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RESEARCH ARTICLE

Imitating accelerated expansion of the Universe by matter inhomogeneities: corrections of some misunderstandings

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Abstract A number of misunderstandings about modeling the apparent accelerated expansion of the Universe and about the 'weak singularity' are clarified: (1) Of the five definitions of the deceleration parameter given by Hirata and Seljak (HS), only q_1 is a correct invariant measure of acceleration/deceleration of expansion. The q_3 and q_4 are unrelated to acceleration in an inhomogeneous model. (2) The averaging over directions involved in the definition of q_4 does not correspond to what is done in observational astronomy. (3) HS's equation (38) connecting q_4 to the flow invariants gives self-contradictory results when applied at the centre of symmetry of the Lemaître–Tolman (L–T) model. The intermediate equation (31) that determines $q_{3'}$ is correct, but approximate, so it cannot be used for determining the sign of the deceleration parameter. Even so, at the centre of symmetry of the L–T model, it puts no limitation on the sign of $q_{3'}(0)$. (4) The 'weak singularity' of Vanderveld et al. is a conical profile of mass density at the centre—a perfectly acceptable configuration.

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(5) The so-called 'critical point' in the equations of the 'inverse problem' for a central observer in an L–T model is a manifestation of the apparent horizon (AH)—a common property of the past light cones in zero-lambda L–T models, perfectly manageable if the equations are correctly integrated.

Keywords Exact solutions · Cosmology · Inhomogeneous cosmological models

1 Motivation

Vanderveld et al. [1] (abbreviated as VFW) claimed that there was a contradiction between two results concerning the putative accelerated expansion of the Universe. Their reasoning was, in brief, this (italics mark quotations): On the one hand, Hirata and Seljak [2] (abbreviated as HS) claimed to have proved that in a perfect fluid cosmological model that is geodesic, rotation-free and obeys the strong energy condition $\rho + 3p \geq 0$, a certain generalisation of the deceleration parameter q must be non-negative. But on the other hand, Iguchi et al. [3] (hereafter INN) did obtain simulated acceleration in Lemaître–Tolman (L–T) models [4,5] with $\Lambda = 0$ that obey HS's conditions. This *contradiction* is resolved by showing that L–T models that simulate accelerated expansion also contain a *weak singularity*, and in this case the derivation of HS breaks down. In addition to this, there are *other singularities that tend to arise* in L–T models, and VFW *have failed to find any singularity-free models that agree with observations.*

This leaves the impression that physically acceptable inhomogeneous models are unable to account for observations. It is shown here that this reasoning is not correct. In brief, our theses are the following:

1. Of the five definitions of the deceleration parameter given by HS, only q_1 is a correct invariant measure of deceleration of fluid expansion. Their $q_{3'}$ derives from a Taylor expansion of the luminosity distance–redshift relation, $D_L(z)$, in powers of z, done separately for each direction. Their q_4 is based on the angular average of a Taylor expansion of $z(D_L)$, and is conceptually different from what is done in observational practice. Although all these definitions reduce to the familiar q in FLRW models, they have different values and distinct meanings in an inhomogeneous model. In particular $q_{3'}$ and q_4 are not measures of deceleration in an inhomogeneous model.
2. HS's q_4 does not correctly represent the observers' deceleration parameter q. When observers assume that our real Universe is in the FLRW class, q is the same in all directions. But when they consider inhomogeneous and anisotropic models, they have to measure the distance–redshift relation for each direction separately rather than average the measured results over directions, as this would mean destroying useful information. Thus, HS's q_4 that represents the result of such averaging does not correspond to the observations actually done in relation to this quantity.
3. HS's reasoning is correct (in the approximate sense) up to their eq. (31), which allows one to calculate $q_{3'}$ at the centre of symmetry in a Lemaître–Tolman (L–T) model. But, at a spherical centre, the final result is the opposite to what they

intended—no limitation on the sign of $q_4(0)$ follows (see our Sects. 4 and 6). Thus, there is no contradiction involved in a decelerating inhomogeneous model imitating observational relations of an accelerating FLRW model. Even so, it is incorrect to use an approximate equation to decide whether some quantity is positive or negative.
4. What VFW call a *weak singularity* is not a singularity.[1]
5. It is also not true that *other singularities* invalidate the L–T models. The differential equations for the *inverse problem*—given observational data functions, calculate the LT model that would give them—are indeed singular at the apparent horizon (AH). VFW refer to this location as a *critical point*, and to this phenomenon as a *pathology*. In truth, this is simply the reconvergence of the observer's past null cone towards the Big Bang, that is a consequence of the decelerating cosmic expansion when lambda is zero, long known in the FLRW case, e.g [7–9]. Though some authors (INN and VFW among them) were unable to propagate their solutions through the 'critical point', this difficulty was overcome by Lu and Hellaby [10] and in fact the AH relation was used to provide extra information by McClure and Hellaby [11,12].

Since these misunderstandings have propagated into the literature, it is essential to resolve these issues.

But the most important point to be stressed is this: what one wants to reproduce, in connection with the distant type Ia supernovae, is not the accelerated expansion of the Universe, but the observed luminosity distance–redshift relation. The apparent dimming of supernovae (first reported by Riess et al. [13] and Perlmutter et al. [14]) stems from a comparison between observations and the Einstein–de Sitter model, the 'old' standard model of the Universe. The first proposed explanation, originating from the assumption that the Universe should be Friedmannian, is an accelerated expansion. However, this acceleration is not the phenomenon to be explained, but a component of a particular explanation—one that assumes homogeneity on the scale in question.

2 The Lemaître–Tolman (L–T) model

Since in the following we often refer to the L–T model, we summarise here the basic facts about it in the case $\Lambda = 0$. For more extended expositions see Refs. [15,16]. The metric of the L–T model is:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{{R_{,r}}^2}{1 + 2E(r)}\mathrm{d}r^2 - R^2(t,r)(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2), \tag{2.1}$$

where $E(r)$ is an arbitrary function, and $R(t,r)$ is determined by the integral of the Einstein equations:

$${R_{,t}}^2 = 2E(r) + 2M(r)/R, \tag{2.2}$$

[1] Moreover, their definition of 'weak singularity' does not agree with the meaning of the term 'singularity' in relativity, and is not supported by anything in the Tipler paper [6] they cite.

$M(r)$ being another arbitrary function. Equation (2.2) has the same algebraic form as one of the well-known Friedmann equations, except that here it contains arbitrary functions of r in place of arbitrary constants. The Friedmann limit follows when $E = -kr^2/2$, $M = M_0 r^3$ and $R = rS(t)$ where k, M_0 and $S(t)$ are the corresponding Friedmann constants and scale factor. The solution of (2.2) may be written as

$$t - t_B(r) = \int \frac{\mathrm{d}R}{\pm\sqrt{2E(r) + 2M(r)/R}}, \tag{2.3}$$

where $t_B(r)$ is one more arbitrary function called the bang-time function; in the Friedmann limit it is constant. The $+$ sign applies for an expanding region, $-$ applies for a collapsing region. The mass density is

$$8\pi G\rho = \frac{2M_{,r}}{R^2 R_{,r}}. \tag{2.4}$$

The pressure is zero, and so the matter (dust) particles move on geodesics.

The equations determining the L–T model are covariant with the transformations $r \to r' = f(r)$. Therefore, we may use such a transformation to give one of the functions (M, E, t_B) a handpicked form, provided the chosen function is monotonic in the range under investigation.

