right]\Phi_1^2\right.$$
$$+ \left[-\frac{2^{3/2}\Phi\Psi}{3\sqrt{3(\varepsilon - k)}} \pm \frac{2}{3}\left(\Phi\frac{\Psi_{,r}}{\Psi} + \Phi_{,r}\right)\right]\Phi_1$$
$$\left.\pm \frac{2}{3}\Phi\left(\Phi\frac{\mathcal{E}_{,rr}}{\mathcal{E}} - \Phi_{,rr}\right)\right\} = 0. \quad \text{(C74)}$$

In this expression, we do the same series of substitutions that we did in the large polynomial that resulted from (C66): we express $\mathcal{E}_{,rr}$ through $\mathcal{E}_{,r}$ using (C5), $\mathcal{E}_{,r}$ through $\Phi_{,r}$ and Φ_1 using (C67), then x through $\mathcal{E}$ and y using (C4), and multiply the whole expression by $\mathcal{E}$. The result is:

$$\left[\mp 1 + \frac{2^{3/2}\Phi\Psi_{,t}}{\sqrt{3(\varepsilon - k)}\Psi}\right]\mathcal{E}\Phi_1{}^2 + \left[-\frac{2^{3/2}\Phi\Psi}{3\sqrt{3(\varepsilon - k)}}\right.$$
$$\left.\pm \frac{2}{3}\left(\Phi\frac{\Psi_{,r}}{\Psi} + \Phi_{,r} + \Phi\frac{S_{,r}}{S}\right)\right]\mathcal{E}\Phi_1$$
$$\mp \frac{2}{3}\Phi\Phi_{,rr}\mathcal{E} \mp \frac{2}{3}\Phi^2\left[\left(\frac{P_{,r}}{S}\right)_{,r}\left(-\frac{S\mathcal{E}}{P} + \frac{Cy}{2P} + \frac{D_3}{P}\right)\right.$$
$$\left.+ \left(\frac{S_{,r}}{S}\right)_{,r}\mathcal{E} + \frac{S_{,r}\Phi_{,r}}{S\Phi}\mathcal{E}\right] = 0. \quad \text{(C75)}$$

We then use the fact that $(\Phi_1, y, \mathcal{E})$ are linearly independent and require that each coefficient of an independent function

is zero. There is only one term containing y, with the coefficient $C(P_{,r}/S)_{,r}$. But C cannot be zero, as explained below (C3). Thus $(P_{,r}/S)_{,r} = 0$ is the unique implication of this (it will be seen from the following that we need not consider whether this condition is consistent with the other equations that P and S must obey). This means:

$$P_{,r} = \alpha_0 S, \quad \text{(C76)}$$

where α_0 is an arbitrary constant. Taking this into account, and taking the term independent of Φ_1 in (C75) we get:

$$\Phi_{,rr} + \Phi\left(\frac{S_{,r}}{S}\right)_{,r} + \frac{S_{,r}\Phi_{,r}}{S} = 0. \quad \text{(C77)}$$

Using (C76), this is easily integrated with the result:

$$\Phi = \frac{\chi_1(t)P}{\alpha_0 S} + \frac{\chi_2(t)}{S}, \quad \text{(C78)}$$

where $(\chi_1(t), \chi_2(t))$ are arbitrary functions of t. Both appear as integration "constants" of (C77). Using such Φ in the definition of Ψ, (6.4), we get:

$$\Psi\Phi = \frac{\gamma(t)}{S}, \qquad \gamma(t) \overset{\text{def}}{=} \chi_2\chi_{1,t} - \chi_1\chi_{2,t}. \quad \text{(C79)}$$

But with $\Psi = \gamma(t)/(S\Phi)$ the last three terms in the coefficient of Φ_1 in (C75) sum up to zero, and what remains of that coefficient is the equation $\Phi\Psi = 0$. The only solution of this can be $\Psi = 0$, but we know it leads to the Friedmann model.

Consequently, the coefficient of $x_{,r}$ in (C16) is always nonzero.

Since (C68) and (C70) were, in their respective branches of the calculation, among the necessary conditions for the existence of RLPs, we conclude that the special quasihyperbolic Szekeres solution defined by (C1) does not contain any RLPs except (possibly) when it becomes axially symmetric or G_3/S_2, but these cases are considered in Appendices F and G.

APPENDIX D: PROOF THAT $(\Phi_1, y, \mathcal{E})$ ARE LINEARLY INDEPENDENT

We take the equation

$$\alpha\Phi_1 + \beta y + \gamma\mathcal{E} = 0 \quad \text{(D1)}$$

with constant coefficients α, β, γ and prove that it implies $\alpha = \beta = \gamma = 0$.

