Alternative methods of describing structure formation in the Lemaitre-Tolman model

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We describe several new ways of specifying the behavior of Lemaître-Tolman (LT) models, in each case presenting the method for obtaining the LT arbitrary functions from the given data, and the conditions for existence of such solutions. In addition to our previously considered "boundary conditions," the new ones include: a simultaneous big bang, a homogeneous density or velocity distribution in the asymptotic future, a simultaneous big crunch, a simultaneous time of maximal expansion, a chosen density or velocity distribution in the asymptotic future, only growing or only decaying fluctuations. Since these conditions are combined in pairs to specify a particular model, this considerably increases the possible ways of designing LT models with desired properties.

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I. AIM AND MOTIVATION

The Lemaître-Tolman (LT) metric is the most widely used model of cosmic inhomogeneity, being suitable for both large and small scale inhomogeneities. As the simplest inhomogeneous solution of Einstein's equations, it is relatively easy to work with, but retains the full nonlinearity of the field equations.

However, because of this nonlinearity, generating models with a specific desired evolution was not all that easy, and involved a large measure of guesswork in choosing the arbitrary functions of the model, as well as repeated numerical evolution to check if satisfactory results had been obtained.

The usefulness of the LT metric is greatly increased if specific models can be designed to have certain properties that satisfy observational or theoretical requirements. Previous papers [1–3], (hereafter papers I, II, and III) have shown how to generate LT models that evolve from a given (spherically symmetric) density or velocity profile at time t_1 to a second given density or velocity profile at time t_2.

We here extend the methods by which models may be constructed. We seek to solve for the arbitrary functions $E(M)$ and $t_B(M)$ that determine the LT model that evolves according to the following requirements, in various combinations, where the first two were considered previously:

(a) a density profile $\rho_i(M)$ is given at time t_i,
(b) a velocity profile $(R_{,t})_i(M)$ is given at time t_i,
(c) both a velocity and a density profile are given at the same time,
(d) the bang time is simultaneous,
(e) the crunch time is simultaneous,
(f) the time of maximum expansion is simultaneous,
(g) the model becomes homogeneous at late times,
(h) only growing modes are present,
(i) only decaying modes are present,
(j) a velocity profile $(R_{,t})(M)$ is given at late times,
(k) a time-scaled density profile $t^3\rho(M)$ is given at late times.

It requires two of the above conditions to specify an LT model (except for condition (c), which does not need a companion), and in various contexts different combinations may be useful. We work them out here in a systematic way for future reference. (In contrast, [4] shows how to determine the LT functions from observational data on the past null cone.)

II. OUTLINE OF THE MODEL

The Lemaître-Tolman (LT) model [5,6] is a spherically symmetric, nonstatic solution of the Einstein equations that is inhomogeneous in the radial direction. The matter source is a perfect fluid with zero pressure, i.e. dust, and the coordinates are comoving with the matter particles. The metric is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{{R_{,r}}^2}{1+2E(r)}\mathrm{d}r^2 - R^2(t,r)(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2), \tag{2.1}$$

where $E(r)$ is an arbitrary function that determines the local curvature of constant t slices, and $R_{,r}$ is the r derivative of the areal radius $R(t,r)$.

With $\Lambda = 0$ assumed, the evolution equation for R is

$$ {R_{,t}}^2 = \frac{2M}{R} + 2E, \tag{2.2}$$

where $M(r)$ is a second arbitrary function, representing the total gravitational mass within the comoving matter shell at constant r. Note that in this equation $E(r)$ has a second interpretation as the local energy per unit mass of the dust

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particles, and thus it also determines the type of evolution (see below). The density is

$$\kappa\rho = \frac{2M_{,r}}{R^2 R_{,r}} \quad \text{where } \kappa = \frac{8\pi G}{c^4}. \tag{2.3}$$

The evolution equation (2.2) has three types of solution:

Elliptic, $E < 0$:

$$\begin{aligned} R(t, r) &= \frac{M}{(-2E)}(1 - \cos\eta), \\ \eta - \sin\eta &= \frac{(-2E)^{3/2}}{M}(t - t_B(r)), \end{aligned} \tag{2.4}$$

where η is a parameter;

Parabolic, $E = 0$:

$$R(t, r) = \left[\frac{9}{2}M(t - t_B(r))^2\right]^{1/3}, \tag{2.5}$$

Hyperbolic, $E > 0$:

$$\begin{aligned} R(t, r) &= \frac{M}{2E}(\cosh\eta - 1), \\ \sinh\eta - \eta &= \frac{(2E)^{(3/2)}}{M}(t - t_B(r)), \end{aligned} \tag{2.6}$$

where $t_B(r)$ is the third arbitrary function, representing the local time at which the big bang occurs. The parabolic evolution is the $E \to 0$ limit of the other two evolutions, obtained by noting that $\eta/\sqrt{-2E}$ remains finite. It is perfectly possible to have adjacent elliptic and hyperbolic regions in one model, the evolution being parabolic on the boundary where $E = 0$, but in general $E' \neq 0$.

Elliptic models have both a big bang $t = t_B$ and a big crunch

$$t_C(r) = t_B + T(r), \tag{2.7}$$

where the lifetime of each world line is

$$T(r) = \frac{2\pi M}{(-2E)^{3/2}}. \tag{2.8}$$

The instant of maximum expansion is

$$t_{MX}(r) = t_B + \frac{\pi M}{(-2E)^{3/2}}, \tag{2.9}$$

at which moment the maximum areal radius

$$R_{MX}(r) = \frac{M}{(-E)} \tag{2.10}$$

is reached.

The homogeneous case is obtained by setting

$$E \propto M^{2/3}, \qquad t_B = \text{constant} \tag{2.11}$$

and it is a Friedmann model, i.e. a Robertson-Walker (RW) model with zero pressure.

In the following, we will use the notation

$$a = R/M^{1/3}, \tag{2.12}$$

$$x = (-2E)/M^{2/3}, \qquad E < 0, \tag{2.13}$$

$$x = (2E)/M^{2/3}, \qquad E \geq 0, \tag{2.14}$$

$$b = R_{,t}/M^{1/3}. \tag{2.15}$$

The parametric solutions (2.4) and (2.6) may be written, for the expanding hyperbolic (HX), expanding elliptic (EX), collapsing elliptic (EC), and collapsing hyperbolic (HC) cases as

$$\text{HX: } t = t_B + x^{-3/2}\{\sqrt{(1 + xa)^2 - 1} - \operatorname{arcosh}(1 + xa)\}, \quad E > 0, \quad t > t_B. \tag{2.16}$$

$$\text{EX: } t = t_B + x^{-3/2}\{\arccos(1 + xa) - \sqrt{1 - (1 + xa)^2}\}, \quad E < 0, \quad 0 \leq \eta \leq \pi, \tag{2.17}$$

$$\text{EC: } t = t_B + x^{-3/2}\{2\pi - \arccos(1 + xa) + \sqrt{1 - (1 + xa)^2}\}, \quad E < 0, \quad \pi \leq \eta \leq 2\pi, \tag{2.18}$$

$$\text{HC: } t = t_C - x^{-3/2}\{\sqrt{(1 + xa)^2 - 1} - \operatorname{arcosh}(1 + xa)\}, \quad E > 0, \quad t < t_C. \tag{2.19}$$