An incoming radial null geodesic is given by the differential equation

$$\frac{\mathrm{d}t}{\mathrm{d}r} = -\frac{R_{,r}}{\sqrt{1 + 2E(r)}}, \tag{2.5}$$

and its solution is denoted $t = \widehat{t}(r)$. The general solution determines the past null cone (PNC) of the observer situated at the centre of symmetry. We use the hat $\widehat{\ }$ and the subscript $\wedge$ to indicate evaluation on the PNC. Then the redshift along the PNC, $z(r)$, is given by [16,17]:

$$\frac{1}{1+z}\,\frac{\mathrm{d}z}{\mathrm{d}r} = \left[\frac{R_{,tr}}{\sqrt{1+2E}}\right]_{\wedge}, \tag{2.6}$$

and the area and luminosity distances are

$$D_A = \widehat{R}, \quad D_L = (1+z)^2\widehat{R}. \tag{2.7}$$

Equations (2.5) and (2.6) can be solved numerically once $M(r)$, $E(r)$ and $t_B(r)$ are specified.

An L–T model may possibly have a curvature singularity at the centre of symmetry. The singularity will be absent when there is no point mass there, and the mass density is finite. This will happen when the conditions given below are obeyed (see Ref. [16] for the derivation).

Let $r = r_c$ be the radial coordinate of the centre of symmetry, where $R(t, r_c) = 0$ for all $t > t_B(r_c)$, and let $\rho(t, r_c) \overset{\text{def}}{=} \alpha(t) < \infty$. We choose the r-coordinate so that

$r_c = 0$ and

$$M = M_0 r^3, \tag{2.8}$$

where $M_0 =$ constant. Then, in a neighbourhood of $r = 0$, the functions behave as follows:

$$R = \beta(t)r + \mathcal{R}_1(t, r), \quad E = \gamma r^2 + \mathcal{R}_2(r), \quad t_B = \tau + \mathcal{R}_0(r), \tag{2.9}$$

γ and τ being constants, where the symbols $\mathcal{R}_a$ will denote quantities with the property

$$\lim_{r\to 0} \frac{\mathcal{R}_a}{r^a} = 0. \tag{2.10}$$

3 The definitions of deceleration by HS and their physical meaning

Hirata and Seljak [2] give in their paper five alternative definitions of the deceleration parameter, three of which we briefly recall here for reference. In all cases it is assumed that the spacetime is filled with a perfect fluid obeying Einstein's equations with zero cosmological constant. HS assume that the fluid moves geodesically, but for the beginning we shall drop this assumption.

Their first definition of the deceleration parameter is:

$$q_1 \stackrel{\text{def}}{=} -1 - u^\mu H_{1;\mu}/{H_1}^2, \quad H_1 \stackrel{\text{def}}{=} u^\mu{}_{;\mu}/3, \tag{3.1}$$

where H_1 is the (local) Hubble parameter and u^μ is the velocity field of the fluid. Then, using these definitions, the Raychaudhuri equation may be rewritten as follows:

$${H_1}^2 q_1 = \frac{1}{3}\left[-\dot{u}^\gamma{}_{;\gamma} + \sigma^{\mu\nu}\sigma_{\mu\nu} - \omega^{\mu\nu}\omega_{\mu\nu}\right] + \frac{4\pi G}{3}(\rho + 3p), \tag{3.2}$$

where $\dot{u}^\mu$ is the acceleration of the fluid, $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are the shear and rotation tensors, respectively, ρ is the mass density and p is the pressure. It follows that with $\dot{u}^\mu = 0 = \omega_{\mu\nu}$ and $\rho + 3p \geq 0$, q_1 must be non-negative. This result applies in particular to the L–T models (where $\dot{u}^\mu = 0 = \omega_{\mu\nu}$ and $p = 0$) and to the FLRW models (where $\dot{u}^\mu = 0 = \omega_{\mu\nu} = \sigma_{\mu\nu}$).

The definition (3.1) is invariant, applies in all geometries, is a generalization of the deceleration parameter long used in the FLRW geometries, and measures the relative change of the expansion scalar along the flow lines of the perfect fluid. With $q_1 \geq 0$ we have $u^\mu H_{1;\mu} \leq -{H_1}^2 < 0$, so the expansion slows down toward the future of the comoving observers. This is the case in all cosmological models that obey the assumptions listed above.

The quantities $q_{3'}$ and $H_{3'}$ refer to a Taylor expansion of the luminosity distance-redshift relation $z(D_L)$:

$$z = H_{3'} D_L - \tfrac{1}{2} {H_{3'}}^2 (1 - q_{3'}) {D_L}^2 + \mathcal{O}({D_L}^3), \tag{3.3}$$

where $H_{3'}$ and $q_{3'}$ are functions of the angle of observation. The definitions of H_4 and q_4 follow by averaging (3.3) over the full solid angle 4π at each fixed D_L:

$$\langle z \rangle_{4\pi} = H_4 D_L - \frac{1}{2} {H_4}^2 (1 - q_4) {D_L}^2 + \mathcal{O}({D_L}^3). \tag{3.4}$$

These equations have the same algebraic form as the corresponding equation in an FLRW model.

4 Problems with HS's equation (38)

By considering a bundle of null geodesics converging to a point in space, Hirata and Seljak [2] derived an equation relating the quantities in (3.4) to the invariants of flow of a perfect fluid filling the spacetime. Flanagan's (hereafter F, [18]) equation is more general, and it reads, in HS's notation,

$${H_4}^2 q_4 = \frac{4\pi}{3}(\rho + 3p) + \frac{1}{3}\left[\dot{u}_\alpha \dot{u}^\alpha + \frac{7}{5}\sigma_{\alpha\beta}\sigma^{\alpha\beta} - \omega_{\alpha\beta}\omega^{\alpha\beta} - 2{\dot{u}^\alpha}_{;\alpha}\right]. \tag{4.1}$$

(HS's result is the subcase $\dot{u}^\alpha = 0 = \omega_{\alpha\beta}$.)

Equation (4.1) has the appearance of being exact and covariant. Unfortunately, it is neither. As we show below, its derivations, both by HS and by Flanagan, contain approximations (some not explicitly spelled out) and involve averaging over directions, which produces coordinate-dependent results at locations corresponding to coordinate singularities. Approximations may be good for some purposes, but not when one intends to show that some quantity is positive.

Below we note instances where HS and F introduced approximations in their reasoning.

4.1 HS's equation (26)

We have no objections to the reasoning of HS up to their eq. (25). Having established the relation between the luminosity distance D_L and the redshift z (their (24)), and between the radiation amplitude and the expansion scalar θ_n of the radiation field (their (25)), HS then copy a relation between θ_n ($\hat{\theta}$ in HS's notation) and the affine parameter v from Ref. [19] (HS's eq. (26)). This requires a comment.

To obtain this relation one considers another family of rays, originating at the centre of a source (call it a star for brevity). The star is assumed spherical and having the radius R. One of the rays hits the observer. The values of the affine parameter along this ray are $v = 0$ at the observer, v_s at the surface of the star and $(v_s + \Delta v_s)$ at the centre of the star. The relation is then

$$\theta_n = \frac{2}{v - (v_s + \Delta v_s)} + \mathcal{O}[v - (v_s + \Delta v_s)]. \tag{4.2}$$

In Ref. [19] this is obtained as an approximate solution of the Raychaudhuri equation for null geodesics, which is

$$k^\gamma \theta_{n,\gamma} + 2({\sigma_n}^2 - {\omega_n}^2) + \frac{1}{2}{\theta_n}^2 = -R_{\rho\gamma} k^\rho k^\gamma, \tag{4.3}$$

where $\theta_n = {k^\mu}_{;\mu}$, σ_n and ω_n are, respectively, the expansion, shear and rotation of the null congruence $k^\mu = \mathrm{d}x^\mu/\mathrm{d}v$ and $R_{\rho\gamma}$ is the Ricci tensor in spacetime.

Equation (4.2) without the second term on the right is the exact solution of (4.3) in the case $\sigma_n = \omega_n = 0 = R_{\rho\gamma} k^\rho k^\gamma$. Thus the approximation involved in (4.2) is that the contributions to $k^\gamma \theta_{,\gamma}$ from shear, rotation and lensing by matter are negligible compared to ${\theta_n}^2$. These are additional assumptions about light propagation in spacetime (σ_n and ω_n negligible) and about the spacetime itself ($R_{\rho\gamma} k^\rho k^\gamma$ negligible).