We substitute for Φ_1 from (4.1), then multiply the equation by $2S\mathcal{E}$ and use (C2) to eliminate x^2. We thus obtain a polynomial of degree 1 in x and degree 2 in y, which we denote by $\mathcal{P}$. We take the second derivative of $\mathcal{P}$ by xy. The result is:

$$2P(\beta S - \gamma C) = 0. \quad \text{(D2)}$$

We discard the solution $P = 0$ because this implies constant S (see (B11) and the remarks above (B22)), and then the metric acquires a G_3/S_2 symmetry group—these are discussed in Appendix G. We also discard the case $\beta \neq 0$ because then S, and consequently P, are constant, again leading to the G_3/S_2 case. So finally, the implication of (D2) is

$$\beta = 0 = \gamma C. \tag{D3}$$

With this, we go back to $\mathcal{P}$ and take its second derivative by y. The result is $-4\gamma P^2/S = 0$, and the solution of this is $\gamma = 0$. We again go back to $\mathcal{P}$ with $\beta = \gamma = 0$, and take its first derivative by y. The result is:

$$\alpha C(\Phi_{,r} + \Phi S_{,r}/S) = 0. \tag{D4}$$

When the expression in parentheses vanishes, Φ becomes a product of the form $R(t)/S(r)$, where $R(t)$ is an arbitrary function. Such form of Φ defines the Friedmann limit, in which we know that all null geodesics are RLPs, so we discard this case. Thus, we follow the case $\alpha C = 0$.

Putting this, together with $\beta = \gamma = 0$, in $\mathcal{P}$ and taking the derivative of the result by x we obtain

$$2\alpha(-P\Phi_{,r} + P_{,r}\Phi - \Phi P S_{,r}/S) = 0. \tag{D5}$$

If the expression in parentheses should vanish, then the solution is $\Phi = R(t)P(r)/S(r)$, which again leads back to the Friedmann model. Thus, finally, (D5) implies $\alpha = 0$, which completes the proof.

APPENDIX E: THE RLPS WITH $P = 0$ AND $x = 0$ ALONG THE GEODESIC.

We can leave aside the case when $Q_{,r} = 0$ because then the metric is axially symmetric from the beginning.

It is useful to turn this case back to that of Appendix B by the transformation (B14) and (B15) with $a \neq 0 \neq b$. After the transformation we have $x' = -by'/a$, i.e. $dx'/dr \neq 0$ if $dy'/dr \neq 0$, and $\tilde{P} = -b\tilde{Q}/a$, i.e. $\tilde{P}_{,r} \neq 0$ if $\tilde{Q}_{,r} \neq 0$. Thus the new $\tilde{P}$ and $\tilde{Q}$ obey (B8) with $D_0 = 0$ from the beginning, and the RLP condition reduces to (B11) alone, with (P, Q) replaced by $(\tilde{P}, \tilde{Q})$. The rest of the reasoning of Appendix B then applies, unchanged, to $(\tilde{P}, \tilde{Q})$, with the same result:

Corollary 3:

RLPs with $P = 0$ and $x = 0$ along the geodesic may exist only in the special case when the coordinates may be transformed so that $Q = 0$ as well, i.e. the metric is axially symmetric, or has a 3-dimensional symmetry group.

APPENDIX F: THE AXIALLY SYMMETRIC CASE $P = Q = 0$: ONLY THE AXIAL GEODESICS $x_{,r} = y_{,r} = 0$ ARE RLPS

We know from Ref. [12] that in the quasispherical case $\varepsilon = +1$ null geodesics on which x and y are constant exist

only when the Szekeres model is axially symmetric. Then coordinates may be chosen so that $P = Q = 0$, and the constant-(x, y) null geodesics have $x = y = 0$, i.e. intersect each $t = $ constant space on the symmetry axis.

In this appendix we show that the statement above applies also with $\varepsilon = 0$ and $\varepsilon = -1$, that the constant-(x, y) null geodesics are RLPs, and that other RLPs may exist only when the Szekeres spacetime has more symmetries.

1. Constant-(x, y) null geodesics exist only in the axially symmetric case

This thesis was proven in Ref. [12] for $\varepsilon = +1$. The assumption made there in the proof, that $\mathcal{E} > 0$, does not hold for $\varepsilon = 0$ and $\varepsilon = -1$, so we first verify what happens when $\mathcal{E} = 0$.

It is seen from (4.10) and (4.11) that constant (x, y) imply $E_{12} = E_{13} = 0$ along the geodesic. With $\mathcal{E} = 0$ (4.4) and (4.5) then imply that either $\mathcal{E}_{,r} = 0$ at all r, which means a G_3/S_2 symmetry (discussed in Appendix G), or $\mathcal{E}_{,x} = \mathcal{E}_{,y} = 0$ along the geodesic, which means P and Q being constant, i.e. axial symmetry. Thus, $\mathcal{E} = 0$ along a constant-(x, y) null geodesic implies axial symmetry anyway.[14]

The equations of Sec. 3.3.1 in Ref. [12] that are imposed on (P, Q, S) by the condition of constant (x, y) along the geodesic become subcases of our (B8) and (B11) for a general ε. As shown in our Appendix B, they imply axial symmetry for any ε. This is true even for the special solution discussed in Appendix C, as we now show.

When $x_{,r} = y_{,r} = 0$ along a null geodesic, as stated above, (4.10) and (4.11) imply $E_{12} = E_{13} = 0$ along this geodesic. Then (C1) implies that either (i) $P_{,r} = 0$, or (ii) $x = 0$ and $y^2 + D_3 = 0$, or (iii) $y = 0$ and $x^2 - D_3 = 0$. Case (i) is axially symmetric. Case (ii) implies $\mathcal{E} = 0$ along the geodesic, and this was discussed above. Case (iii) implies $D_3 \geq 0$. However, the solution of Appendix C has $D_3 \leq 0$ by definition. So the only subcase to consider here is $D_3 = 0 \Rightarrow x = 0$ along this geodesic. But then we have again $\mathcal{E} = 0$, which completes the proof.