Alternatively, from (2.2) and (2.13) or (2.14) we have $b^2 = 2/a \pm x$, so substituting for a in the above gives

$$\text{HX: } t = t_B + x^{-3/2}\left\{\sqrt{\left(\frac{b^2 + x}{b^2 - x}\right)^2 - 1} - \operatorname{arcosh}\left(\frac{b^2 + x}{b^2 - x}\right)\right\}, \tag{2.20}$$

$$\text{EX: } t = t_B + x^{-3/2}\left\{\arccos\left(\frac{b^2 + x}{b^2 - x}\right) - \sqrt{1 - \left(\frac{b^2 + x}{b^2 - x}\right)^2}\right\}, \tag{2.21}$$

$$\text{EC: } t = t_B + x^{-3/2}\left\{2\pi - \arccos\left(\frac{b^2 + x}{b^2 - x}\right) + \sqrt{1 - \left(\frac{b^2 + x}{b^2 - x}\right)^2}\right\}, \tag{2.22}$$

$$\text{HC: } t = t_C - x^{-3/2}\left\{\sqrt{\left(\frac{b^2 + x}{b^2 - x}\right)^2 - 1} - \operatorname{arcosh}\left(\frac{b^2 + x}{b^2 - x}\right)\right\}. \tag{2.23}$$

There are also two borderline cases, the parabolic case, and the elliptic case which has reached maximum expansion at time t. Since the above expressions are not numerically well behaved near these borderlines, we will present series solutions for near-parabolic (nP), and for near-maximum-expansion (nEM) models.

The time reversed solutions, in which the hyperbolic and parabolic solutions are collapsing (HC and nPC), are also possible, but not as relevant for cosmology. Thus they will only be given for certain cases to make a complete listing of possibilities.

It is convenient in what follows to use $M(r)$ as the radial coordinate (i.e. $r = M(r)$), since we are not really considering vacuum, or extrema in the spatial section (''bellies'' and ''necks''). Thus we can integrate (2.3) along a constant time slice, $t = t_i$, to obtain

$$R_i^3 = a_i^3 M = \int_0^M \frac{6}{\kappa \rho_i(\tilde{M})} \mathrm{d}\tilde{M}. \tag{2.24}$$

This equation tells us that, if we have a density profile $\rho_i(M) > 0$ as a function of mass given at a particular time t_i, then a straightforward integration gives us $R_i(M)$ on that time slice. Ways of coping with regions of zero density were discussed in [2].

Ideally, we seek LT models that have regular origins, and are free of shell crossings and surface layers [7].

See [8,9] as well as [1–3] for more details on LT models.

A. Profile to profile solutions

As background to the present work, we very briefly summarize the results of the previous papers.

Paper I showed that, if a spherically symmetric density profile is given at two different times, i.e.

$$\rho = \rho_1(M) > 0 \quad \text{at} \quad t = t_1, \tag{2.25}$$

$$\rho = \rho_2(M) > 0 \quad \text{at} \quad t = t_2, \tag{2.26}$$

then there always exists a LT model that evolves from one to the other. The formulas for the arbitrary functions $E(M)$ and $t_B(M)$ were given, as well as the conditions that determine which type of evolution applies at each M value. Since the formula for $E(M)$ can only be given implicitly, a numerical code was written to implement this algorithm. An example of the formation of an Abell cluster with a realistic density profile at the presentday, starting from a small fluctuation at recombination, was calculated and its evolution plotted.

Although the formulas given were for $t_2 > t_1$ and for an expanding model at t_1, it is easy to adapt to the time-reverse scenario. A key step in the solution process is converting a given density profile $\rho_i(M)$ to an areal radius profile $R_i(M)$ via Eq. (2.24). Thus, if $R_i(M)$ were given instead, this is easily incorporated into the method.

Since the two density profiles are arbitrary, it is entirely possible that the resulting LT model could develop shell crossings somewhere, but it is easy to check for this once $E(M)$ and $t_B(M)$ are known. A second possibility to check for is whether $2E$ has reached -1, which indicates a maximum in the spatial sections, $R_{,r} = 0$ has been reached. Assuming both given densities are finite and nonzero at this point, it would be a regular comoving maximum, $R_{,r}(t, M_{\max}) = 0$ for all t. Beyond this locus, a regular model would have M and R decreasing with increasing radial distance, so, in order to make further progress, one would have to replace $\rho(M)$ with $\rho(2M_{\max} - M)$. But this eventuality was not included in the numerics, as models of objects that large were not contemplated.

Paper II considered the possibility that velocity profiles might be given

$$R_{,t} = (R_{,t})_i(M) > 0 \quad \text{at} \quad t = t_i, \tag{2.27}$$

and again showed how to find the LT model that evolves between two such profiles or between one density and one velocity profile. Several numerical examples were given, illustrating different possibilities, including an improved Abell cluster model and others that showed how radical changes in the density profile are possible, and highlight the fact that both density and velocity fluctuations at recombination play a significant role. There was also a model of the development of a void, but the density and velocity fluctuations at recombination could not both be made small enough in any of the models tried. Improved void models were constructed in [10], using a realistic presentday density profile based on observations, and various density or velocity profiles at last scattering consistent with cosmic microwave background (CMB) observations, as well as a variety of model parameters, particularly Ω_{matter} and Ω_Λ, H_0, k. It was found that in each model there was always some inconsistency with observations: either the void density was not low enough, or the initial velocity was too high, or the density profile at the void wall was too steep and developed a shell crossing. A much improved consistency with observations was achieved by Bolejko [11] by including an inhomogeneous radiation component in the density distribution after last scattering.

Paper III used the above methods to generate a model of a galaxy with a central black hole evolving from a small fluctuation at recombination. The final density profile was made a good fit to presentday observations with data from M87, whose central black hole could be as massive as $3 \times 10^9 M_\odot$. The initial fluctuations at recombination were well within observational limits.[1] Two possibilities were considered for the central black hole. The first supposed that it formed by collapse of matter during the evolution of the model, and for the particular model chosen, we found the black hole formed about 4×10^8 years before the present. The second supposed that it was a full Schwarzschild-

[1]The amplitudes of the density and of the velocity perturbation were within the limits set by observations done at the scale of $\approx 1°$. However, the scale appropriate for a single galaxy is $\approx 0.004°$, and at this scale there are no observational data at all.