However, HS's (38) is used by VFW as if it were exact. When $\sigma_n = 0 = R_{\rho\gamma} k^\rho k^\gamma$ hold exactly, the Goldberg–Sachs theorem [20] applies, which says that the spacetime must be algebraically special, with k^μ being the double principal null congruence of the Weyl tensor. The assumption $R_{\rho\gamma} k^\rho k^\gamma = 0$, when imposed on the L–T spacetime, reduces it to the Schwarzschild solution. Therefore, HS's equation (38), when treated as exact, involves strong assumptions that don't apply to the L–T model, or even the FLRW model.

4.2 The two errors that cancelled each other

The two errors shown below have no meaning for the final result because (in the approximation in which HS worked) they cancelled out in the end, but they might be confusing. HS's equation (27) should actually read

$$\Delta v_s = -\frac{R}{1+z}, \tag{4.4}$$

i.e. it is the inverse of what they wrote. Similarly, their equation (28) should actually read

$$\frac{A(v_s)}{A(0)} = 1 + \frac{v_s}{\Delta v_s} \tag{4.5}$$

(where A is the radiation amplitude), i.e. again the inverse of their result. Since HS assumed $|\Delta v_s| \ll |v_s|$, the above is approximately equal to

$$\frac{A(v_s)}{A(0)} = \frac{v_s}{\Delta v_s} = -v_s \frac{1+z}{R}, \tag{4.6}$$

which results in HS's eqs. (29)–(31) being right again. Since we will later refer to their (31), we copy it here:

$$z = -K_{ij}n^i n^j D_L - 2\left(K_{ij}n^i n^j\right)^2 (D_L)^2 + \frac{1}{2}\left(\dot{K}_{ij} + K_{ij|k}n^k + 4{K_i}^k K_{kj}\right) n^i n^j (D_L)^2 + \mathcal{O}({D_L}^3). \tag{4.7}$$

The meaning of the symbols in (4.7) is as follows. Since HS work under the assumption that $\omega = 0$ for the cosmic fluid, comoving and synchronous coordinates exist in which the metric $g_{\alpha\beta}$ has the properties $g_{00} = 1$, $g_{0i} = 0$, where $x^0 = t$ is the time coordinate and x^i, $i = 1, 2, 3$, are the space coordinates. Then $h_{ij} = -g_{ij}$ is the metric of a space of constant t, and K_{ij} is the second fundamental form of this space; in these coordinates $K_{ij} = -(1/2)(\partial/\partial t)h_{ij} \overset{\text{def}}{=} -(1/2)\dot{h}_{ij}$. Then the geodesic null vector k^α can be normalised at the observer so that $k^0 = 1$ and its space components $k^i \overset{\text{def}}{=} n^i$ form a 3-dimensional unit vector, $h_{ij}n^i n^j = 1$.

4.3 The problem with averaging

There is a problem with averaging products like $n^i n^j$ and $n^i n^j n^k$ at coordinate singularities in space, and the centre of symmetry in L–T is a singularity of the spherical coordinates. For example, a general unit vector in Euclidean space attached at a point $\mathcal{O}$, when referred to Cartesian coordinates and parametrised by spherical angles (α, β), has the components $(n^x, n^y, n^z) = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$. In this case, the averages[2] are $\langle n^i n^j\rangle_{4\pi} = (1/3)\delta^{ij}$, $\langle n^i n^j n^k\rangle_{4\pi} = 0$, $\langle n^i n^j n^k n^l\rangle_{4\pi} = (1/5)\delta^{(ij}\delta^{kl)}$. The first and last of these are special cases (in Euclidean space) of the covariant equations

$$\langle n^i n^j\rangle_{4\pi} = (1/3)h^{ij}, \tag{4.8}$$

$$\langle n^i n^j n^k\rangle_{4\pi} = 0, \tag{4.9}$$

$$\langle n^i n^j n^k n^l\rangle_{4\pi} = (1/5)h^{(ij}h^{kl)} \equiv (1/15)\left(h^{ij}h^{kl} + h^{ik}h^{jl} + h^{il}h^{jk}\right). \tag{4.10}$$

However, these results of averaging change when we transform to spherical coordinates centred at $\mathcal{O}$. When the components of the unit vector n^i in Euclidean space are referred to the spherical coordinates, they are $(n^1, n^2, n^3) = (1, 0, 0)$ (with $(x^1, x^2, x^3) = (r, \alpha, \beta)$), and then the averages are $\langle n^i n^j\rangle_{4\pi} = \delta^i_1\delta^j_1$, $\langle n^i n^j n^k\rangle_{4\pi} = \delta^i_1\delta^j_1\delta^k_1$. Explicit calculation done in Sect. 5 for the L–T model will show that the general formulae

[2] The averages are defined by

$$\langle n^i n^j\rangle_{4\pi} = \frac{1}{4\pi}\int_0^\pi \mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\beta n^i n^j \sin\alpha,$$

and similarly for $< n^i n^j n^k >_{4\pi}$.

(4.8)–(4.10), in particular (4.9), lead to incorrect results when applied at the centre of symmetry.

Since (4.1) is supposed to follow from (4.7) by this kind of averaging, the argument above invalidates (4.1) at the centre of symmetry of the L–T model even as an approximate equation.

5 The sign of $q_{3'}$ can be any

We will now use (4.7) to estimate HS's deceleration parameter $q_{3'}$ at the centre of the L–T model without averaging over directions. Identifying the coefficient of D_L in (4.7) with the $H_{3'}$ of (3.3) we find from (3.3):

$$q_{3'} = 1 + \frac{1}{{H_{3'}}^2}\left[-4\left(K_{ij}n^in^j\right)^2 + \left(\dot{K}_{ij} + 4{K_i}^k K_{kj}\right)n^in^j + K_{ij|k}n^in^jn^k\right]. \tag{5.1}$$

In the coordinates of (2.1) we have, with $(x^1, x^2, x^3) = (r, \vartheta, \varphi)$:

$$(h_{11}, h_{22}, h_{33}) = \left(\frac{{R_{,r}}^2}{1+2E}, R^2, R^2\sin^2\vartheta\right), \tag{5.2}$$

$$(K_{11}, K_{22}, K_{33}) = \left(-\frac{R_{,r}R_{,tr}}{1+2E}, -RR_{,t} - RR_{,t}\sin^2\vartheta\right), \tag{5.3}$$

$$K_{11|1} = \frac{R_{,rr}R_{,tr} - R_{,r}R_{,trr}}{1+2E}, \tag{5.4}$$

$$K_{12|2} = K_{22|1} = \frac{K_{13|3}}{\sin^2\vartheta} = \frac{K_{33|1}}{\sin^2\vartheta} = R_{,t}R_{,r} - RR_{,tr}, \tag{5.5}$$

the components not listed are zero.

At a general point of an L–T manifold, the unit spatial vector n^i must obey $h_{ij}n^in^j = 1$, and so, in consequence of (5.2), its components are constrained by

$$n^1 = \pm\frac{\sqrt{1+2E}}{R_{,r}}\sqrt{1 - R^2\left[\left(n^2\right)^2 + \sin^2\vartheta\left(n^3\right)^2\right]}. \tag{5.6}$$

At the centre of symmetry each vector n^i is necessarily radial, so $n^2 = n^3 = 0$, and (5.6) gives

$$n^1(0) = \frac{\sqrt{1+2E}}{R_{,r}}, \tag{5.7}$$

which is finite by (2.9) and (A.4).