2. Constant-(x, y) null geodesics are RLPs

As stated above, along null geodesics of constant (x, y) we have $E_{12} = E_{13} = 0$. Then (6.2) and (6.3) are fulfilled identically, which means that such geodesics are RLPs.

3. Other RLPs may exist in the axially symmetric case only with higher symmetries

We will now show that, in the axially symmetric case, (6.2) and (6.3) may have other solutions than constant (x, y)

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[14]Moreover, as shown in Ref. [20], the location $\mathcal{E} = 0$ is infinitely far from any point within the spacetime, i.e. does not in fact belong to the spacetime.

only when the spacetime has more symmetries than just the axial.

The whole reasoning and calculation is closely analogous to the one presented in Appendix C for the special Szekeres solution. We proved in Appendix B that in the axially symmetric case coordinates may be chosen so that $P = Q = 0$ and the candidate RLP has $x = 0$. Then:

$$\mathcal{E} = \frac{x^2 + y^2}{2S} + \frac{1}{2}\varepsilon S,$$

$$E_{12} = \varepsilon x S_{,r}/S,$$

$$E_{13} = \varepsilon y S_{,r}/S. \tag{F1}$$

Note that with $\varepsilon = 0$, this axially symmetric Szekeres solution is in fact plane symmetric. Thus, it will be considered together with other G_3/S_2 symmetric solutions in Appendix G. From here on, in the present appendix we assume $\varepsilon \neq 0$, i.e. $\varepsilon = \pm 1$.

With (F1) obeyed, (6.2) and (6.5) are fulfilled identically along $x = 0$. From (5.15) we obtain:

$$\frac{y_{,r}^2}{\mathcal{E}^2} = \frac{t_{,r}^2}{\Phi^2} - \frac{\Phi_1^2}{(\varepsilon - k)\Phi^2}, \tag{F2}$$

and (C12) applies unchanged. We then multiply (6.3) by $\Phi\Phi_1^2 t_{,r}/[(\varepsilon - k)\Psi]$ and use (F2) and (C12)–(C15) and $x = 0$ in the result. As before, τ factors out and is cancelled, and we obtain an equation almost identical to (C16), with the same definitions of (B_3, c_2, c_3, c_4), but with $y_{,r}$ in place of $x_{,r}$, and with the definition of B_1 changed to:

$$B_1 \overset{\text{def}}{=} \frac{2\varepsilon y S_{,r}}{(\varepsilon - k)S}. \tag{F3}$$

We proceed in strict analogy to Appendix B. From (4.9) using (F2) we again obtain (C22), with $y_{,r}$ in place of $x_{,r}$, and with the same definitions of $(c_5, \ldots, c_8)$, but with the definition of A changed to:

$$A \overset{\text{def}}{=} \frac{2\varepsilon y \Phi S_{,r}}{S\mathcal{E}^2 \Phi_1}. \tag{F4}$$

Then we square the current analogue of (C16) and use (F2) to eliminate $y_{,r}^2$ from the result. We obtain an 8th degree polynomial in $t_{,r}$, but this time the coefficient of $t_{,r}^8$ is

$$\alpha_1 = \mathcal{E}^2/\Phi^2 \tag{F5}$$

and is sure to be nonzero. Dividing the polynomial by α_1 we obtain (C38) again, but with the definitions of some of the coefficients changed as follows:

$$a_3 \overset{\text{def}}{=} 2c_3 + c_2^2 - \frac{\Phi_1^2}{\varepsilon - k} - \frac{4\varepsilon^2 \Phi^2 S_{,r}^2 y^2}{(\varepsilon - k)^2 S^2 \mathcal{E}^2}, \tag{F6}$$

$$a_5 \overset{\text{def}}{=} 2c_2 c_4 + c_3^2 - (2c_3 + c_2^2)\frac{\Phi_1^2}{\varepsilon - k} + \frac{12\varepsilon^2 \Phi^2 \Phi_1^2 S_{,r}^2 y^2}{(\varepsilon - k)^3 S^2 \mathcal{E}^2}, \tag{F7}$$

$$a_7 \overset{\text{def}}{=} c_4^2 - (2c_2 c_4 + c_3^2)\frac{\Phi_1^2}{\varepsilon - k} - \frac{18\varepsilon^2 \Phi^2 \Phi_1^4 S_{,r}^2 y^2}{(\varepsilon - k)^4 S^2 \mathcal{E}^2}, \tag{F8}$$

the remaining ones are the same as given by (C32), (C34), (C36), and (C37).

Now we differentiate the current analogue of (C30) by r along the null geodesic by the rule (B1). This time, however, $x = 0$ along our candidate RLP, so no coefficient depends on x. We then use our analogue of (C30) to eliminate $t_{,r}^{10}$, $t_{,r}^9$ and $t_{,r}^8$ from the result. The equation that emerges is an analogue of (C38) with $y_{,r}$ in place of $x_{,r}$, with the same definitions of $(b_1, \ldots, b_8)$, and with the definitions of $(\beta_1, \ldots, \beta_8)$ changed to

$$\beta_1 \overset{\text{def}}{=} 2c_{2,y} - 2c_2 A, \tag{F9}$$

$$\beta_2 \overset{\text{def}}{=} a_{3,y} - 2a_3 A, \tag{F10}$$

$$\beta_3 \overset{\text{def}}{=} a_{4,y} - 3a_4 A, \tag{F11}$$

$$\beta_4 \overset{\text{def}}{=} a_{5,y} - 4a_5 A, \tag{F12}$$

$$\beta_5 \overset{\text{def}}{=} a_{6,y} - 5a_6 A, \tag{F13}$$

$$\beta_6 \overset{\text{def}}{=} a_{7,y} - 6a_7 A, \tag{F14}$$

$$\beta_7 \overset{\text{def}}{=} a_{8,y} - 7a_8 A, \tag{F15}$$

$$\beta_8 \overset{\text{def}}{=} a_{9,y} - 8a_9 A, \tag{F16}$$

where for A the definition (F4) must be used.