Kruskal-Szekeres type wormhole with past and future singularities, but filled with matter [12]. Again for the particular model chosen, it was found that the wormhole is only open for 6×10^{-5} s, and the original black hole mass was only $2M_\odot$, with all the rest of the mass accreting onto it over time. By recombination the mass was $2 \times 10^5 M_\odot$, with a horizon 5×10^{-3} AU across, which corresponds to 4×10^{-13} degrees on the CMB sky—much too small to leave an observable imprint (see Paper II for formulae relating the sizes of various objects to the angles that their images fill in the CMB sky).

The subsequent sections complement these results by adding other criteria by which models can be specified, and deriving the solution algorithms that give the LT arbitrary functions in each case.

III. MODELS WITH A SIMULTANEOUS BANG TIME

We here show how to find a LT model that evolves to a given "final" (or "initial" or "middle") density profile starting from a simultaneous bang time

$$t_B' = 0, \qquad (t_B = \text{constant}). \tag{3.1}$$

By (2.2), the value of E is unimportant near $R = 0$, so the bang is Robertson-Walker (RW) like. This condition is known to generate only growing modes, but as discussed below, does not quite cover all possibilities, though the omitted case is not relevant to cosmology in an expanding universe.

A. Density profile given at time t_i

We specify a density profile $\rho_i(M)$ at time t_i, and choose t_B. The function $a_i(M)$ is then determined from $\rho_i(M)$ via (2.24) and (2.12), and the equations to be solved, in the three cases, are (2.17), (2.18), and (2.16) with $t_B =$ constant:

$$\text{HX: } 0 = \sqrt{(1 + a_i x)^2 - 1} - \operatorname{arcosh}(1 + a_i x) - x^{3/2}(t_i - t_B) \stackrel{\text{def}}{=} \psi_{BDH}(x), \tag{3.2}$$

$$\text{EX: } 0 = \arccos(1 - a_i x) - \sqrt{1 - (1 - a_i x)^2} - x^{3/2}(t_i - t_B) \stackrel{\text{def}}{=} \psi_{BDX}(x), \tag{3.3}$$

$$\text{EC: } 0 = 2\pi - \arccos(1 - a_i x) + \sqrt{1 - (1 - a_i x)^2} - x^{3/2}(t_i - t_B) \stackrel{\text{def}}{=} \psi_{BDC}(x). \tag{3.4}$$

Here the age of the model $(t_i - t_B)$ is a free parameter that must be specified along with a_i. Equations (3.3) and (3.2) may be solved numerically for $x(M)$ and hence $E(M)$. In addition, we give series expansions for solutions that are near to an expanding parabolic model (nPX), and near to maximum expansion in an elliptic model (nEM) at t_i, because the above solutions would encounter numerical difficulties close to the parabolic and maximum expansion borderlines:

$$\text{nPX:} \qquad x \approx \frac{20}{3a_i}\left(-\frac{d\tau_P}{\tau_P} + \frac{25 d\tau_P^2}{14\tau_P^2}\right), \tag{3.5}$$

$$\text{nEM:} \qquad x \approx \frac{2}{a_i} - \frac{d\tau_N^2}{a_i^4} + \frac{3\pi d\tau_N^3}{2^{5/2} a_i^{11/2}}, \tag{3.6}$$

where

$$\tau_i = t_i - t_B, \tag{3.7}$$

$$\tau_P = \frac{\sqrt{2}}{3} a_i^{3/2}, \tag{3.8}$$

$$d\tau_P = \tau_i - \tau_P, \tag{3.9}$$

$$\tau_N = \pi\left(\frac{a_i}{2}\right)^{3/2}. \tag{3.10}$$

$$d\tau_N = \tau_i - \tau_N. \tag{3.11}$$

One may think of τ_P as the time it would take a parabolic model to reach a_i, and τ_N as the time to maximum expansion if a_i were the maximum a value. But note that $d\tau_N$ is not the time since maximum expansion, because when $d\tau_N \neq 0$, the model is not exactly at maximum expansion, so a_i and τ_N are less than their maximum values. Note too that (2.14) is used to define x for the near-parabolic series, so a negative x indicates a slightly elliptic model.

1. Existence of solutions

We next consider existence for each M value point by point, as the solution type does not have to be the same at each point. Since the argument is very similar to that of previous papers, we will only here consider one case, and otherwise just summarize the conditions.

We take $(t_i - t_B) > 0$, and, for the EX case (3.3), we calculate

$$\psi_{BDX}(0) = 0, \qquad \lim_{x \to 2/a_i} \psi_{BDX} = \pi - (2/a_i)^{2/3}(t_i - t_B),$$
$$\frac{\mathrm{d}\psi_{BDX}}{\mathrm{d}x} = \psi_{BDX,x} = \sqrt{x}\left[\frac{a_i^{3/2}}{\sqrt{2 - a_i x}} - \frac{3}{2}(t_i - t_B)\right],$$
$$\psi_{BDX,x}(0) = 0, \qquad \lim_{x \to 2/a} \psi_{BDX,x} = \infty. \tag{3.12}$$

Clearly $\psi_{BDX,x}$ is the product of a positive term ($\sqrt{x}$) and a monotonically increasing term (in square brackets), and therefore it can only change sign if the second term is negative at $x = 0$, i.e.

$$\frac{\sqrt{2} a_i^{3/2}}{3} < (t_i - t_B). \tag{3.13}$$

Only in this case does ψ_{BDX} descend below 0 before rising to the limit (3.12), and only if this limit is positive, i.e. if

$$(t_i - t_B) < \pi\left(\frac{a_i}{2}\right)^{3/2}, \tag{3.14}$$

does it create a root at $x > 0$. Otherwise ψ_{BDX} either rises monotonically from the initial 0 at $x = 0$, or it descends from 0 and then rises to a negative limiting value. Therefore (3.13) and (3.14) are the necessary and sufficient conditions for a growing mode, still-expanding, elliptic solution to exist. The complete set of conditions is

$$\text{HX:} \qquad (t_i - t_B) < \frac{\sqrt{2}a_i^{3/2}}{3}; \tag{3.15}$$

$$\text{PX:} \qquad (t_i - t_B) = \frac{\sqrt{2}a_i^{3/2}}{3}; \tag{3.16}$$

$$\text{EX:} \qquad \frac{\sqrt{2}a_i^{3/2}}{3} < (t_i - t_B) < \pi\left(\frac{a}{2}\right)^{3/2}; \tag{3.17}$$

$$\text{EM:} \qquad (t_i - t_B) = \pi\left(\frac{a}{2}\right)^{3/2}; \tag{3.18}$$

$$\text{EC:} \qquad \pi\left(\frac{a}{2}\right)^{3/2} < (t_i - t_B). \tag{3.19}$$

The parabolic and maximum expansion borderlines are sufficiently obvious that they will not be listed in subsequent sets of existence conditions.