However, (5.6) and (5.7) imply that at the centre of symmetry the angle-average of (4.7) is discontinuous. Namely, as long as the vector n^i is attached off the centre, its $n^1 = n^r$ component may point in the direction of increasing r (in which case we take

(5.6) with the $+$ sign) or toward decreasing r (in which case we take $-$ in (5.6)). Then the average comes out zero. But when the vector n^i is attached at the centre of symmetry, n^1 may point only toward increasing r, so $n^1 > 0$ on the whole sphere, while $n^2 = n^3 = 0$. Thus, $(n^1)^3 > 0$ everywhere on the sphere, while all other components of $n^i n^j n^k$ are zero. Consequently, $\langle (n^1)^3 \rangle_{4\pi} > 0$, while all the remaining averages are zero. Thus, in the spherical coordinates, $\langle n^i n^j n^k \rangle_{4\pi}$ is discontinuous at the centre of symmetry.

Let us apply (5.1) at the centre of symmetry in the L–T model. Using (5.2)–(5.7) and

$$H_{3'} = -K_{ij} n^i n^j \underset{r \to 0}{\longrightarrow} \lim_{r \to 0} \left(\frac{R_{,tr}}{R_{,r}} \right) \tag{5.8}$$

in (5.1) we find

$$q_{3'}(0) = \lim_{r \to 0} \left\{ -\frac{R_{,r} R_{,ttr}}{{R_{,tr}}^2} + \sqrt{1+2E} \left(\frac{R_{,rr}}{R_{,r} R_{,tr}} - \frac{R_{,trr}}{{R_{,tr}}^2} \right) \right\}. \tag{5.9}$$

In order to clearly see the contribution of the inhomogeneous part of the geometry, we will separate R and E into the part that survives in the Friedmann limit and the part that disappears in that limit. Note that the whole calculation will be exact, and our definitions for M, E and R will ensure they automatically obey the conditions of regularity at the centre (2.9).

We define:

$$R = r(S + \mathcal{P}), \quad \lim_{r \to 0} \mathcal{P} = 0, \tag{5.10}$$

$$E = r^2(-k/2 + F), \quad \lim_{r \to 0} F = 0, \tag{5.11}$$

where k is a constant (the familiar FLRW curvature index, so it may be of any sign, also zero) and $S(t)$, $F(r)$ and $\mathcal{P}(t, r)$ are functions (S being the FLRW scale factor). The terms $r\mathcal{P}$ and $r^2 F$ represent non-Friedmannian contributions to R and E, respectively; with $\mathcal{P} = F = 0$ the L–T model reduces to that of Friedmann. In calculating the limits at $r \to 0$ we will assume that the derivatives of F and t_B do not diverge too fast, that is

$$\lim_{r \to 0} \left[r \left(F_{,r}\, , {F_{,r}}^2, F_{,rr}\, , t_{B,r}, t_{B,rr} \right) \right] = (0, 0, 0, 0, 0). \tag{5.12}$$

Since some of the functions will be complicated, we introduce the following abbreviations:

$$V \overset{\text{def}}{=} \sqrt{\frac{2M_0}{S + \mathcal{P}} - k + 2F}, \tag{5.13}$$

$$\mathcal{L} \overset{\text{def}}{=} \frac{2(S + \mathcal{P})F_{,r}}{k - 2F} - V \left[\frac{3F_{,r}}{k - 2F} (t - t_B) + t_{B,r} \right] \tag{5.14}$$

(it can be verified that $\mathcal{L} = \mathcal{P}_{,r}$), and the subscript 0 will denote the limit at $r \to 0$, thus

$$V_0 = \sqrt{\frac{2M_0}{S} - k}, \tag{5.15}$$

$$\mathcal{L}_0 = \frac{2SF_{,r}(0)}{k} - V_0 \left[\frac{3F_{,r}(0)}{k} (t - t_B(0)) + t_{B,r}(0) \right]. \tag{5.16}$$

The formulae for the derivatives of R and their limits at $r \to 0$ are given in Appendix A. We see from there that all the terms entering $q_{3'}$ in (5.1) have nonzero and nondivergent limits at $r \to 0$.

For $H_{3'}$ we get from (5.8):

$$\lim_{r \to 0} H_{3'} = \frac{V_0}{S}. \tag{5.17}$$

Using (5.2)–(5.17) and (A.1)–(A.8) in (5.9) we now obtain:

$$\lim_{r \to 0} q_{3'} = \frac{M_0}{S{V_0}^2} - \frac{2F_{,r}(0)}{{V_0}^3} \\ - \frac{3M_0/S - k}{S{V_0}^2} \left\{ 2t_{B,r}(0) + \frac{2F_{,r}(0)}{kV_0} \left[-2S + 3V_0 (t - t_B(0)) \right] \right\}. \tag{5.18}$$

The terms proportional to $F_{,r}(0)$, $t_{B,r}(0)$ and $F_{,r}(0)/k$ can each have any sign, and so each one can make $q_{3'}(0)$ negative. Note that $F_{,r}(0)$ (which comes from the non-Friedmannian contribution to the energy function) alone can cause $q_{3'}(0) < 0$ (when $t_{B,r} = 0$), and so can $t_{B,r}(0)$ alone (when $F = 0$).

At the centre of spherical symmetry, $q_{3'} = q_3 = q_4$. This confirms that HS's result is not correct at $r = 0$, and a negative q_4, as found by INN, is perfectly possible.

The reason why HS concluded that $q_4 \geq 0$ is that they found $\langle K_{ij|k} n^i n^j n^k \rangle_{4\pi} = 0$ while averaging (5.1) which is not true at a spherical centre in spherical coordinates. In consequence, they dismissed all terms in (5.18) which come from $K_{ij|k} n^i n^j n^k$, i.e. all except the first one. With only the first term, one obviously gets $\lim_{r \to 0} q_{3'} \geq 0$.

Since nothing can be concluded in general about the sign of $q_{3'}(0)$, there would be no contradiction involved in certain L–T models imitating a negative FLRW deceleration parameter even if Eq. (5.1) were exact. But we recall: this equation being approximate, using it to evaluate the sign of $q_{3'}$ cannot lead to an unambiguous conclusion, so the alleged contradiction was a nonexistent problem.

There is one more piece of evidence that HS's method of averaging is not universally correct. Suppose (5.1) is applied at the centre of symmetry of an L–T model. Then each component of the sum in (5.1) is invariant under the group $SO(3)$. Thus, averaging over directions should be an identity operation and change nothing. This is indeed the case when one uses (5.3)–(5.5) for K_{ij} and $K_{ij|k}$, applied at $r = 0$, and (5.7) for $n^i(0)$; actually one gets (5.17)–(5.18) again. However, if one uses HS's prescription for averaging, then the whole term $\langle K_{ij|k} n^i n^j n^k \rangle(0)$ still drops out in spite of being spherically symmetric and nonzero initially.

6 Problems with Flanagan's eq. (5)

Flanagan's paper [18] addressed the question of whether superhorizon perturbations can cause apparent acceleration. He concluded that if their effect on q is negative, it must be too small to *be responsible for the acceleration of the universe.* The paper was not intended to be applied to exact inhomogeneous cosmological models, so the author should not be blamed for the incorrect use made later of his equation. But the HS subcase of that approximate equation was used by VFW as if it were exact.

In this section we follow F's reasoning in order to identify the approximations assumed along the way.

6.1 Comments to F's reasoning up to eq. (13)

The correct form of Flanagan's eq. (9) is:

$$\overline{u}^{\alpha}(x, x') = u^{\alpha}(x) + u^{\alpha\beta}(x)\sigma_{;\beta}(x, x') + \frac{1}{2}u^{\alpha\beta\gamma}(x)\sigma_{;\beta}(x, x')\sigma_{;\gamma}(x, x') + O(s^3). \tag{6.1}$$

The meaning of the symbols is as follows: x is the set of coordinates of the observer at the point P, x' is the set of coordinates of the light source at the point Q, $u^{\alpha}(x)$ is the four-velocity of the cosmic medium at P (the observer is comoving), $\overline{u}^{\alpha}(x, x')$ is the four-velocity of the medium parallel-transported from Q to P along the light ray, $\sigma(x, x')$ is Synge's [21] world function for P and Q, and the coefficients are $u^{\alpha\beta}(x) = -u^{\alpha;\beta}(x)$, $u^{\alpha\beta\gamma}(x) = u^{\alpha;(\beta\gamma)}(x)$.