Here we can assume that the coefficient of $y_{,r}$ in the present analogue of (C16) is nonzero—the explanation given in the paragraphs containing (C72)–(C75) still applies, except that the B_1 given by (F3) cannot vanish for somewhat different reasons.[15] Then we determine $y_{,r}$ from that equation and substitute the result in the current analogue of (C38). After multiplying out to get a polynomial in $t_{,r}$, we again use (C30) to eliminate $t_{,r}^{10}$ and $t_{,r}^9$ (but not $t_{,r}^8$). Then we assume that the coefficient of $t_{,r}^8$, denoted d_1, is nonzero (we will check the case $d_1 = 0$ later), and divide the equation by d_1. In this way we obtain an exact copy of (C55), with the same definitions (C56)–(C64) of the coefficients; but it is to be remembered that some of the symbols in these formulae (namely B_1, B_2,

---

[15]The cases $\varepsilon = 0$ and $S_{,r} = 0$ define metrics of higher symmetry, treated in Appendix G, while $y = 0$ (together with $x = 0$ assumed throughout this appendix) defines an axial geodesic, which we already know is an RLP.

a_3, a_5, a_7 and all of $(\beta_1, \ldots, \beta_8)$) now have different definitions from those in Appendix C.

Consequently, Eqs. (C65) must still hold, and we again choose (C66) to investigate, by exactly the same method as before. By the method of Appendix D we show that $(\Phi_1, y, \mathcal{E})$ are still linearly independent in the present case (i.e. with $\mathcal{E}$ given by (F1), and along the $x = 0$ geodesic). The explanation given under (C67) still applies, with the modification that now x is nowhere present, so does not have to be eliminated. In place of (C68) we now obtain:

$$256\varepsilon^4 \Phi^6 \Psi \Psi_{,t} S_{,r}{}^4 / [S^4(\varepsilon - k)^9] = 0. \qquad \text{(F17)}$$

We recall that we excluded the case $\varepsilon = 0$ (since it is treated in Appendix G), and $\Psi = 0$ because it reduces the Szekeres model to Friedmann. We can also exclude $S_{,r} = 0$ because then the metric acquires a G_3/S_2 symmetry and is also treated in Appendix G. So, as before, the only case left to investigate is $\Psi_{,t} = 0$.

We substitute $\Psi_{,t} = 0$ in the large main polynomial, and in the resulting smaller polynomial we take the coefficient of y^4. The equation that results is:

$$256\varepsilon^3 \mathcal{E} \Phi^5 \Psi^3 S_{,r}{}^3 / [S^4(\varepsilon - k)^9] = 0. \qquad \text{(F18)}$$

The only factors that could vanish here are Ψ and $S_{,r}$, but, as explained above, their vanishing leads to simpler cases of higher symmetry. Thus, (F18) does not include any case that would define any new RLP, apart from those considered elsewhere.

We go back to the paragraph after (F16) to consider the case $d_1 = 0$. The explanation given above (C70) still applies, but this time, in the expression $\mathcal{E}^4 d_1$ calculated by the algebraic program, we take the coefficient of y^4 at $\mathcal{E} = 0$, and obtain:

$$64\varepsilon^4 \Phi^4 \Psi_{,t} S_{,r}{}^4 / [S^4(\varepsilon - k)^4 \Psi] = 0. \qquad \text{(F19)}$$

As explained above, only $\Psi_{,t}$ could possibly be zero, so we substitute $\Psi_{,t} = 0$ in the main large polynomial, and in the resulting expression take the coefficient of y^4. The result is:

$$64\varepsilon^3 \mathcal{E}^5 \Phi^3 \Psi S_{,r}{}^3 / [S^4(\varepsilon - k)^4] = 0, \qquad \text{(F20)}$$

and again does not include any case that would define a new RLP.

So, the final conclusion is:

Corollary 4:

In the axially symmetric Szekeres solutions, apart from cases of higher symmetry, the only RLPs are the axial null geodesics that intersect each 3-space of constant t on the symmetry axis.

APPENDIX G: THERE ARE NO NON-RADIAL RLPS IN ANY G_3/S_2 MODEL.

We will investigate the equation $\chi = 0$ (see (7.1)) and will show that it has no solutions defining nonradial RLPs, unless the model reduces to Friedmann.

Several equations in this appendix follow from the corresponding ones in Appendix C as the special case $\mathcal{E}_{,r} = 0$; they are similar but not identical.