B. Velocity profile given at time t_i

For this scenario, the equations to be solved are (2.21), (2.22), and (2.20) with $t_B =$ constant:

$$\text{HX: } 0 = \sqrt{\left(\frac{b_i^2 + x}{b_i^2 - x}\right)^2 - 1} - \operatorname{arcosh}\left(\frac{b_i^2 + x}{b_i^2 - x}\right) - x^{3/2}(t_i - t_B) \stackrel{\text{def}}{=} \psi_{BVH}(x), \tag{3.20}$$

$$\text{EX: } 0 = +\arccos\left(\frac{b_i^2 - x}{b_i^2 + x}\right) - \sqrt{1 - \left(\frac{b_i^2 - x}{b_i^2 + x}\right)^2} - x^{3/2}(t_i - t_B) \stackrel{\text{def}}{=} \psi_{BVX}(x), \tag{3.21}$$

$$\text{EC: } 0 = 2\pi - \arccos\left(\frac{b_i^2 - x}{b_i^2 + x}\right) + \sqrt{1 - \left(\frac{b_i^2 - x}{b_i^2 + x}\right)^2} - x^{3/2}(t_i - t_B) \stackrel{\text{def}}{=} \psi_{BVC}(x), \tag{3.22}$$

where $b_i = (R_{,t})_i/M^{1/3}$, which are to be solved numerically, and, from series expansions near the borderlines:

$$\text{nPX:} \qquad x \approx \frac{5b_i^2}{6}\left(\frac{d\tau_P}{\tau_P} - \frac{25d\tau_P^2}{28\tau_P^2}\right), \tag{3.23}$$

$$\text{nEM:} \qquad x \approx x_N\left(1 + \frac{16}{3\pi^2}\frac{b_i}{\overline{b}} + \frac{64}{3\pi^4}\frac{b_i^2}{\overline{b}^2}\right), \tag{3.24}$$

where

$$\tau_i = t_i - t_B, \tag{3.25}$$

$$\tau_P = \frac{4}{3b_i^3}, \tag{3.26}$$

$$d\tau_P = \tau_i - \tau_P, \tag{3.27}$$

$$x_N = \left(\frac{\pi}{\tau_i}\right)^{2/3}, \tag{3.28}$$

$$\overline{b} = \frac{2\sqrt{x_N}}{\pi}. \tag{3.29}$$

Here $\overline{b}$ is a kind of average velocity; if the model were exactly at maximum expansion after τ_i, then a would be $a_N = 2/x_N$ and $\overline{b} = \frac{a_N}{\tau_i}$.

1. Existence of solutions

The existence conditions for each case are:

$$\text{HX:} \qquad b_i > 0, \qquad (t_i - t_B) > \frac{4}{3b_i^3}; \tag{3.30}$$

$$\text{EX:} \qquad b_i > 0, \qquad (t_i - t_B) < \frac{4}{3b_i^3}; \tag{3.31}$$

$$\text{EC:} \qquad b_i < 0. \tag{3.32}$$

IV. MODELS THAT BECOME HOMOGENEOUS AT LATE TIMES

We next find a LT model that evolves from a given "initial" profile and approaches a RW model at late times, $t \to \infty$. Only expanding hyperbolic (and parabolic) models have an infinite future, and for these we require

$$2E = KM^{2/3}, \qquad K \text{ a constant.} \tag{4.1}$$

Clearly, this case has only decaying modes. The time reverse of this case—the collapsing hyperbolic model—also satisfies the same condition but was RW-like in the infinite past. See [8] for examples.

This condition may also be applied to elliptic models, and although the inhomogeneities due to $t_{B,r} \neq 0$ do decay initially, other modes can grow as the big crunch is approached because $t_{C,r} \neq 0$. However, (4.1) has the effect of making the lifetime $T(r)$ along the dust world lines a constant (see (2.8)). Now, it is known that the set of necessary and sufficient conditions for no shell crossings in an elliptic model is $\{t_{B,r}/M_{,r} \leq 0, t_{C,r}/M_{,r} \geq 0\}$ [7], but $T(r) =$ const makes it impossible to obey both these inequalities, unless $t_B =$ const and $t_C =$ const, in which case the Friedmann model results. Thus the only elliptic model without shell crossings that satisfies (4.1) is the RW model.[2]

A. Density profile given at time t_i

Applying (4.1) to (2.16) leads to the direct solution

$$\text{HX:}\qquad t_B = t_i - [\sqrt{(1+a_iK)^2-1} - \operatorname{arcosh}(1+a_iK)]/K^{3/2}, \tag{4.2}$$

$$\text{nPX:}\qquad t_B \approx t_i - \sqrt{2a_i^3}\left(\frac{1}{3} - \frac{a_iK}{20} + \frac{3a_i^2K^2}{224}\right). \tag{4.3}$$

Notice that the model is only fully defined once K is given—in other words, the limiting late-time RW model must be fully specified.

1. Existence of solutions

The condition is

$$\text{HX:}\qquad K > 0, \tag{4.4}$$

and it follows that

$$t_i - t_B < \frac{\sqrt{2}}{3}a_i^{3/2}. \tag{4.5}$$

This will keep appearing in what follows as a complementary inequality to the others we shall derive.

B. Velocity profile given at time t_i

Given the velocity distribution $b_i(M)$, the hyperbolic case again has a direct solution for t_B, once (4.1) is applied,

$$\text{HX:}\qquad t_B = t_i - \left[\sqrt{\left(\frac{b_i^2+K}{b_i^2-K}\right)^2 - 1} - \operatorname{arcosh}\left(\frac{b_i^2+K}{b_i^2-K}\right)\right]/K^{3/2}, \tag{4.6}$$

$$\text{nPX:}\qquad t_B \approx t_i - \frac{4}{b_i^3}\left(\frac{1}{3} + \frac{2K}{5b_i^2} + \frac{3K^2}{7b_i^4}\right). \tag{4.7}$$

1. Existence of solutions

$$\text{HX: } b_i > 0, \qquad 0 < K < b_i^2, \qquad (t_i - t_B) > \frac{4}{3b_i^3}. \tag{4.8}$$

V. MODELS WITH A SIMULTANEOUS CRUNCH TIME

For elliptic models, the requirement of only decaying modes is that the crunch time must be simultaneous (i.e. the crunch be RW-like),

$$t_C = \text{constant} = t_B + \frac{2\pi M}{(-2E)^{3/2}} = t_B + \frac{2\pi}{x^{3/2}}. \tag{5.1}$$

H and P models with a simultaneous crunch time obviously have no bang, and are therefore collapsing at all times.