The quantity in (6.1) is apparently only needed to calculate the redshift, which F writes as

$$1 + z = \frac{\overline{u}^{\alpha}k_{\alpha}}{u^{\alpha}k_{\alpha}}, \tag{6.2}$$

where k_{α} is the tangent vector to the ray at P. Usually the numerator is taken simply at Q, without parallel-transporting it to P [7]. Nevertheless, it is easy to verify that (6.2) is equal to the usual formula.

F then considers two families of light rays: one converging at P, with the affine parameter s and tangent vectors k normalised so that $k_{\alpha}u^{\alpha} = -1$ at P, and another one diverging from Q, with the affine parameter $\overline{\lambda}$ and tangent vectors l normalised so that $l_{\alpha}u^{\alpha} = -1$ at Q. In consequence of this, and of (6.2), the two affine parameters are related by

$$\overline{\lambda} = s(1 + z). \tag{6.3}$$

6.2 The θ_n approximation

Below his (13), F writes: *We choose the normalization of A so that* $A \approx 1/\overline{\lambda}$ *for* $\overline{\lambda} \to 0$ *near Q*. This is an assumption equivalent to HS's (26)–our (4.2) (A is the same quantity as in (4.5)), and F makes use of it further on (see below our Eq. (6.7)). This *normalization* also defines the units for $\overline{\lambda}$ as the units of distance.

6.3 F's equation (14)

In the paragraph of F's paper that contains eqs. (14)–(16) the quantities referring to P and to Q are mixed up. The text below is our interpretation of this segment of F's reasoning, in which we take care about properly distinguishing between these two points.

From the definition of the energy flux at P one obtains:

$$\begin{aligned}\frac{\mathrm{d}E_P}{\mathrm{d}t\, d^2 A} &= T_{\alpha\beta}(P) u^{\alpha}_P u^{\beta}_P = {A_P}^2 \left(l_\alpha u^\alpha\right)^2_P \\ &= {A_P}^2 \left(k_\alpha u^\alpha\right)^2_P /(1+z)^2 = {A_P}^2/(1+z)^2, \end{aligned} \tag{6.4}$$

whereby a rewriting error was corrected above: F's $(k_\alpha u^\alpha)^2$ should read $(l_\alpha u^\alpha)^2$. The following information was fed into (6.4) at the consecutive equality signs: the definition of energy flux, the definition of the radiation energy-momentum tensor at Q, Eq. (6.2), the normalization of k^α at P.

Integrating the energy flux *at Q* over a sphere of small radius $\overline{\lambda}_Q$, one obtains for the luminosity at Q, again from the definition:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = {A_Q}^2 \left(l_\alpha u^\alpha\right)^2_Q \times 4\pi {\overline{\lambda}_Q}^2 = 4\pi {A_Q}^2 {\overline{\lambda}_Q}^2; \tag{6.5}$$

the simplification occurs because of the normalization assumed at Q. Putting (6.2) and (6.3) into the definition of the luminosity distance (which we denote here by D_L for consistency with the previous sections) one obtains

$$A_Q \overline{\lambda}_Q = \frac{D_L A_P}{1+z}. \tag{6.6}$$

In the next step F uses the assumed normalization of A_Q to write the above as $D_L = (1+z)/A$, but then gives up on it and uses (6.3) to rewrite (6.6) as follows:

$$D_L = (1+z)^2 s_Q A_Q / A_P = (1+z)^2 s_Q / (A_P \overline{\lambda}_Q) \tag{6.7}$$

(the second equality follows by the assumed normalization of A at Q). Then F cites Visser [22] for the result

$$A_P \overline{\lambda}_Q = 1 + O({s_Q}^2). \tag{6.8}$$

Thus finally, up to s^2-terms:

$$D_L \approx (1+z)^2 s_Q, \tag{6.9}$$

which is the same equation as HS's (29). Just as in the HS paper, this equation is approximate, where in addition to HS's (26) (our (4.2)), Flanagan used one more approximation—our (6.8).

From (6.9) and from eqs. (1) and (11) in F's paper, F's equations (15) and (16) follow by simple substitution.

6.4 Choice of averaging procedure

Below his eq. (16), Flanagan notes that there is no unique way to choose the order in which the coefficients in the $D_L(z)$ equation should be averaged over directions, but dismisses this problem by saying:
Thus the different averaging prescriptions give different answers. However the fractional differences are of order σ^2/θ^2, which we have argued above is of order ε and is small.
This is correct in the context of his paper. However, this indicates that there is a further approximation involved in the reasoning, and this one is quite arbitrary and unpredictable. Namely, it depends on the will of the person doing the calculation exactly which quantity he or she wishes to average first. F himself uses $H_0 = \langle A^{-1} \rangle$ to obtain the H_0 of his eq. (4), but $H_0^{-1} = \langle A \rangle$ for computing his J. This is perhaps the strongest indication that it makes no sense to use the resulting final equation for determining the sign of any quantity. Note that the same problem exists for the HS derivation, but it was not mentioned in the HS paper.

6.5 The problem with rotation

The averaging over directions is defined only in a certain 3-space S_3, not in spacetime. Without fixing S_3, the directional angles of the light rays are not defined. When the rotation of the cosmic fluid is zero, as in HS's paper, S_3 is the 3-space orthogonal to the fluid's flow lines.

However, when the cosmic fluid has nonzero rotation, such S_3 do not exist—not even locally, because the 3-volume elements locally orthogonal to the flow lines, when followed around any flow-tube, refuse to connect up to a 3-space. One could consider a 3-space of constant time-coordinate, which is not orthogonal to the flow lines, but then the metric in this space is not the $h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ assumed by F. Thus, in the presence of rotation, the whole calculation should be reformulated. Without that, for the averaging over angles considered by F there exists no space in which it occurs.

7 VFW's *weak singularity* is not a singularity

In the paragraph containing their eq. (2.18) VFW say:
We expand the density (2.5) to second order in r as

$$\rho(r,t) = \rho_0(t) + \rho_1(t)r + \rho_2(t)r^2 + \mathcal{O}(r^3). \tag{2.18}$$

The weak singularity occurs when $\rho_1(t)$ is nonzero, in which case the gravitational field is singular since $\Box\mathcal{R} \to \pm\infty$ as $r \to 0$, where $\mathcal{R}$ is the Ricci scalar. In other words, second derivatives of the density diverge at the origin, independent of where observers may be located. This is true both in flat spacetime and in the curved LTB metric when we have a density profile of the form (2.18). The singularity is weak according to the classification scheme of the literature on general relativity [28]. (Ref. [28] in this quotation is [6].)

In truth, whether a curvature singularity is there is decided by curvature alone, and not by secondary constructs like the d'Alembertian of the curvature scalar. We do not know about any physical or geometrical interpretation of this quantity, and VFW do not mention any, nor have they shown that their 'weak singularity' causes any problems at the origin. As we show below, the curvature of the L–T model at the centre of symmetry is nonsingular provided that ρ is finite there.

For the metric (2.1), the orthonormal tetrad components of the curvature tensor, in the basis defined by $e^0 \overset{\text{def}}{=} \mathrm{d}t$, $e^1 \overset{\text{def}}{=} R_{,r}\mathrm{d}r/\sqrt{1+2E}$, $e^2 \overset{\text{def}}{=} R\mathrm{d}\vartheta$, $e^3 \overset{\text{def}}{=} R\sin\vartheta\mathrm{d}\varphi$, are

$$\begin{aligned} R_{0101} &= \frac{2M}{R^3} - \frac{M_{,r}}{R^2 R_{,r}}, \qquad R_{0202} = R_{0303} = -\frac{M}{R^3}, \\ R_{1212} &= R_{1313} = \frac{M}{R^3} - \frac{M_{,r}}{R^2 R_{,r}}, \qquad R_{2323} = -\frac{2M}{R^3}. \end{aligned} \tag{7.1}$$

The quantities given above are all scalars, so any scalar polynomial in the curvature components will be a polynomial in the quantities given in (7.1). If these are nonsingular, then there will be no scalar polynomial curvature singularity.

