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NONSINGULAR COLLAPSE OF SPHERICALLY SYMMETRIC CHARGED DUST

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In spherically symmetric charged dust, two kinds of singularity may be present: the Big Bang/Crunch (BB/BC), and shell crossings. The BB/BC is avoided when the charge density ρ_e and the mass-energy density ϵ obey $|\rho_e| < \sqrt{G}\epsilon/c^2$. However, shell crossings then usually appear. This note shows how to avoid also the shell crossings for one cycle of collapse/expansion, and gives an example of a configuration that really avoids it. The configuration goes through the tunnel between the singularities in the maximally extended Reissner – Nordström spacetime.

Keywords: cosmology; Einstein-Maxwell equations; singularities

1. Spherically symmetric charged dust

In the spherically symmetric spacetimes, in comoving coordinates, the solution of the Einstein–Maxwell equations for a charged dust source has the metric

$$ds^2 = \mathrm{e}^{C(t,r)}\mathrm{d}t^2 - \mathrm{e}^{A(t,r)}\mathrm{d}r^2 - R^2(t,r)\left[\mathrm{d}\vartheta^2 + \sin^2(\vartheta)\mathrm{d}\varphi^2\right]. \tag{1}$$

The solution of the Maxwell equations is

$$F^{01} = -F^{10} = Q(r)\mathrm{e}^{-(A+C)/2}/R^2, \qquad Q,_r = (4\pi/c)\rho_e \mathrm{e}^{A/2}R^2, \tag{2}$$

where $Q(r)$ (the electric charge within the r-surface) is an arbitrary function and ρ_e is the density of the electric charge; other F^{ij}-s are zero.

The Einstein equations then imply

$$\frac{\kappa}{2}\epsilon R^2 \mathrm{e}^{A/2} = \frac{G}{c^4}\, N,_r\,, \tag{3}$$

where ϵ is the energy density and $N,_r$ is an arbitrary function of integration (N is the energy equivalent to the sum of rest masses within the r-surface);

$$\frac{\rho_e Q}{c\epsilon} \equiv \frac{QQ,_r}{N,_r} = QQ,_N\,, \qquad \mathrm{e}^{-A/2}R,_r = \Gamma(r) - QQ,_N/R, \tag{4}$$

where $\Gamma(r)$ is an arbitrary function of integration;

$$C,_r = 2\frac{\mathrm{e}^{A/2}}{R^2}QQ,_N\,, \tag{5}$$

$$\mathrm{e}^{-C}R,_t{}^2 = \Gamma^2 - 1 + \frac{2M(r)}{R} + \frac{Q^2\left(Q,_N{}^2 - G/c^4\right)}{R^2} - \frac{1}{3}\Lambda R^2, \tag{6}$$

where Λ is the cosmological constant and $M(r)$ is an arbitrary function – the *effective mass* that drives the evolution. It is a combination of active gravitational mass and charge that *need not be positive.*

The Einstein equations imply the following relation in addition:

$$\frac{G}{c^4}\Gamma N,_r = (M + QQ,_N \Gamma)\,,_r \,. \tag{7}$$

The quantity $\mathcal{M} \overset{\text{def}}{=} M + QQ,_N \Gamma$ is the active gravitational mass. Thus Γ is a measure of the gravitational mass defect/excess.

The set (5) – (6) defines the functions $C(t,r)$ and $R(t,r)$ implicitly.

The matching conditions between this metric and the Reissner–Nordström metric then imply $e = \sqrt{G}Q(r_b)/c^2$, $m = (M + QQ,_N \Gamma)_{r=r_b}$, where e and m are the R–N charge and mass parameters, and $r = r_b$ is the matching hypersurface.

For a derivation of all these results see Refs. 1 and 2 (based on the original source, Ref. 3). These references also contain the derivations of all the results given in the following sections.