We will use all equations adapted to the special case discussed in Sec. VII, i.e. $\zeta = \psi = \xi = \eta = \mathcal{E}_{,r} = dy/dr = 0$. From (5.15) we find

$$\frac{x_{,r}{}^2}{\mathcal{E}^2} = \frac{t_{,r}{}^2}{\Phi^2} - \frac{\Phi_{,r}{}^2}{(\varepsilon - k)\Phi^2}. \qquad \text{(G1)}$$

Then, using (G1) in (5.16) we obtain

$$\tau_{,r} t_{,r} = \frac{\tau \Phi_{,r} \Psi}{\varepsilon - k} + \frac{\tau \Phi_{,t}}{\Phi} t_{,r}{}^2, \qquad \text{(G2)}$$

where Ψ is defined by (6.4)—as seen from (2.9) this is a coefficient of shear, whose vanishing defines the Friedmann limit.

We now substitute (G1) and (G2) in $\chi = 0$, where χ is given by (7.1). We multiply the result by $\Phi \Phi_{,r}{}^2 t_{,r} / [(\varepsilon - k)\Psi]$, and cancel τ that factors out. The result is:

$$W_1 \stackrel{\text{def}}{=} t_{,r}{}^3 + c_2 t_{,r}{}^2 + c_3 t_{,r} + c_4 = 0, \qquad \text{(G3)}$$

where

$$c_2 \stackrel{\text{def}}{=} \frac{2\Phi(\Phi_{,r} \Psi_{,t} - \Psi^2)}{(\varepsilon - k)\Psi} \qquad \text{(G4)}$$

$$c_3 \stackrel{\text{def}}{=} \frac{\Phi(\Phi_{,r} \Psi_{,r} - \Phi_{,rr} \Psi) - 2\Phi_{,r}{}^2 \Psi}{(\varepsilon - k)\Psi} \qquad \text{(G5)}$$

$$c_4 \stackrel{\text{def}}{=} \frac{2\Phi \Phi_{,r}{}^2 \Psi}{(\varepsilon - k)^2}. \qquad \text{(G6)}$$

Adapting (4.9) to the G_3/S_2 case, eliminating $x_{,r}$ with use of (G1) and using (G3) to eliminate $t_{,r}{}^3$ we obtain:

$$t_{rr} = c_6 t_{,r}{}^2 + c_7 t_{,r} + c_8, \qquad \text{(G7)}$$

where

$$c_6 \stackrel{\text{def}}{=} \frac{2\Psi_{,t}}{\Psi} + \frac{\Phi_{,t}}{\Phi}, \qquad \text{(G8)}$$

$$c_7 \stackrel{\text{def}}{=} \frac{\Psi_{,r}}{\Psi} - \frac{\Phi_{,r}}{\Phi} + \frac{k_{,r}}{2(\varepsilon - k)}, \qquad \text{(G9)}$$

$$c_8 \stackrel{\text{def}}{=} \frac{\Phi_{,r} \Psi}{\varepsilon - k}. \qquad \text{(G10)}$$

Now we differentiate (G3) along a null geodesic (since the equation must hold all along it), by the rule given in (B1), and use (G7) to eliminate $t_{,rr}$. The resulting equation is of 4th degree in $t_{,r}$. We eliminate the 4th power of $t_{,r}$ by using (G3). In the end, we obtain an equation of degree 3 in $t_{,r}$, which we write symbolically as follows:

$$d_1 t_{,r}{}^3 + d_2 t_{,r}{}^2 + d_3 t_{,r} + d_4 = 0. \qquad \text{(G11)}$$

The expressions for the coefficients in (G11) are:

$$d_1 = c_{2,t} + 3c_7 - c_2 c_6, \tag{G12}$$

$$d_2 = c_{2,r} + c_{3,t} + 3c_8 + 2c_2 c_7 - 2c_3 c_6, \tag{G13}$$

$$d_3 = c_{3,r} + c_{4,t} + 2c_2 c_8 + c_3 c_7 - 3c_4 c_6, \tag{G14}$$

$$d_4 = c_{4,r} + c_3 c_8. \tag{G15}$$

For the beginning, let us assume that $d_1 \neq 0$. For further considerations it will be more convenient to write (G11) as follows:

$$W_2 \stackrel{\text{def}}{=} t_{,r}^3 + \delta_2 t_{,r}^2 + \delta_3 t_{,r} + \delta_4 = 0, \tag{G16}$$

where $\delta_i \stackrel{\text{def}}{=} d_i / d_1$, $i = 2, 3, 4$.

Equation (G3) is equivalent to $\chi = 0$ in (7.1). Thus (G3), just like (7.1), defines the collection of RLPs together with the conditions of their existence. Every solution of (G3) and (7.1) is a candidate RLP, and every RLP must obey (G3) and (7.1). Then, (G16) is the condition that the solutions of (G3) are preserved along the null geodesics, thus every solution of (G3) must be a solution of (G16) and vice versa. But if (G3) and (G16) have the same set of solutions, then they must be identical, i.e. their respective coefficients must be equal. Thus, the necessary conditions for the existence of RLPs are:

$$c_i = \delta_i \Longleftrightarrow c_i d_1 - d_i = 0, \qquad i = 2, 3, 4. \tag{G17}$$