A. Density profile given at time t_i

The equations to be solved are (2.17) and (2.18) with (5.1) and $t_C = \text{constant}$:

$$\text{EX: } 0 = -2\pi + \arccos(1-a_ix) - \sqrt{1-(1-a_ix)^2} + x^{3/2}(t_C - t_i) \stackrel{\text{def}}{=} \psi_{CDX}(x), \tag{5.2}$$

$$\text{EC: } 0 = -\arccos(1-a_ix) + \sqrt{1-(1-a_ix)^2} + x^{3/2}(t_C - t_i) \stackrel{\text{def}}{=} \psi_{CDC}(x), \tag{5.3}$$

$$\text{HC: } 0 = \sqrt{(1+a_ix)^2-1} - \operatorname{arcosh}(1+a_ix) - x^{3/2}(t_C - t_i) \stackrel{\text{def}}{=} \psi_{CDH}(x) \tag{5.4}$$

with the equations for the borderline cases being:

$$\text{nPC:}\qquad x \approx \frac{20}{3a_i}\left(-\frac{d\tau_P}{\tau_P} + \frac{25d\tau_P^2}{14\tau_P^2}\right), \tag{5.5}$$

$$\text{nEM:}\qquad x \approx \frac{2}{a_i} - \frac{d\tau_N^2}{a_i^4} + \frac{3\pi d\tau_N^3}{2^{5/2}a_i^{11/2}}, \tag{5.6}$$

where

$$\tau_i = t_C - t_i \tag{5.7}$$

while (3.8)–(3.11) define τ_P, τ_N, and $d\tau_N$. Again the remaining lifetime of the model, $(t_C - t_i)$ is a free parameter that must be specified along with a_i, and as before, Eqs. (5.2)–(5.4) are to be solved numerically for x. The functions E and t_B then follow from (2.14) or (2.13) and (5.1).

1. Existence of solutions

The complete set of conditions is

$$\text{EX:}\qquad \pi\left(\frac{a_i}{2}\right)^{3/2} < (t_C - t_i), \tag{5.8}$$

$$\text{EC:}\qquad \frac{\sqrt{2}}{3}a_i^{3/2} < (t_C - t_i) < \pi\left(\frac{a_i}{2}\right)^{3/2}, \tag{5.9}$$

$$\text{HC:}\qquad (t_C - t_i) < \frac{\sqrt{2}}{3}a_i^{3/2}. \tag{5.10}$$

The first inequality in (5.9) means that the time difference between t_i and t_C is larger than it would be in a collapsing

[2]Nevertheless, part of the evolution of nonhomogeneous elliptic models obeying (4.1) will be free of shell crossings and may be of interest.

parabolic model—this is seen from (2.5). If $(t_C - t_i)$ is smaller, then the model collapsing from the given density distribution to the big crunch in the time interval $(t_C - t_i)$ would have to be hyperbolic. This is consistent with the time reverse of Eq. (4.5).

B. Velocity profile given at time t_i

These models are found by solving

$$\text{EX: } 0 = -2\pi + \arccos\left(\frac{b_i^2 - x}{b_i^2 + x}\right) - \sqrt{1 - \left(\frac{b_i^2 - x}{b_i^2 + x}\right)^2} + x^{3/2}(t_C - t_i) \stackrel{\text{def}}{=} \psi_{CVX}(x), \tag{5.11}$$

$$\text{EC: } 0 = -\arccos\left(\frac{b_i^2 - x}{b_i^2 + x}\right) + \sqrt{1 - \left(\frac{b_i^2 - x}{b_i^2 + x}\right)^2} + x^{3/2}(t_C - t_i) \stackrel{\text{def}}{=} \psi_{CVC}(x), \tag{5.12}$$

$$\text{HC: } 0 = \sqrt{\left(\frac{b_i^2 + x}{b_i^2 - x}\right)^2 - 1} - \text{arcosh}\left(\frac{b_i^2 + x}{b_i^2 - x}\right) - x^{3/2}(t_C - t_i) \stackrel{\text{def}}{=} \psi_{CVH}(x), \tag{5.13}$$

or, for the borderline cases, calculating

$$\text{nPX:} \qquad x \approx \frac{5b_i^2}{6}\left(\frac{d\tau_P}{\tau_P} - \frac{25 d\tau_P^2}{28\tau_P^2}\right), \tag{5.14}$$

$$\text{nEM:} \qquad x \approx x_N\left(1 + \frac{16}{3\pi^2}\frac{b_i}{\overline{b}} + \frac{64}{3\pi^4}\frac{b_i^2}{\overline{b}^2}\right), \tag{5.15}$$

where

$$\tau_i = t_C - t_i, \tag{5.16}$$

while (3.26), (3.28), and (3.29) define τ_P, x_N, and $\overline{b}$.

1. Existence of solutions

$$\text{EX:} \qquad b_i > 0, \qquad t_C > t_i; \tag{5.17}$$

$$\text{EC:} \qquad b_i < 0, \qquad (t_C - t_i) < \frac{4}{3b_i^3}; \tag{5.18}$$

$$\text{HC:} \qquad b_i < 0, \qquad (t_C - t_i) > \frac{4}{3b_i^3}, \tag{5.19}$$

where the last equation above is for collapsing hyperbolic models. By (5.1), $x^{3/2}(t_C - t_i) = 2\pi - x^{3/2}(t_i - t_B)$, and using this in (5.11) leads to an equivalent existence condition for the EX case, $b_i > 0$, $(t_i - t_B) > \frac{4}{3b_i^3}$. This means the conditions (4.8) and (5.17) are mutually exclusive.

VI. GROWING AND DECAYING MODES

For pure decaying modes, hyperbolic (and parabolic) models that are expanding must become RW-like at late times, whereas for elliptic models and for hyperbolic and parabolic models that are collapsing, they must become RW like at the crunch.

Conversely, for pure growing modes, hyperbolic models that are collapsing must have been RW like in the distant past, while elliptic models and expanding hyperbolic and parabolic models must have been RW like at the bang.

This is summarized in the following table:

	Only growing modes	Only decaying modes
HX, PX	$t_{B,r} = 0$	$2E = KM^{2/3}$
E	$t_{B,r} = 0$	$t_{C,r} = 0$
PC, HC	$2E = KM^{2/3}$	$t_{C,r} = 0$

VII. MODELS WITH A SIMULTANEOUS TIME OF MAXIMUM EXPANSION

Not infrequently, authors seeking a manifestly regular initial condition for an inhomogeneous matter distribution, have required a finite density distribution and zero initial velocity. This is achieved in a LT model if the moment of maximum expansion occurs at the same time along all world lines.

Naturally only elliptic (EX and EC) models have a moment of maximum expansion. In general, the time of maximum expansion (2.9) is different for each world line. The condition that it be simultaneous is $t_{MX} =$ constant, i.e.

$$t_{SMX} = \text{constant} = t_B + \frac{\pi M}{(-2E)^{3/2}} = t_B + \frac{\pi}{x^{3/2}}. \tag{7.1}$$

This needs to be combined with another condition to obtain a solution.