2. Prevention of the Big Crunch by electric charge

(a) When $E < 0$, solutions without a BC singularity exist if and only if the following conditions are obeyed

$$M^2 \geq 2EQ^2\left(Q,_N{}^2 - G/c^4\right), \tag{8}$$

$$Q,_N{}^2 < G/c^4 \qquad \text{and} \qquad M > 0. \tag{9}$$

(b) When $E = 0$, a singularity is avoided if and only if (9) applies.

(c) When $E > 0$, there will be no BC singularity if and only if (8) and one of the two following conditions applies:

(c_1) Equation (9), or

(c_2) $M < 0$ and $R > R_+$ initially, where

$$R_+ = -\frac{M}{2E} + \frac{1}{2E}\sqrt{M^2 - 2EQ^2\left(Q,_N{}^2 - G/c^4\right)}$$

The surface of the charged sphere obeys the equation of radial motion of a charged particle in the Reissner–Nordström spacetime. For such a particle, if the ratio of its charge q to its mass μ obeys $(q/\mu)^2 < 1$, then the reversal of fall to escape can occur only inside the inner R–N horizon, at $R < r_- = m - \sqrt{m^2 - e^2}$. Thus, the surface of a collapsing sphere must continue to collapse until it crosses the inner horizon, and can bounce at $R < r_-$. Then, however, it cannot re-expand back into the same spacetime region from which it collapsed, as this would require motion backward in time. The surface would thus continue through the tunnel between the singularities and re-expand into another copy of the asymptotically flat region.

Unfortunately, Ori [4,5] proved that if $Q,_N{}^2 < G/c^4$ holds throughout the volume, then a shell crossing is unavoidable. If the charged dust is matched to the Reissner–Nordström solution, then Ori's result [4,5] prevents going through the tunnel between the singularities in the maximally extended R–N spacetime.

The only situations in which both BB/BC and shell crossing singularities could possibly be avoided are these:

1. When $\lim_{r\to r_c} Q,_N{}^2 = G/c^4$, while $Q,_N{}^2 < G/c^4$ elsewhere.
2. When $Q,_N{}^2 > G/c^4$, $E > 0$ and $M < 0$.

An example of a bounce of the first kind will be given in Sec. 5.

3. Regularity conditions at the center

The set $R = 0$ in charged dust consists of the Big Bang/Crunch singularity (which we showed to be avoidable) and of the center of symmetry, which may or may not be singular. The conditions for the absence of the central singularity are as follows:

$$\lim_{r\to r_c} R/\mathcal{M}^{1/3} = \beta(t) \neq 0; \qquad \lim_{r\to r_c} \Gamma^2(r) = 1 \Longrightarrow \lim_{r\to r_c} E(r) = 0; \tag{10}$$

$$\lim_{r\to r_c} 2E/\mathcal{M}^{2/3} = \lim_{r\to r_c} \left(\Gamma^2(r) - 1\right)/\mathcal{M}^{2/3} = \text{const}, \tag{11}$$

and this constant may be zero.

4. Conditions for a nonsingular bounce of a weakly charged dust

From now on we assume $Q,_N{}^2 < G/c^4$ (weakly charged dust). Such a configuration can bounce singularity-free through the R–N wormhole if the following necessary conditions are obeyed in a neighbourhood of the center:

(1) $E < 0$;

(2) $E \geq -1/2$;

(3) $\lim_{r\to r_c} F_1/\mathcal{M}^{1/3} = 0$, where $F_1 \overset{\text{def}}{=} 1 - (c^4/G)\left(Q,_N{}^2 + QQ,_{NN}\right)$;

(4) $\Gamma,_\mathcal{M} < 0$;

(5) $Q,_N{}^2 < G/c^4$ at $N > 0$ and $Q,_N{}^2 = G/c^4$ at $N = 0$;

(6) $M \equiv \mathcal{M} - QQ,_N\Gamma > 0$;

(7) $M^2 - 2EQ^2\left(Q,_N{}^2 - G/c^4\right) > 0$;

(8) $1 - (c^4/G)\left(Q,_N{}^2 + QQ,_{NN}\right) > \frac{\Gamma\Gamma,_\mathcal{M}}{2E} M$ (a necessary condition for the inequality below to be obeyed);

(9) $1-(c^4/G)\left(Q,_N{}^2 + QQ,_{NN}\right) > \frac{\Gamma\Gamma,_\mathcal{M}}{2E}\left[M + \sqrt{M^2 - 2EQ^2\left(Q,_N{}^2 - G/c^4\right)}\right]$.