As with the previous calculations, we employed the algebraic program ORTOCARTAN [26,27]. We consider (G17) with $i = 4$, the simplest one. In it, we substitute (G4)–(G15) and multiply the result by $(\Psi/\Phi_{,r})$ to get a polynomial in Φ, Ψ and their derivatives ($\Phi_{,r}$ factors out in the original expression). The resulting expression is simple enough to be shown here:

$$W_3 = \frac{-\Phi\Phi_{,r}\Psi^2 k_{,r} + 8\Phi^2\Phi_{,r}\Psi^2\Psi_{,t} + 4\Phi\Phi_{,t}\Phi_{,r}^2\Psi\Psi_{,t}}{(\varepsilon - k)^3}$$
$$+ \frac{4\Phi^2\Phi_{,r}^2\Psi\Psi_{,tt} - 12\Phi^2\Phi_{,r}^2\Psi_{,t}^2}{(\varepsilon - k)^3}$$
$$+ \frac{3\Phi\Phi_{,r}\Psi\Psi_{,r} - 6\Phi_{,r}^2\Psi^2 - 3\Phi\Phi_{,rr}\Psi^2}{(\varepsilon - k)^2} = 0. \tag{G18}$$

We assume $\Lambda = 0$ and take the $k > 0$ model for the beginning. The solution of (2.3) can then be written as

$$\Phi(t, r) = \frac{M}{k}(1 - \cos\eta), \qquad \eta - \sin\eta = \frac{k^{3/2}}{M}[t - t_B(r)], \tag{G19}$$

where η is a parameter (dependent on t and r), and $t_B(r)$ is an arbitrary function, the bang time. We introduce the abbreviations:

$$I_M \stackrel{\text{def}}{=} \frac{3k_{,r}}{2k} - \frac{M_{,r}}{M}, \qquad D_M \stackrel{\text{def}}{=} \frac{k^2 t_{B,r}}{M}. \tag{G20}$$

The derivatives of Φ and Ψ can then be written as

$$\Phi_{,r} = \left(\frac{M}{k}\right)_{,r}(1 - \cos\eta) + \frac{MI_M}{k}\frac{\sin\eta(\eta - \sin\eta)}{1 - \cos\eta}$$
$$- \frac{MD_M}{k^{3/2}}\frac{\sin\eta}{1 - \cos\eta}, \tag{G21}$$

$$\Phi_{,t} = \frac{\sqrt{k}\sin\eta}{1 - \cos\eta}, \tag{G22}$$

$$\Phi_{rr} = \left(\frac{M}{k}\right)_{,rr}(1 - \cos\eta) + \left(\frac{M}{k}\right)_{,r}\sin\eta\left[2I_M\frac{\eta - \sin\eta}{1 - \cos\eta}\right.$$
$$\left. - \frac{D_M}{\sqrt{k}(1 - \cos\eta)}\right] + \frac{M(I_M)_{,r}}{k}\frac{\sin\eta(\eta - \sin\eta)}{1 - \cos\eta}$$
$$+ \frac{MI_M^2}{k}\frac{(2\sin\eta - \sin\eta\cos\eta - \eta)(\eta - \sin\eta)}{(1 - \cos\eta)^2}$$
$$- \frac{MI_M D_M}{k^{3/2}}\frac{3\sin\eta - \sin\eta\cos\eta - 2\eta}{(1 - \cos\eta)^2}$$
$$- (\sqrt{k}t_{B,r})_{,r}\frac{\sin\eta}{1 - \cos\eta} - \frac{MD_M^2}{k^2(1 - \cos\eta)^2}, \tag{G23}$$

$$\Psi = \sqrt{k}I_M\frac{3\sin\eta - \eta\cos\eta - 2\eta}{(1 - \cos\eta)^2} + D_M\frac{2 + \cos\eta}{(1 - \cos\eta)^2}, \tag{G24}$$

$$\Psi_{,r} = (\sqrt{k}I_M)_{,r}\frac{3\sin\eta - \eta\cos\eta - 2\eta}{(1 - \cos\eta)^2}$$
$$+ I_M\frac{4\cos\eta + \eta\sin\eta(5 + \cos\eta) - 4 - 4\sin^2\eta}{(1 - \cos\eta)^4}$$
$$\times [\sqrt{k}I_M(\eta - \sin\eta) - D_M] + (D_M)_{,r}\frac{2 + \cos\eta}{(1 - \cos\eta)^2}$$
$$- D_M\frac{\sin\eta(5 + \cos\eta)}{(1 - \cos\eta)^4}\left[I_M(\eta - \sin\eta) - \frac{D_M}{\sqrt{k}}\right], \tag{G25}$$

$$\Psi_{,t} = \frac{k^2 I_M}{M}\frac{4\cos\eta + \eta\sin\eta(5 + \cos\eta) - 4 - 4\sin^2\eta}{(1 - \cos\eta)^4}$$
$$- \frac{k^{3/2}D_M}{M}\frac{\sin\eta(5 + \cos\eta)}{(1 - \cos\eta)^4}, \tag{G26}$$

$$\Psi_{,tt} = \frac{k^{7/2}I_M}{M^2(1 - \cos\eta)^5}(-23\eta - 19\eta\cos\eta + 2\eta\sin^2\eta$$
$$+ 33\sin\eta + 9\sin\eta\cos\eta)$$
$$- \frac{k^3 D_M}{M^2}\frac{-19\cos\eta + 2\sin^2\eta - 23}{(1 - \cos\eta)^5}. \tag{G27}$$

From here on, the intermediate expressions become so large that we cannot reproduce them here; we only describe how the calculation is done. We substitute (G21)–(G27) in (G18) and multiply the result by $(1 - \cos\eta)^6$ to get a polynomial in $(1 - \cos\eta)$ (of 6th degree). It is also a polynomial of 4th degree in η. The function η is the only one in this polynomial that depends on t, the coefficients of η, $(1 - \cos\eta)$ and their powers depend only on r. Thus we treat η and $(1 - \cos\eta)$ as independent variables, and the coefficients of their different powers must vanish separately.