Let $r = r_c$ be the radial coordinate corresponding to the centre of symmetry $R = 0$. The requirement of no point-mass at the centre implies $M(r_c) = 0$. It can be seen that with $M(r_c) = R(t, r_c) = 0$ all we need to make the curvature nonsingular at $r = r_c$ is a finite limit of M/R^3 at $r = r_c$. We have, using (2.4)

$$\lim_{r\to r_c} \frac{M}{R^3} = \lim_{r\to r_c} \frac{M_{,r}}{3R^2 R_{,r}} = (4/3)\pi G\rho(t, r_c), \tag{7.2}$$

so the curvature is finite at $r = r_c$ if and only if $\rho(t,r)$ is.

As regards geodesic completeness, the geodesic equation is well behaved through the origin. Geodesics passing through the origin must be purely radial, and radial geodesics stay radial. The only Christoffel symbols that diverge at $R = 0$ are $\Gamma^{\vartheta}{}_{r\vartheta} = \Gamma^{\varphi}{}_{r\varphi} = R_{,r}/R$, but these do not appear in the equations of radial geodesics, so cannot cause any problem.

The condition $\Box\mathcal{R} \rightarrow \pm\infty$ is not a criterion for a singularity in any accepted sense. The paper of Tipler [6] contains a proposal of a definition of a *strong singularity*, but does not mention a 'weak singularity' anywhere and contains no 'classification scheme'. For a singularity to be 'weak' (i.e. not strong), it first has to be a singularity. An accurate description of the feature that VFW had in mind is 'the density profile is not C^1 through the origin'.

Therefore, VFW's *weak singularity* is not a singularity, so its presence is no reason to dismiss models containing it. There is nothing wrong with a jump in the density gradient at the centre. Since our Galaxy is centrally concentrated and has a central black hole, a pointed density profile is a good approximation. The NFW density profile [23] for galaxy clusters is divergent at the centre. A jump in the density and its gradient exists, for example, at the surface of the Earth, thus we deal with a 'weak singularity' in everyday life.

8 Transcritical solutions are generic

A large part of VFW's paper is devoted to considering the *inverse problem* and arguing that only FLRW models have rays that cross the apparent horizon—what they call the *critical point*. (Their problem is inverse to the 'direct problem' first considered in the literature in Ref. [24].) We here dispel the impression they convey, that *transcritical* solutions are well-nigh impossible to find for non-homogeneous models that account for observations. We emphasise (a) that all L–T models with a decelerating expansion phase have apparent horizons that are crossed by large families of light rays, and (b) no cosmological model has been shown to solve the inverse problem (as distinct from the fitting problem) relative to real data.

VFW correctly point out that the differential equations (DEs) of the *inverse problem*, i.e. the DEs determining the L–T model that reproduces given observational data, contain terms that would diverge unless a certain condition is met. They call such loci *critical points*, and they correctly claim that such points are generic to these DEs. This was actually pointed out by Mustapha, Hellaby and Ellis [25], solved by Lu and Hellaby [10], discussed as an observational feature in [11], and used to advantage by McClure and Hellaby [12]. These papers develop a numerical procedure for extracting the metric of the cosmos from observational data.[3]

In their analysis of the inverse problem, VFW consider only flat L–T models, which are a subset of measure zero in the family of all possible L–T models. They say that FLRW models provide examples of *transcritical* solutions (i.e. those that do not cause any divergence of the DEs), but they fail to find any other viable solution.[4] They then conclude that physically reasonable solutions *must be very exceptional indeed*, and although they have not tried the $E \neq 0$ case, they opine that *generic solutions with*

[3] This problem persists even in the fluid-ray coordinates (observer coordinates): in [29], the differential equations (59)–(65) contain $\partial C/\partial z$ in the denominator, and on the observer's past null cone $w = w_0$, this is $\partial \widehat{R}/\partial z$ here, so it is again zero at the apparent horizon.

[4] An L–T model is fully specified by two physical functions. We agree that a random combination such as $E = 0$ and an arbitrarily chosen $D_L(z)$, specified via $r_{FRW}(z)$ or $V(z)$, may well produce an L–T model with unrealistic features.

$E(r) \neq 0$ would have all the *singularities* and *observational problems* they encountered, and in this case too transcritical solutions do *not appear to be likely.*

In assessing the likelihood of finding a 'transcritical solution' it is important to understand the physics and geometry of the problem. For any *given* cosmological metric, the DEs that determine the path of the light ray and the variation of z and $D_A(z)$ along it (for example (2.5), (8.2), (2.6)) are free of critical points. However, the path followed by an incoming light ray depends on the geometry it passes through, and this affects the observations, z, $D_A(z)$ etc. All known expanding cosmologies with a deceleration phase have a past apparent horizon (AH^-); relative to a given worldline (observer), it is the locus where incoming light rays overcome the cosmic expansion and start to make progress inwards. The maximum in the area distance $D_{Am} = \widehat{R}_m$ occurs where an observer's PNC crosses her past apparent horizon. The fact that $\mathrm{d}\widehat{R}/\mathrm{d}z = 0$ at this maximum is the cause of the critical behaviour. This behaviour is a generic feature of the L–T and Friedmann models. We express this in the following theorems. We consider only models that: (i) have a big bang and after 14 billion years are still expanding, so any recollapse is to the future, and times well beyond the present are not considered; (ii) have no shell crossings [26]; (iii) are large, e.g. if the curvature is positive, then any spatial maximum where $M_{,r} = 0 = R_{,r}$ is well beyond observational range, which means $M_{,r} > 0$ and $R_{,r} > 0$; (iv) if $\Lambda > 0$ its effect is only discernable in fairly distant observations. We will call such a model an rLTc: a realistic L–T cosmology.

Theorem 1 *In every rLTc, every light ray arriving at the origin is a transcritical ray.*

Proof The areal radius $R(t, r)$ obeys

$$R_{,t} = \ell\sqrt{\tfrac{2M}{R} + 2E + \tfrac{\Lambda R^2}{3}}, \tag{8.1}$$

where $\ell = +1$ if R is increasing (expansion) and -1 if it is decreasing (contraction). Worldlines with $2E \geq -(9M^2\Lambda)^{1/3}$ are ever-expanding and the rest are recollapsing. Those that recollapse reach their maximum R where $6M + 6ER + \Lambda R^3 = 0$ and this maximum is $\leq (3M/\Lambda)^{1/3}$.

The incoming radial light rays satisfy (2.5), so the variation of R along a ray is

$$\begin{aligned}\frac{\mathrm{d}\widehat{R}}{\mathrm{d}r} &= \left[R_{,t}\frac{\mathrm{d}\widehat{t}}{\mathrm{d}r} + R_{,r}\right]_\wedge = \left(-\ell\frac{\sqrt{2M/\widehat{R} + 2E + \Lambda\widehat{R}^2/3}}{\sqrt{1+2E}} + 1\right)\widehat{R_{,r}} \\ &= \frac{-\widehat{R_{,r}}}{\sqrt{1+2E}}\frac{\mathrm{d}\widehat{R}}{\mathrm{d}t}.\end{aligned} \tag{8.2}$$

The past apparent horizon is the locus where $\mathrm{d}\widehat{R}/\mathrm{d}r = 0$ for all rays, which implies

$$6M = 3R - \Lambda R^3 \tag{8.3}$$

and the past apparent horizon has $\ell = +1$. For a given $M > 0$, (8.3) has either two, one or zero solutions in the range $R > 0$. The smaller R solution corresponds to the regular ($\Lambda = 0$) AH, and the larger one corresponds to the de Sitter horizon, but the two merge on the $3M\sqrt{\Lambda} = 1$ worldline.