In addition to that, all the regularity conditions of sec. 3 must be obeyed.

Conditions (1) – (9) must hold in a *neighbourhood* of the center. *At* the center, the left-hand sides of conditions (1) and (6 – 9) must have zero limits.

5. An example

In order to prove that conditions (1) – (9) from the previous section are not mutually contradictory, we shall now provide an example of a model that obeys them.

Since the functions appearing in the inequalities are rather complicated, the proof that the inequalities are all obeyed was given mainly by numerical graphs (see Ref. 2). However, it was verified by exact methods that in a neighbourhood U of the center the functions indeed behave in the desired way.

Choose:

$$Q(N) = q\frac{\sqrt{G}N_0}{c^2}\, p(x), \qquad p(x) = x/(1+x)^2, \tag{12}$$

where $q = \pm 1$, $x \stackrel{\text{def}}{=} N/N_0$ and N_0 is a constant. Then

$$F_1(x) \stackrel{\text{def}}{=} 1 - \frac{c^4}{G}\left({Q,_N}^2 + QQ,_{NN}\right) = 1 - \frac{3x^2 - 6x + 1}{(1+x)^6}. \tag{13}$$

We choose now the function $E(N)$

$$2E = -\frac{bx^{5/3}}{1+bx^{5/3}} \quad \Longrightarrow \quad \Gamma(x) = \frac{1}{\sqrt{1+bx^{5/3}}} \quad \Longrightarrow \tag{14}$$

$$\mathcal{M}(x) = \frac{GN_0}{c^4}\int_0^x \frac{\mathrm{d}x'}{\sqrt{1+bx'^{5/3}}} \stackrel{\text{def}}{=} \frac{GN_0}{c^4}\mu(x). \tag{15}$$

We have now:

$$\begin{aligned} M &\equiv \mathcal{M} - QQ,_N\Gamma = \frac{GN_0}{c^4}\, F_2(x), \\ F_2(x) &\stackrel{\text{def}}{=} \mu(x) - \frac{x(1-x)}{(1+x)^5\sqrt{1+bx^{5/3}}}. \end{aligned} \tag{16}$$

It is easy to verify that $F_2 > 0$ for all $x > 0$, so condition (6) is fulfilled.

Condition (7) is equivalent to

$$F_3(x) > 0, \qquad F_3(x) \stackrel{\text{def}}{=} {F_2}^2(x) - F_6(x), \tag{17}$$

where

$$F_6(x) \stackrel{\text{def}}{=} 2Ep^2\left(1 - {p,_x}^2\right) = \frac{bx^{11/3}}{\left(1+bx^{5/3}\right)(1+x)^4}\left[1 - \frac{(1-x)^2}{(1+x)^6}\right]. \tag{18}$$

Condition (8) becomes:

$$F_4(x) > 0, \qquad \text{where} \qquad F_4(x) \stackrel{\text{def}}{=} F_1(x) - \frac{5}{6x\sqrt{1+bx^{5/3}}}\, F_2(x). \tag{19}$$

Finally, condition (9) can be written as

$$F_5(x) > 0, \qquad \text{where } F_5(x) \stackrel{\text{def}}{=} F_1(x) - \frac{5}{6x\sqrt{1+bx^{5/3}}}\left[F_2(x) + \sqrt{F_3(x)}\right]. \tag{20}$$

Since there exists a neighbourhood U of $x = 0$ in which the functions $F_1, \ldots, F_5$ are positive for $x > 0$, we can cut away a finite ball of the charged dust with a sufficiently small radius x_0 and match the charged dust to the Reissner – Nordström solution at $x = x_0$. In this way, we obtain a finite charged body of dust that can go through a minimal size and bounce without encountering any singularity.