In W_3 transformed in this way we now take the coefficient of η^4, the resulting equation is a polynomial of degree 4 in $(1 - \cos\eta)$:

$$\frac{M^2 I_M{}^4}{(\varepsilon - k)^3}[9(\varepsilon/k - 1)(1 - \cos\eta)^4 - 78(\varepsilon/k - 1)(1 - \cos\eta)^3$$
$$+ (216\varepsilon/k - 276)(1 - \cos\eta)^2$$
$$- (189\varepsilon/k - 381)(1 - \cos\eta) - 144] = 0. \qquad (G28)$$

The term independent of $(1 - \cos\eta)$ clearly shows that $I_M = 0$ is a unique solution of this, independently of the value of k.

So, we substitute $I_M = 0$ in the main large polynomial and in the resulting smaller polynomial we take the

term independent of $(1 - \cos\eta)$. The equation that results is:

$$- 144M^2 D_M{}^4/[k^2(\varepsilon - k)^3] = 0. \qquad (G29)$$

Here the unique solution is $D_M = 0$. But with $I_M = D_M = 0$ we get $\Psi = 0$ from (G24), i.e. the Friedmann limit. Thus, $c_4 d_1 - d_4 = 0$, which is one of the necessary conditions for the existence of RLPs, can in this case be fulfilled only when the Szekeres model trivializes to Friedmann.

The calculation above was done for $k > 0$. The calculation with $k < 0$ is essentially the same and need not be done separately—it is enough to replace (k, η) in all equations with $(-\tilde{k}, i\tilde{\eta})$.

When $k = 0$, we have necessarily $\varepsilon = +1$ and the calculation must be done separately. Then we have:

$$\Phi = \left(\frac{9M}{2}\right)^{1/3}(t - t_B)^{2/3},$$
$$\Psi = \frac{2}{3}\left(\frac{9M}{2}\right)^{1/3} t_{B,r}(t - t_B)^{-4/3}. \qquad (G30)$$

With the r-coordinate chosen so that $M = M_0 r^3$, where M_0 is a constant, this simplifies W_3 in (G18) to

$$W_4 \stackrel{\text{def}}{=} -(64/9)M_0{}^2 r^6 t_{B,r}{}^4(t - t_B)^{-4} - (14/3) \times 36^{1/3} M_0{}^{4/3} r^4 t_{B,r}{}^4(t - t_B)^{-10/3} + (256/3)M_0{}^2 r^5 t_{B,r}{}^3(t - t_B)^{-3}$$
$$+ 14 \times 36^{1/3} M_0{}^{4/3} r^3 t_{B,r}{}^3(t - t_B)^{-7/3} - 112 M_0{}^2 r^4 t_{B,r}{}^2(t - t_B)^{-2} - 3 \times 36^{1/3} M_0{}^{4/3} r^2 t_{B,r}{}^2(t - t_B)^{-4/3}$$
$$+ 3 \times 36^{1/3} M_0{}^{4/3} r^3 t_{B,rr} t_{B,r}(t - t_B)^{-4/3} = 0. \qquad (G31)$$

Now the independent variables are t and r, and t appears always in the combination $(t - t_B)$. Thus different powers of $(t - t_B)$ are linearly independent, and their coefficients must vanish separately. Whichever term we take, except for the last two, the result is always the same:

$$t_{B,r} = 0 \qquad (G32)$$

(because $M_0 = 0$ is the vacuum, i.e. Schwarzschild, limit of the L-T model). This guarantees that all the terms in (G31) vanish. However, as seen from (G30), $t_{B,r} = 0$ means $\Psi = 0$, i.e. zero shear (see (6.4) and (2.9)), i.e. the Friedmann limit. Thus, there are no nonradial RLPs also when $k = 0$, which completes the proof in the case $d_1 \neq 0$.

We go back to (G15), where we assumed $d_1 \neq 0$ and proceed from there on to consider the case $d_1 = 0$. Instead of (G18) we now get:

$$W_5 = \frac{-6\Phi\Phi_{,r}\Psi_{,t}{}^2/\Psi^2 + 2\Phi\Phi_{,r}\Psi_{,tt}/\Psi}{\varepsilon - k}$$
$$+ \frac{4\Phi\Psi_{,t} + 2\Phi_{,t}\Phi_{,r}\Psi_{,t}/\Psi + (3/2)k_{,r}}{\varepsilon - k}$$
$$- 3\frac{\Phi_{,r}}{\Phi} + 3\frac{\Psi_{,r}}{\Psi} = 0. \qquad (G33)$$

As before, we begin by considering the case $k > 0$. We multiply W_5 by $\Psi^2\Phi(1 - \cos\eta)^6$ and substitute for Φ and Ψ from (G19)–(G27). What results is a polynomial of degree 6 in $(1 - \cos\eta)$ and of degree 3 in η. Taking the coefficient of η^3 we obtain:

$$W_6 = \frac{MI_M{}^3 \sin\eta}{\varepsilon - k}[-6(\varepsilon - k)(1 - \cos\eta)^3$$
$$+ 45(\varepsilon - k)(1 - \cos\eta)^2 - 81(\varepsilon - k)(1 - \cos\eta)$$
$$+ 30k(1 - \cos\eta) - 36k] = 0. \qquad (G34)$$

Looking at the term independent of $(1 - \cos\eta)$ we see that the unique solution of this is $I_M = 0$.