Near the origin, $R(t, 0) = 0\ \forall t$, regularity [16,27,28,30] requires $R \sim M^{1/3}$ and $2E \sim M^{2/3}$, so by (8.1) $R_{,t} \to 0$ and by (8.2) $\mathrm{d}\widehat{R}/\mathrm{d}r \to R_{,r}$ there. This means $6M < 3R - \Lambda R^3$ for incoming rays near the origin. Following the rays outwards and back in time, (2.5) shows r and therefore M is increasing, and (8.2) shows $\widehat{R}$ is increasing near the origin. Provided $R_{,r}$ stays positive as assumed, (2.5) may be integrated all the way to the big bang; and here we have $M > 0$ and $R \to 0$, so $6M > 3R - \Lambda R^3$. Therefore the rays have crossed AH$^-$. This argument applies to every ray arriving at the origin before the big crunch, if there is a crunch.

The DEs for the general L–T 'inverse problem' are given in [10,12], and it is evident that the critical points are where $\mathrm{d}\widehat{R}/\mathrm{d}r = 0 = \mathrm{d}\widehat{R}/\mathrm{d}z$. If a ray crosses the past AH, then by definition it (a) has $\mathrm{d}\widehat{R}/\mathrm{d}r = 0$ there, and therefore (b) it is transcritical. □

If $\Lambda > 0$ then there could[5] be worldlines with $M > 1/(3\sqrt{\Lambda})$. These worldlines never encounter a solution to (8.3), so every incoming ray crossing them has $\mathrm{d}\widehat{R}/\mathrm{d}r < 0$ and $\mathrm{d}\widehat{R}/\mathrm{d}t > 0$. In ever-expanding models, AH$^-$ asymptotically approaches $R = \sqrt{3/\Lambda}$ (and $M = 0$) towards the future, and it separates incoming light rays that always have increasing R from those that reach the origin. Therefore there are rays emerging from the bang that neither cross AH$^-$, nor reach the origin.

Theorem 2 *Every incoming radial light ray in every $\Lambda = 0$ L–T model with a big bang is a transcritical ray.*

Proof When $\Lambda = 0$, (8.3) becomes the familiar

$$R = 2M \tag{8.4}$$

and such apparent horizons have been well studied in [31,32]. In this case, every worldline encounters AH$^-$, which is at $R > 0$ except at an origin, and every light ray starts on the big bang. Therefore, of the rays emerging from the bang and arriving at the crunch or $R = \infty$, only those starting at an origin may be said to have not crossed the past AH. Models usually have one origin, but closed models normally have two. The rays emerging from the local origin are outgoing, not incoming, and the rays emerging from an antipodal origin may eventually be locally incoming, but in an rLTc this would happen long after the present, if at all. □

Theorem 3 *Transcritical rays of inhomogeneous models that reproduce the observed $D_A(z)$ do exist.*

Proof See e.g. [33,34]. The point here is that if a chosen model produces a $D_A(z)$ and $D_L(z)$ that fits the observations for a given PNC, out past a maximum in D_A, then that PNC is necessarily transcritical. □

Theorem 4 *Any reasonable $D_A(z)$ can be reproduced by a $\Lambda = 0$ L–T model.*

[5] In $E < 0$ models, it is possible the maximum M does not reach $1/(3\sqrt{\Lambda})$.

Proof This is a special case of the theorem in [25], which claims any pair of reasonable observational functions, $D_A(z)$ or $D_L(z)$ and $\mu n(z)$ can be reproduced by an L–T model. Here $n(z)$ is the number density of sources (e.g. galaxies) in redshift space, and μ is the mean mass per source, which may also vary with z. Essentially the two arbitrary functions in an L–T model allow one to reproduce two observational relations. In fact [10–12] have demonstrated how to solve the 'inverse problem' numerically for both observational functions, and shown that the conditions at the AH do impose a significant constraint that holds only at a single z value. But with only $D_L(z)$ given, the two L–T functions allow the observations and the constraint to be fitted easily. □

Together these results show that there is a plentiful supply of transcritical solutions amongst the L–T models, and that there is a well defined procedure for extracting an L–T model that fits the observed $D_A(z)$.

More important, however, is the fact that VFW do not try to find an L–T model that reproduces real observational data, but the one that reproduces a chosen idealised function for $D_L(z)$.

VFW write as if FLRW models are known to solve the *inverse problem*. To date the 'inverse problem' has never even been attempted with real data. What is usually done is to solve the fitting problem: choose your favourite type of cosmological metric and find the free functions and parameters that make it best fit the data.

Even models that are arbitrarily close to reproducing the observations will fail this test. For example, take the 'observational' relations of a given FLRW model, add small random or systematic errors, and use this 'data' as input to the inverse problem. The DEs of the inverse problem will return the given FLRW model quite closely, but (8.3) will not be *exactly* satisfied where $\mathrm{d}\widehat{R}/\mathrm{d}z = 0$, so they will still diverge (see [12]). The chances of an FLRW light ray being exactly transcritical relative to real observational data are distinctly less than for an L–T model.

The failure of a proposed solution to be transcritical does not indicate any physical problem with the underlying L–T or FLRW model; at worst it indicates that a particular ray in a particular model does not (exactly) reproduce the given observational function(s).

As discussed in [10–12,34,35], when attempting to reproduce *two* observational functions with an L–T model, the AH constraints *do* provide a significant challenge. In overcoming this challenge it was shown [11,12] the AH can give us useful observational information.

9 Conclusions

We have corrected a number of inaccurate statements appearing in the HS [2], Flanagan [18] and VFW [1] papers. At least two of these misunderstandings are widespread and therefore it is worth putting the record straight.

The first error-correction to be made is that a conical density profile at the origin is not a singularity in any accepted sense, and there are no physical problems associated with it. In some situations a conical profile may be a good model of structures that are strongly centrally concentrated.

Secondly, critical points in the inverse problem are generic. When solving the 'inverse problem' as defined in Sect. 8 for a realistic L–T or FLRW model, and finding that the differential equations diverge at some redshift, one must be aware that this is an inevitable consequence of the apparent horizon, where the diameter distance is maximum. The chances that real observational data *exactly* satisfy the full transcriticality conditions for any given model, homogeneous or not, are essentially zero. Such divergent behaviour does not indicate any physical problem with the underlying L–T or FLRW model, and transcritical solutions in a variety of models that closely reproduce the observed $D_L(z)$ do exist. Actually this point seems not to be well understood and other examples of this misunderstanding can be found in the literature (see, e.g. [3,36]).

Our point three is that one should be careful when extending FLRW definitions of some functions to other cosmological models. In most such cases, when inhomogeneities are present, the FLRW parameters do not mean the same thing as in the homogeneous case.

It is very common to get acceleration and supernova dimming mixed up. The interpretation of the supernova dimming as an accelerated expansion (which is due to the use of FLRW models) has firmly taken root, which is why some have focussed on acceleration in the fluid expansion. In the L–T models, the 'acceleration' defined from $z(D_L)$ using (3.3) exhibits a behaviour different from the q_1 of the fluid flow defined by (2.1), even though they coincide for FLRW models. We hope our paper will help establish the difference between 'accelerated expansion' and 'supernova dimming' in the observed magnitude–redshift relation.

We summarise and discuss different misunderstandings related to inhomogeneous cosmology in Table 1.