It can be verified that in our example $(G/c^4)Q^2 < \mathcal{M}^2$ for all $x > 0$, provided $b < 25.3$. Thus, if a finite sphere is cut out of this configuration and matched to the Reissner – Nordström solution, the exterior R–N metric will have $e^2 < m^2$, and horizons will exist in it. The reversal of collapse to expansion can occur only within the inner R–N horizon. Our example is consistent with this: the maximal radius R_- achieved by a given $\mathcal{M}$ shell, the radius R_{H+} of the outer horizon, the radius of the inner horizon, R_{H-}, and the minimal radius, R_+, obey $R_+ < R_{H-} < R_{H+} < R_-$ for all values of $x = N/N_0$.

Fig. 1 shows the numerically calculated period (in the coordinate time t) of the full cycle of collapse/expansion as a function of mass. Fig. 2 shows a collection of curves $R(\mathcal{M}, t)$ corresponding to different values of $\mathcal{M}$. The configuration is time-symmetric with respect to the instant $t = 0$.

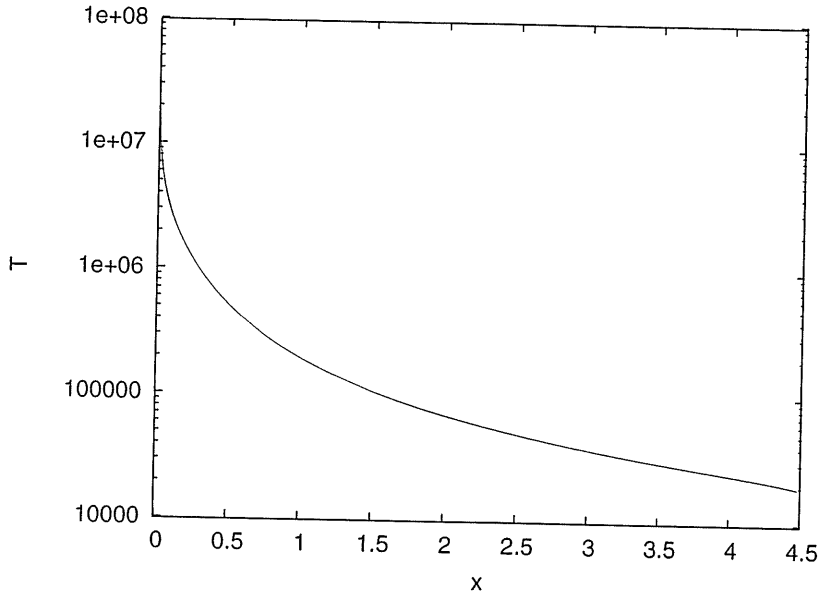

Fig. 1. The period (in the time coordinate) as a function of $x = N/N_0$.

Each mass shell avoids shell crossings throughout the first expansion phase after the time-symmetric bounce, but then experiences crossings after going through the maximal size. In order to avoid shell crossings in the whole volume for the whole expansion phase of the outermost shell, the radius of the dust ball cannot be too large. If it is very large, then the period of oscillations of the outermost shells will also become very large. Then, the time by which first shell crossings appear inside the ball will become a small fraction of the duration of the expansion phase of the outer surface, i.e. shell crossings will appear before the surface of the ball emerges from inside the outer horizon.