So we substitute $I_M = 0$ in the main polynomial, and in the resulting expression we take the term independent of $(1 - \cos\eta)$. The resulting equation is:

$$\frac{36\sin\eta MD_M{}^3}{\sqrt{k}(\varepsilon - k)} = 0. \qquad (G35)$$

The unique solution of this is $D_M = 0$, which, together with $I_M = 0$, leads back to the Friedmann limit. Thus, no

RLPs exist in nontrivial Szekeres spacetimes in this case, either.

The argument given before, that the result for $k < 0$ follows by the substitution $(k, \eta) \rightarrow (-\tilde{k}, i\tilde{\eta})$, is still valid. So we now consider $d_1 = 0$ with $k = 0$. We substitute $k = 0$ (and, as is necessary, $\varepsilon = +1$) in (G33), multiply the result by Ψ, substitute then for Ψ and Φ from (G30), and obtain:

$$
\begin{aligned}
W_7 &= (16/9)M_0 r^3 t_{B,r}{}^2 (t - t_B)^{-3} \\
&\quad + 2 \times 36^{1/3} M_0{}^{1/3} r t_{B,r}{}^2 (t - t_B)^{-7/3} \\
&\quad - (56/3)M_0 r^2 t_{B,r} (t - t_B)^{-2} \\
&\quad + 36^{1/3} M_0{}^{1/3} r t_{B,rr} (t - t_B)^{-4/3} = 0.
\end{aligned} \quad \text{(G36)}
$$

The coefficients of independent powers of $(t - t_B)$ have to vanish separately, as explained before. Whichever term we take, except for the last one, the result is always the same:

$$
t_{B,r} = 0 \quad \text{(G37)}
$$

and this implies the Friedmann limit in the same way as explained after (G32). This also guarantees that the whole of (G36) is fulfilled.

Thus, in every case considered, the assumption that nonradial RLPs could exist leads to either the Friedmann limit or the Schwarzschild limit. The final conclusion is that the only RLPs in the G_3/S_2 models are the radial null geodesics. $\qquad\square$

APPENDIX H: A DETAILED DESCRIPTION OF THE MODEL PRESENTED IN SEC. VIII.

The algorithm used in the calculations discussed in Sec. VIII consists of the following steps:

(1) First we set the observer at R_0 (the present-day areal distance) and consider sources which are, at the present instant, away from the observer by 1 Gly (≈ 306.6 Mpc).
(2) To calculate the evolution of the model one needs to follow the following points:
(i) The radial coordinate is chosen to be the areal radius at the present instant: $\bar{r} = \Phi(t_0, r)$. However, to simplify the notation we will omit the bar and denote the new radial coordinate by r.
(ii) The chosen asymptotic cosmic background is an open Friedmann model, i.e. $\Omega_m = 0.3$ and $\Lambda = 0$. The background density is then given by

$$
\rho_b = \Omega_m \times \rho_{cr} = 0.3 \times \frac{3H_0^2}{8\pi G}(1 + z)^3, \quad \text{(H1)}
$$

where the Hubble constant is $H_0 = 72 \text{ km s}^{-1} \text{ Mpc}^{-1}$.
(iii) The initial time t_0 is calculated from the following formula for the background Friedmann Universe

$$
t(z) = \int_0^\infty \frac{H_0^{-1}(1 + \tilde{z})^{-1} d\tilde{z}}{\sqrt{\Omega_m(1 + \tilde{z})^3 + (1 - \Omega_m)(1 + \tilde{z})^2}}, \quad \text{(H2)}
$$

(iv) The age of the universe is assumed to be everywhere the same: $t_B = 0$.
(v) The function $M(r)$ follows from (2.5), where the present-day density is

$$
\rho(t_0, r) = \rho_0[1 + \delta - \delta \exp(-r^2/\sigma^2)]
$$

(vi) Because of the assumed spherical symmetry $e^\nu = 1$.
(vii) The function $k(r)$ can be calculated from (2.6).
(viii) Then the evolution of the model can be calculated from Eq. (2.3).
(3) We then find a null geodesic that joins the observer and the source. The angle between the direction towards the source and the direction towards the origin, at the present instant, is denoted as γ.
(4) The null geodesics are found in the following manner:
(i) Because of spherical symmetry we may set one of the angular components of the null vector to zero. We set $k^\phi = 0$.
(ii) The second angular component, k^θ, follows from $R^2 k^\theta = J = \text{const} = R_0 \sin\gamma$, where $R_0 = R(t_0, r_0)$, i.e. at the observer's position. This relation is a consequence of (4.10) and (4.11) and was derived in [12] (see equation (3.26) in [12]).
(iii) The radial component is evaluated from (4.8) with $E_{12} = 0 = E_{13} = \mathcal{E}_{,r}$, and $\Sigma = \mathcal{E}^2 (d\theta/dr)^2$.
(iv) The time component of the null vector is found from $k^\alpha k_\alpha = 0$.
(5) We then find two other null geodesics: one that will reach the observer in 1 Gy time, the other one that arrived at the observer's position 1 Gy ago. Because of the non-RLP effect these geodesics approach the observer at angles that are different from γ.
(6) The difference between these angles allows us to evaluate the rate of change of the angle γ.

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