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Appendix A: A Behaviour of derivatives of $R(t, r)$ in the neighbourhood of the symmetry centre

The formulae below are obtained using the coordinate r introduced in (2.8), the symbols introduced in (5.10)–(5.11), and eq. (2.2). We show the intermediate expressions in order to demonstrate that all terms that could cause divergencies cancel out before the limit is taken. The first two equations follow trivially from (2.2).

$$R_{,t} = \sqrt{\frac{2M}{R} + 2E} = rV \underset{r\to 0}{\longrightarrow} 0, \tag{A.1}$$

$$R_{,tt} = -\frac{M}{R^2} = -\frac{M_0 r}{(S + \mathcal{P})^2} \underset{r\to 0}{\longrightarrow} 0, \tag{A.2}$$

To evaluate the behaviour of $R_{,r}$ and $R_{,rr}$ in the neighbourhood of $r = 0$ we use the following equation [37]:

Table 1 Summary of misconceptions and misunderstandings related to inhomogeneous cosmology

Misconceptions and misunderstandings	Corrections and clarifications
Weak singularity	
$\Box\mathcal{R} \to \infty$ is a singularity. Models with $\Box\mathcal{R} = \infty$ are unphysical	$\Box\mathcal{R} \to \infty$ is not a curvature singularity and has no physical interpretation. Many objects have $\Box\mathcal{R} = \infty$, among them the Earth at its surface
Deceleration parameter	
There are general theorems that prohibit $q_0 < 0$ where $q_0 \equiv q(z = 0)$, presented in Refs. [18,2]	There are two distinctive deceleration parameters: q^{obs} based on a Taylor expansion of the luminosity distance and the invariantly defined q^{inv} which measures the acceleration of expansion. If $\Lambda = 0 = \omega_{ab} = \dot{u}^{\alpha}$ and $\rho + 3p > 0$ then $q^{inv} > 0$, however for the same case q^{obs} may be negative. F's & HS's equations relating q^{obs} to the flow invariants are approximate and coordinate-dependent, thus they do not exist as covariant laws. The approximately correct intermediate relation does not exclude $q^{obs} < 0$ in the L–T model
There is a local singularity in models with $q_0 < 0$	$\Box\mathcal{R} \to \infty$ when $q_0 < 0$ and the observer is at the centre of spherical symmetry. However, this is not a singularity. Away from the the centre of spherical symmetry $q_0 < 0$ does not imply divergence of $\Box\mathcal{R}$
Other singularities arise in models with $q(z) < 0$. In such models there is a class in which $q(z) = -1 + \frac{1+z}{H(z)}\frac{dH(z)}{dz} \to \infty$ at some z, where $H(z) = \left[\frac{d}{dz}\left(\frac{D_L}{1+z}\right)\right]^{-1}$	q defined in this way diverges when $\frac{d}{dz}\left(\frac{D_L}{1+z}\right) = 0$ but this is not a singularity. In fact, if one applies these definitions to a zero-Λ dust FLRW model, one obtains regions where $q < 0$ when $\Omega_{k0} > 0.6$ This is because such $H(z)$ only applies when $k = 0$
Inverse problem	
In solving for the model that gives a selected $D_L(z)$, one encounters a 'pathology' or 'critical point', beyond $z \sim 1$ and the solution generally breaks down there	The 'critical point' is the apparent horizon, where the diameter distance is maximum. It is a general property of expanding decelerating $\Lambda = 0$ models such as L–T and FLRW, long known in the FLRW case. This point requires special treatment, but has useful properties [10–12]
Only FLRW models have "transcritical" solutions	"Transcritical" rays, that cross the apparent horizon, are generic in L-T and FLRW models. With small errors in observational data, all L-T and FLRW models will technically fail the apparent horizon conditions, but this can be fixed [12]. Arbitrary choices such as $E = 0$ and a given $D_L(z)$ may well fail to be transcritical
Fitting observations	
Inhomogeneous models are exposed to singularities. Non-singular models are exceptional and rigid, hence unable to account for observations	Inhomogeneous models like the L–T models include the FLRW models as a subcase. Thus, if the FLRW models are considered good enough for cosmology, then the L–T models can only be better: they constitute an *exact perturbation* of the FLRW background, and can reproduce the latter as a limit with an arbitrary precision
$q_0^{obs} < 0$ is essential to account for observations	Derivation of q^{obs} in an inhomogeneous model involves approximations—one of them linearity, so the sign of q^{obs} cannot be determined. One can have a very good fit to observations with $q_0^{obs} > 0$ [38,39]

$$R_{,r} = \left(\frac{M_{,r}}{M} - \frac{E_{,r}}{E}\right) R + \left[\left(\frac{3}{2}\frac{E_{,r}}{E} - \frac{M_{,r}}{M}\right)(t - t_B) - t_{B,r}\right] R_{,t}. \tag{A.3}$$

The above applies also with $E = 0$ if we neglect the $E_{,r}/E$ terms (as verified by direct calculation). From (A.3) we obtain:

$$R_{,r} = S + \mathcal{P} + r\mathcal{L} \underset{r\to 0}{\longrightarrow} S. \tag{A.4}$$

The next equations below follow by consecutively differentiating (A.1)–(A.3) by r and using (A.3)–(A.4) in the result:

$$R_{,ttr} = -\frac{M_{,r}}{R^2} + \frac{2MR_{,r}}{R^3} = -\frac{M_0}{(S+\mathcal{P})^2} + \frac{2M_0 r\mathcal{L}}{(S+\mathcal{P})^3} \underset{r\to 0}{\longrightarrow} -\frac{M_0}{S^2}, \tag{A.5}$$

$$R_{,tr} = \frac{1}{R_{,t}}\left(\frac{M_{,r}}{R} - \frac{MR_{,r}}{R^2} + E_{,r}\right) = V + \frac{r}{V}\left[F_{,r} - \frac{M_0\mathcal{L}}{(S+\mathcal{P})^2}\right] \underset{r\to 0}{\longrightarrow} V_0, \tag{A.6}$$

$$\begin{aligned} R_{,trr} = & -\frac{1}{{R_{,t}}^3}\left(\frac{M_{,r}}{R} - \frac{MR_{,r}}{R^2} + E_{,r}\right)^2 + \frac{1}{R_{,t}}\left(\frac{M_{,rr}}{R} - \frac{2M_{,r}R_{,r}}{R^2} - \frac{MR_{,rr}}{R^2}\right. \\ & \left. + \frac{2M{R_{,r}}^2}{R^3} + E_{,rr}\right) = \frac{1}{V}\left\{-\frac{M_0}{(S+\mathcal{P})^2}R_{,rr} + 2F_{,r} + r\left[F_{,rr} + \frac{2M_0\mathcal{L}^2}{(S+\mathcal{P})^3}\right]\right\} \\ & -\frac{r}{V^3}\left[F_{,r} - \frac{M_0\mathcal{L}}{(S+\mathcal{P})^2}\right]^2 \underset{r\to 0}{\longrightarrow} \frac{1}{V_0}\left[-\frac{M_0}{S^2}R_{,rr}(0) + 2F_{,r}(0)\right], \end{aligned} \tag{A.7}$$

$$\begin{aligned} R_{,rr} = & \frac{1}{-k/2+F}\left[-(S+\mathcal{P}) + \frac{3}{2}V(t-t_B)\right]\left[2F_{,r} + r\left(F_{,rr} - 2\frac{{F_{,r}}^2}{-k/2+F}\right)\right] \\ & -V\left[2t_{B,r} + r\left(t_{B,rr} + \frac{F_{,r}\,t_{B,r}}{2(-k/2+F)}\right)\right] \\ & +\frac{r}{V}\left[\frac{3}{2}\frac{F_{,r}}{-k/2+F}(t-t_B) - t_{B,r}\right]\left[F_{,r} - \frac{M_0\mathcal{L}}{(S+\mathcal{P})^2}\right] \\ & \underset{r\to 0}{\longrightarrow} \quad -\frac{2F_{,r}(0)}{k}\left[-2S + 3\left(t - t_B(0)\right)V_0\right] - 2t_{B,r}(0)V_0. \end{aligned} \tag{A.8}$$

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