The evolution of our configuration is summarised in the Penrose diagram in Fig. 3. The diagram is written into the background of the Penrose diagram for the maximally extended Reissner–Nordström spacetime (thin lines). C is the center of symmetry, Sb is the surface of the charged ball, S_{RN} is the Reissner–Nordström singularity. The interior of the body is encompassed by the lines C, E, Sb and B; no

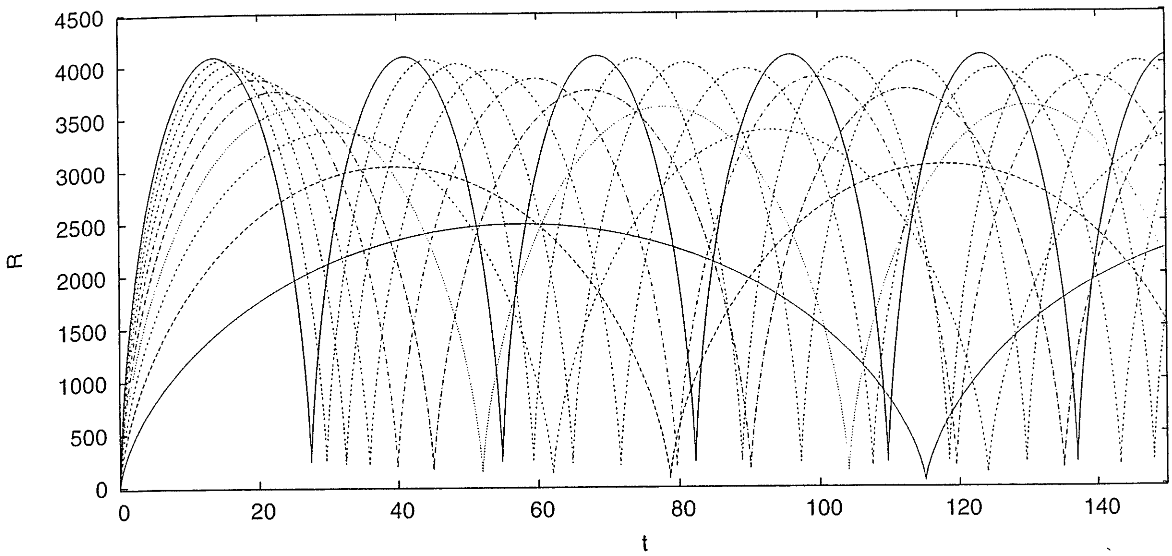

Fig. 2. The curves $R(\mathcal{M}, t)$ corresponding to several values of $\mathcal{M}$. The mass increases uniformly from $x = 0.01$ on the lowest curve to $x = 0.1$ on the highest curve. The bounce is always smooth and at a nonzero value of R.

singularity occurs within this area. Lines B and E connect the points in spacetime where the shell crossings occur at different mass shells. N1 (N2) are the past- (future-) directed null geodesics emanating from the points in which the shell crossings reach the surface of the body (compare Fig. 2). The line Sb should be identified with the uppermost curve in Fig. 2. The top end of Sb is where the corresponding curve in Fig. 2 first crosses another curve, the middle point of Sb is at $t = 0$ in Fig. 2.

It would be interesting to have an example of a configuration that can pulsate for ever, avoiding shell crossings in all of its collapse/expansion phases. Whether such a permanently pulsating singularity-free configuration exists at all is a problem to be investigated in the future.

6. Can such an object exist in the real Universe?

It is clear that the smaller the net charge, the greater the chance that such an object might exist. For our object, the absolute value of the charge first increases from zero in the center to the maximum $\sqrt{G}N_0/(4c^2)$ achieved at $x = 1$, and then keeps decreasing all the way to zero as $x \to \infty$.