A. Density profile given at time t_i

The equations to be solved, with $t_{SMX} =$ constant, are

$$\text{EX: } 0 = -\pi + \arccos(1 - a_i x) - \sqrt{1 - (1 - a_i x)^2} + x^{3/2}(t_{SMX} - t_i) \stackrel{\text{def}}{=} \psi_{SDX}(x), \tag{7.2}$$

$$\text{EC: } 0 = \pi - \arccos(1 - a_i x) + \sqrt{1 - (1 - a_i x)^2} + x^{3/2}(t_{SMX} - t_i) \stackrel{\text{def}}{=} \psi_{SDC}(x). \tag{7.3}$$

As above, these equations are solved numerically for x, and E and t_B then follow from (2.13) and (7.1).

1. Existence of solutions

The complete set of conditions is

$$\text{EX:} \qquad t_{SMX} > t_i; \tag{7.4}$$

$$\text{EC:} \qquad t_{SMX} < t_i. \tag{7.5}$$

B. Density profile given at a simultaneous time of maximum expansion

Although this is a special case of Section VII A, it has an explicit solution. Given $\rho = \rho_{SMX}(M)$ at $t = t_{SMX}$, we calculate $R_{SMX}(M)$ with (2.24), then write (2.10) as

$$(-E) = \frac{M}{R_{SMX}} \quad \Leftrightarrow \quad x = \frac{2}{a_{SMX}} \tag{7.6}$$

and use it in (7.1), giving the direct solution

$$t_B = t_{SMX} - \pi\sqrt{\frac{a_{SMX}^3}{8}}, \tag{7.7}$$

$$E = -\frac{M^{2/3}}{a_{SMX}}. \tag{7.8}$$

Solutions obviously exist for $R_{SMX} > 0$ and $M > 0$.

C. Velocity profile given at time t_i

The two elliptic cases are found from,

$$\text{EX: } 0 = -\pi + \arccos\left(\frac{b_i^2 - x}{b_i^2 + x}\right) - \sqrt{1 - \left(\frac{b_i^2 - x}{b_i^2 + x}\right)^2} + x^{3/2}(t_{SMX} - t_i) \stackrel{\text{def}}{=} \psi_{SVX}(x), \tag{7.9}$$

$$\text{EC: } 0 = \pi - \arccos\left(\frac{b_i^2 - x}{b_i^2 + x}\right) + \sqrt{1 - \left(\frac{b_i^2 - x}{b_i^2 + x}\right)^2} + x^{3/2}(t_{SMX} - t_i) \stackrel{\text{def}}{=} \psi_{SVC}(x). \tag{7.10}$$

1. Existence of solutions

The complete set of conditions is

$$\text{EX:} \qquad b_i > 0, \qquad t_{SMX} > t_i; \tag{7.11}$$

$$\text{EC:} \qquad b_i < 0, \qquad t_{SMX} < t_i. \tag{7.12}$$

The conditions $b_i > 0$ and $b_i < 0$ follow directly from the assumptions that the model is expanding or collapsing, respectively. The other ones are conditions for solvability of the corresponding equations.

VIII. MODELS WITH GIVEN DENSITY AND VELOCITY PROFILES AT THE SAME TIME

For this case, we are given $R_{,t}(M, t_i)$ and $\rho(M, t_i)$ which provide us with $b_i(M)$ and $a_i(M)$ via Eqs. (2.24), (2.12), and (2.15). We then solve (2.2) for E,

$$2E = (R_{,t})_i^2 - \frac{2M}{R_i} \leftrightarrow \pm x = \frac{2E}{M^{2/3}} = b_i^2 - \frac{2}{a_i}, \tag{8.1}$$

and obtain t_B from one of (2.16)–(2.19). The equation for t_B is sensitive to the sign of b_i (i.e, $R_{,t}$), even though the E equation is not. This method is given in [10].

The simple solution (8.1) does also follow as the limit $t_2 \to t_1$ from the considerations of Paper II, though merely substituting $t_2 = t_1$ leads to degenerate equations. See Appendix A for a proof.

The borderline cases need no special treatment, as there are no numerical difficulties arising from being close to them. The model is parabolic if $b_i^2 = 2/a_i$, and is at maximum expansion (at t_i) if $b_i = 0$.

1. Existence of solutions

Equation (8.1) always has a solution, but, for the purposes of determining t_B, the various types of solutions exist if

$$\text{HX:} \qquad b_i^2 > \frac{2}{a_i} \quad \text{and} \quad b_i > 0; \tag{8.2}$$

$$\text{EX:} \qquad b_i^2 < \frac{2}{a_i} \quad \text{and} \quad b_i > 0; \tag{8.3}$$

$$\text{EC:} \qquad b_i^2 < \frac{2}{a_i} \quad \text{and} \quad b_i < 0; \tag{8.4}$$

$$\text{HC:} \qquad b_i^2 > \frac{2}{a_i} \quad \text{and} \quad b_i < 0. \tag{8.5}$$

IX. MODELS WITH A GIVEN VELOCITY PROFILE AT LATE TIMES

By "late times" we mean the asymptotic future, i.e. the limit $\eta \to \infty$ and $t \to \infty$, so this section applies only to expanding hyperbolic (HX) models. The time reverse—a collapsing hyperbolic (HC) model with a given density or velocity profile in the infinite past—follows in an obvious way.

From (2.6) and (2.2) we find

$$\lim_{\eta\to\infty} \frac{R}{t - t_B} = \sqrt{2E}, \tag{9.1}$$

$$\lim_{t\to\infty} R_{,t} = \sqrt{2E}, \tag{9.2}$$

which gives simply

$$E = \frac{R_{,t_{\rm late}}^2}{2} \Leftrightarrow x = b_{\rm late}^2. \tag{9.3}$$

This always exists, and fully determines $E(M)$, leaving $t_B(M)$ free to be determined by a second requirement.

A. Density profile given at time t_i

Since x is known from (9.3) and t_i, ρ_i, and a_i are finite, we find t_B from:

HX: $$t_B = t_i - [\sqrt{(1 + a_i b_{\mathrm{late}}^2)^2 - 1} - \mathrm{arcosh}(1 + a_i b_{\mathrm{late}}^2)]/b_{\mathrm{late}}^3, \tag{9.4}$$

nPX: $$t_B \approx t_i - \tau_P\left(1 - \frac{3a_i b_{\mathrm{late}}^2}{20} + \frac{9a_i^2 b_{\mathrm{late}}^4}{224}\right), \tag{9.5}$$

which are well defined for all $b_{\mathrm{late}} \geq 0$, but imply (4.5).