As seen from Fig. 1, the period of oscillations first decreases with mass, up to $x \approx 4.0$, then begins to increase. We cannot take the radius of our object larger than that because if it is large, then the period will be large, and shell crossings will appear inside the object before its surface emerges from the outer horizon. Thus, the largest total mass that we can assume corresponds to $x \approx 4.0$. At $x = 4$, the charge is $|Q| = 0.16\sqrt{G}N_0/c^2$, while the active mass, in physical units, is $c^2\mathcal{M}/G = 1.68N_0/c^2$. The ratio of charge to mass is thus $G|Q|/(c^2\mathcal{M}) = 0.095\sqrt{G}$ in electrostatic units. This makes 0.82×10^{-14} coulombs per gram. For the whole Earth, this ratio is 0.502×10^{-25} C/g [6]. However, for a neutron star of 1 solar

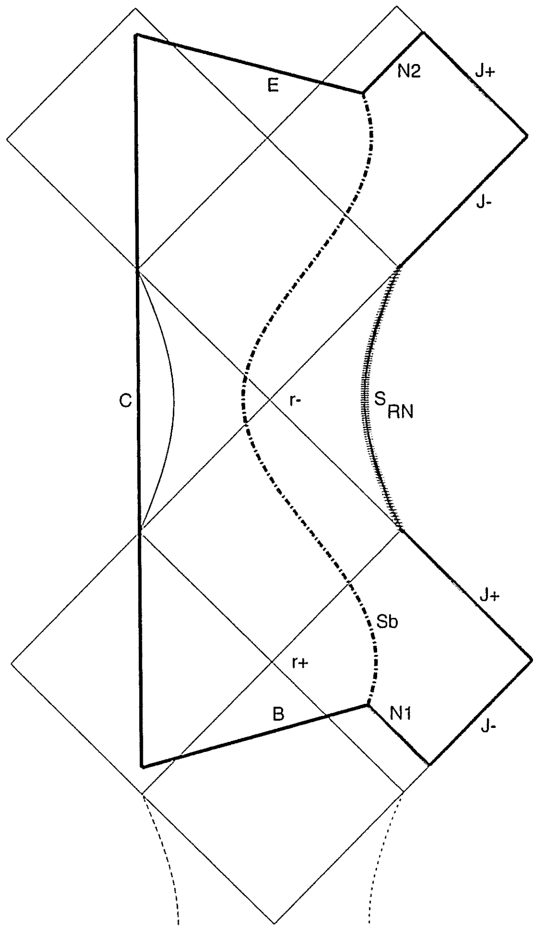

Fig. 3. A schematic Penrose diagram for the configuration defined by eqs. (12) and (14). See explanation in the text.

mass, the authors of Ref. 7 found that the total charge might be 10^{20} C, which makes 5.03×10^{-14} C/g – 6 times as much as in our object. Thus, the possibility to find a real object with charge and mass similar to our example is not outlandish.

7. Conclusions and possible further research

We have shown that it is possible to set up such initial conditions for a charged dust sphere of a finite radius that its outer surface completes one full cycle of pulsation, while no singularity appears either at the surface or anywhere inside it.

With the most favourable value of mass, the ratio of total charge to total mass of our dust ball is 6 times smaller than the corresponding theoretically estimated maximum for charged neutron stars (see our Sec. 6 and Ref. 7).

Other choices of the arbitrary functions are possible that might improve some of the characteristics of our model. For example, it would be desirable to avoid the matching to the R–N spacetime, so that the dust distribution can extend over the whole space (infinite or closed) – then the solution could be investigated as a possible model of the Universe with a localised charged object in it. Another desirable generalisation would be to make *all* bounces time-symmetric, so that the model oscillates singularity-free for ever.

To avoid the artificially limited volume, one might try to match the model to a spacetime different from R–N. For example, it could be a charged dust ball with the charge density becoming strictly zero at a certain distance from the center. Then, in the outer region, $Q,_N = 0$, which is sufficient to prevent the BB/BC singularity (see Refs. 1 and 2). Since the difficulty in avoiding shell crossings is close to the center, and the $Q,_N = 0$ region would not extend to the center, chances are that shell crossings could be avoided as well. We do not know if such a configuration exists.

One way to avoid the limited time interval would be to make the bounce at minimal radius simultaneous in t for all constant-mass shells. This means that the period of oscillations (measured by the time-coordinate t) would have to be independent of mass. We have not investigated this condition.

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