1. Existence of solutions

The model is expanding if

$$b_{\mathrm{late}} > 0, \tag{9.6}$$

and the above solution is well defined provided

$$a_i > 0. \tag{9.7}$$

B. Velocity profile given at time t_i

The equation for t_B is

HX: $$t_B = t_i - \left[\sqrt{\left(\frac{b_i^2 + b_{\mathrm{late}}^2}{b_i^2 - b_{\mathrm{late}}^2}\right)^2 - 1} - \mathrm{arcosh}\left(\frac{b_i^2 + b_{\mathrm{late}}^2}{b_i^2 - b_{\mathrm{late}}^2}\right)\right]/b_{\mathrm{late}}^3, \tag{9.8}$$

nPX: $$t_B \approx t_i - \tau_P\left(1 + \frac{6b_{\mathrm{late}}^2}{5b_i^2} + \frac{9b_{\mathrm{late}}^4}{7b_i^4}\right). \tag{9.9}$$

1. Existence of solutions

This solution is well defined (for an expanding model) provided

$$b_i > b_{\mathrm{late}} > 0. \tag{9.10}$$

This does mean that not all choices of $b_{\mathrm{late}}(M)$ and $b_i(M)$ are possible, but it should be easy to choose a suitable b_i after b_{late} is fixed.

C. Simultaneous bang time specified

As soon as the (constant) value of t_B is given, we have all the LT arbitrary functions $E(M)$ and $t_B(M)$.

1. Existence of solutions

Existence only requires that the model be expanding,

$$b_{\mathrm{late}} > 0. \tag{9.11}$$

X. MODELS WITH A GIVEN DENSITY PROFILE AT LATE TIMES

From (2.6) and (2.3) we obtain, using $\partial R/\partial M = R_{,r}/M_{,r}$,

$$\frac{\partial R/\partial M}{(t - t_B)} = \sqrt{2E}\left[\left(\frac{1}{M} - \frac{dE/dM}{E}\right)\frac{(\cosh\eta - 1)}{(\sinh\eta - \eta)} + \left(\frac{3dE/dM}{2E} - \frac{1}{M}\right)\frac{\sinh\eta}{(\cosh\eta - 1)}\right] - \frac{(2E)^2 dt_B/dM}{M}\frac{\sinh\eta}{(\cosh\eta - 1)(\sinh\eta - \eta)}, \tag{10.1}$$

$$\kappa\rho(t - t_B)^3 = \frac{2}{\left(\frac{R}{(t-t_B)}\right)^2\left(\frac{\partial R/\partial M}{(t-t_B)}\right)}. \tag{10.2}$$

Assuming dt_B/dM is finite,[3] we find that

$$\lim_{\eta\to\infty}\frac{\partial R/\partial M}{(t - t_B)} = \frac{dE/dM}{\sqrt{2E}}, \tag{10.3}$$

which with (9.1) leads to

$$\lim_{\eta\to\infty}[\kappa\rho(t - t_B)^3] = \frac{2}{\sqrt{-2E}dE/dM}. \tag{10.4}$$

Clearly $\rho(t - t_B)^3$ freezes out—becomes time independent—and

$$\lim_{\eta\to\infty}\kappa\rho(t - t_B)^3 = \frac{6}{\frac{\mathrm{d}}{\mathrm{d}M}((2E)^{3/2}))}. \tag{10.5}$$

Thus, if we specify the late-time limit $[\rho(t - t_B)^3]_{\mathrm{late}}(M)$, we have

$$x_{\mathrm{late}}^{3/2} = \frac{(2E)^{3/2}}{M} = \frac{1}{M}\int_0^M \frac{6}{\kappa[\rho(t - t_B)^3]_{\mathrm{late}}}\mathrm{d}\tilde{M}. \tag{10.6}$$

This assumes an origin exists, i.e. $E = 0$ at $M = 0$. If not, then we must specify some $E = E_i$ at some $M = M_i$.

Again, (10.6) fully determines $E(M)$, leaving $t_B(M)$ free.

A. Density profile given at time t_i

In this case we find t_B from:

HX: $$t_B = t_i - [\sqrt{(1 + a_i x_{\mathrm{late}})^2 - 1} - \mathrm{arcosh}(1 + a_i x_{\mathrm{late}})]/x_{\mathrm{late}}^{3/2}, \tag{10.7}$$

nPX: $$t_B \approx t_i - \tau_P\left(1 - \frac{3a_i x_{\mathrm{late}}}{20} + \frac{9a_i^2 x_{\mathrm{late}}^2}{224}\right), \tag{10.8}$$

[3] One can consider models in which dt_B/dM becomes infinite while dE/dM does not, either asymptotically or at individual points. However, at such locations there is a permanent zero in the density, and the above limit is not valid. The other cases that give a permanent zero in the density, loci where dE/dM is divergent and dt_B/dM is not, and loci where $M_{,r} = 0$ are not problematic for this limit. (See Paper II for a discussion of regions of zero density.)

which is well defined for all $x_{\rm late} > 0$, but again implies (4.5).

1. Existence of solutions

Equation (10.6) will have a positive solution for x for any positive $[\rho(t - t_B)^3]_{\rm late}(M)$ and the above equation for t_B is well defined for any a_i that is derived from a positive ρ_i.

B. Velocity profile given at time t_i

The equation for t_B is

$$\text{HX:} \qquad t_B = t_i - \left[\sqrt{\left(\frac{b_i^2 + x_{\rm late}}{b_i^2 - x_{\rm late}}\right)^2 - 1} - \mathrm{arcosh}\left(\frac{b_i^2 + x_{\rm late}}{b_i^2 - x_{\rm late}}\right)\right] / x_{\rm late}^{3/2}, \tag{10.9}$$

$$\text{nPX:} \qquad t_B \approx t_i - \tau_P\left(1 + \frac{6x_{\rm late}}{5b_i^2} + \frac{9x_{\rm late}^2}{7b_i^4}\right). \tag{10.10}$$

1. Existence of solutions

The above is well defined provided

$$b_i^2 > x_{\rm late}. \tag{10.11}$$

It should not be hard to choose b_i to satisfy this, once $x_{\rm late}$ is known.

C. Simultaneous bang time specified

Again, setting the constant value of t_B completes the specification of the LT arbitrary functions, and the solution always exists.

D. Late-time velocity profile given

Since specifying the late-time velocity profile via Eq. (9.3), and specifying the late-time density profile via Eq. (10.6) both fix x, this combination is not possible. (Should it happen that the two specifications are consistent, then another ‘‘boundary condition’’ would be required to specify a particular LT model.)

XI. CONCLUSIONS

We have developed several new ways of specifying the ‘‘boundary’’ data needed to uniquely determine the evolution of a Lemaître-Tolman model, and thus provided more options for designing models with particular properties or behaviors. Thus one can now easily generate models that start from an initial stationary state, or have only growing modes, or approach a specified density or velocity profile in the asymptotic future, or approach RW models at late times or diverge from them, etc., as listed in the introduction. The foregoing properties are combined in pairs to fully specify a particular LT model. Although several of the individual properties considered here have previously been used, what is significant here is the equations that result from the many combinations of pairs of properties, and the derivation of existence conditions for the 3 types of solution. Also, our results have been presented in a form that is easily converted into coding for numerical calculations. Current work [13] provides an example of the use of some of the new LT specification methods to create and evolve a model of the Shapley concentration and the great attractor.

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APPENDIX A: DERIVATION OF (8.1) FROM PAPER II RESULTS

Using the results of Sec. IV of Paper II with $t_2 = t_1 = t_i$, and remembering that for the EC case *both* profiles must be specified in the collapsing phase, we obtain

$$\text{HX:} \quad 0 = \sqrt{(1 + a_i x)^2 - 1} - \mathrm{arcosh}(1 + a_i x) - \sqrt{\left(\frac{b_i^2 + x}{b_i^2 - x}\right)^2 - 1} + \mathrm{arcosh}\left(\frac{b_i^2 + x}{b_i^2 - x}\right); \tag{A1}$$

$$\text{EX and EC:} \; 0 = \arccos(1 - a_i x) - \sqrt{1 - (1 - a_i x)^2} + \sqrt{1 - \left(\frac{b_i^2 - x}{b_i^2 + x}\right)^2} - \arccos\left(\frac{b_i^2 - x}{b_i^2 + x}\right). \tag{A2}$$

Denote $1 + a_i x = \cosh u$ and $\frac{b_i^2 + x}{b_i^2 - x} = \cosh v$ for the hyperbolic case. Equation (A1) then is equivalent to $\sinh u - u = \sinh v - v$. But the function $F(y) := \sinh y - y$ is single-valued, so the equation $F(u) = F(v)$ has only one solution, $u = v$. In our case, this solution is $b_i^2 = 2/a_i + x$. Since $x = 2E/M^{2/3}$ in the hyperbolic case, this result is equivalent to (8.1).

The result $b_i^2 = 2/a_i - x$ for the elliptic cases, where $x = -2E/M^{2/3}$, follows in a similar way.

APPENDIX B: CALCULATION OF SERIES EXPANSIONS

Most of the series expansions given above are nontrivial to derive, so a couple of representative calculations are outlined here.

In each case, such as Eqs. (3.2)–(3.4), it is usually possible to do a direct series expansion in powers of some suitable small variable, and then invert it to get a series for x. However, the result is usually not very tidy, and tends to have sign ambiguities. Much better defined and neater results always follow from starting with the parametric evolution equations, (2.4)–(2.6). The calculations were all done to 6th order, using Maple, but only truncated series are written out here.

Let us obtain the near-parabolic series that lies between (3.2) and (3.3). The parabolic limit occurs when $x \to 0$, while R and $\tau = t - t_B$ remain finite. This requires

$$\eta \to 0 \quad \text{and} \quad \frac{\eta}{\sqrt{x}} \to e \tag{B1}$$

so that the new evolution parameter e remains finite for finite τ. Series expansions of (2.6) then give

$$\tau \approx \frac{e^3}{6} + \frac{xe^5}{120} + \frac{x^2e^7}{5040} + \frac{x^3e^9}{362\,880} + \cdots, \tag{B2}$$

$$a \approx \frac{e^2}{2} + \frac{xe^4}{24} + \frac{x^2e^6}{720} + \frac{x^3e^8}{40\,320} + \cdots \tag{B3}$$

Now we invert the above series for τ by writing

$$e \approx e_0 + e_1x + e_2x^2 + e_3x^3 + \cdots, \tag{B4}$$

substituting into (B2), and solving each power of x in turn for the coefficients e_i. We get

$$e \approx (6\tau)^{1/3}\left(1 - \frac{(6\tau)^{2/3}x}{60} + \frac{(6\tau)^{4/3}x^2}{1400} - \frac{(6\tau)^2x^3}{25\,200} + \cdots\right), \tag{B5}$$

which we substitute into (B3),

$$a \approx \frac{(6\tau)^{2/3}}{2}\left(1 - \frac{(6\tau)^{2/3}x}{20} + \frac{3(6\tau)^{4/3}x^2}{2800} - \frac{23(6\tau)^2x^3}{504000} + \cdots\right). \tag{B6}$$

We define τ_P as the time (since the bang) that it would take an exactly parabolic model to expand to a

$$a = \frac{(6\tau_P)}{2}, \tag{B7}$$

and we define $d\tau_P$ as the difference

$$d\tau = \tau - \tau_P. \tag{B8}$$

Then we once more invert the series by writing

$$x \approx x_1 d\tau + x_2 d\tau^2 + x_3 d\tau^3 + \cdots, \tag{B9}$$

substituting into (B6), and solving each power of $d\tau$ in turn for the coefficients x_i, thus obtaining

$$x \approx \frac{20}{3a}\left(-\frac{d\tau_P}{\tau_P} + \frac{25d\tau_P^2}{14\tau_P^2} - \frac{10\,000d\tau_P^3}{3969\tau_P^3} + \cdots\right), \tag{B10}$$

which is (3.5).

As our second example we find the near-maximum-expansion series lying between (3.21) and (3.22). Maximum expansion occurs when $\eta \to \pi$, $b \to 0$, $a \to 2/x$, and $\tau \to \pi/x^{3/2}$. The mean velocity is

$$\frac{a_{\max}}{\tau_{\max}} = \frac{2}{\pi}\left(\frac{\pi}{\tau_{\max}}\right)^{1/3}. \tag{B11}$$

Thus we write

$$\eta \to \pi + e, \tag{B12}$$

expand in powers of this new e,

$$\tau \approx \frac{1}{x^{3/2}}\left(\pi + 2e - \frac{e^3}{6} + \cdots\right), \tag{B13}$$

$$b \approx x\left(\frac{e^2}{4} + \frac{e^4}{24} + \cdots\right), \tag{B14}$$

and invert to get

$$e \approx \frac{2}{\sqrt{x}}\left(b - \frac{b^3}{3x} + \cdots\right). \tag{B15}$$

We then substitute this as well as

$$x \approx x_1 b + x_2 b^2 + x_3 b^3 + \cdots, \tag{B16}$$

into (B13), to get the second series inversion:

$$x \approx \frac{20}{3a}\left(1 + \frac{16b}{3\pi^2\overline{b}} + \frac{64b^2}{3\pi^4\overline{b}^2} - \frac{128(9\pi^2 - 112)b^3}{81\pi^6\overline{b}^3} + \cdots\right), \tag{B17}$$

where

$$\overline{b} = \frac{2}{\pi}\left(\frac{\pi}{\tau_i}\right)^{1/3} \tag{B18}$$

is what the average velocity would be if maximum expansion occurred at τ_i. This gives us (3.24).

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